Catenoids: A Look at Conformal Flatness

by

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More than thirty years ago I proved that a minimal, conformally flat hypersurface in Euclidean space E^{n+1} ($n \geq 4$) is either totally geodesic or (a piece of) a generalized catenoid, i.e. a hypersurface of revolution with a unique profile curve, depending on dimension, which for $n = 2$ is a catenary. In 1999 I. Castro and F. Urbano introduced a Lagrangian catenoid in $\mathbb{C}^n$ and proved that if a minimal, Lagrangian submanifold in $\mathbb{C}^n$ is foliated by round $(n-1)$-spheres, it is homothetic to the Lagrangian catenoid. In view of these two results it is natural to investigate conformally flat, minimal, Lagrangian submanifolds in $\mathbb{C}^n$. If such a submanifold is not totally geodesic, it resembles a Lagrangian catenoid at least to the extent that its Schouten tensor has an eigenvalue of multiplicity one. If the Schouten tensor has at most two eigenvalues, the submanifold is either flat and totally geodesic or is homothetic to the Lagrangian catenoid.

It is well known that a quasi-umbilical submanifold of dimension ≥ 4 of a conformally flat manifold is conformally flat but in general not conversely; e.g. the Lagrangian catenoid of Castro and Urbano is conformally flat but not quasi-umbilical. However a partial converse was given by Moore and Morvan in 1978: If $p \leq \min\{4, n-3\}$, a conformally flat submanifold M^n of Euclidean space E^{n+p} is quasi-umbilical.

Thus we return to the question of a minimal, conformally flat submanifold M^n in Euclidean space E^{n+p}. If $p \leq \min(4, n-3)$ and the Schouten tensor has at most two eigenvalues, then M^n is either totally geodesic or homothetic to a generalized catenoid lying in some $n+1$-dimensional Euclidean space.