Flag Curvature

Miguel Angel Javaloyes

Universidad de Granada

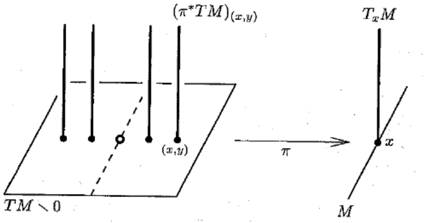
Cedeira 31 octubre a 1 noviembre

Chern Connection

S.S. CHERN (1911-2004)

- $\pi : TM \setminus \{0\} \to M$ is the natural projection

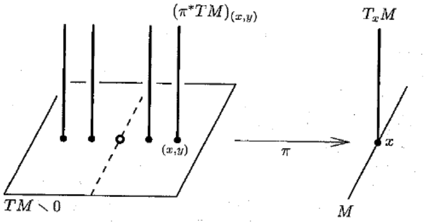
S.S. CHERN (1911-2004)



Chern Connection

- $\pi : TM \setminus \{0\} \to M$ is the natural projection
- now we take the pullback of TM by $d\pi = \pi^*$, that is, $\pi^* TM$

S.S. Chern (1911-2004)

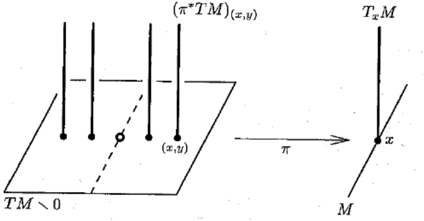


Chern Connection

- $\pi : TM \setminus \{0\} \to M$ is the natural projection
- now we take the pullback of TM by $d\pi = \pi^*$, that is, $\pi^* TM$
- We have a metric over this vector bundle given by $g_{ij}(x, y)dx^i \otimes dx^j$, where

S.S. Chern (1911-2004)

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2(F^2)}{\partial y^i \partial y^j}$$



Chern Connection

Chern Connection

- The *Chern connection* ∇ is the unique linear connection on $\pi^* TM$ whose connection 1-forms ω^i_j satisfy:

Chern Connection

- The *Chern connection* ∇ is the unique linear connection on $\pi^* TM$ whose connection 1-forms ω_j^i satisfy:

$$\mathrm{d}x^j \wedge \omega_j^i = 0 \qquad \textit{torsion free} \tag{1}$$

$$\mathrm{d}g_{ij} - g_{kj}\omega_i^k - g_{ik}\omega_i^k = \frac{2}{F} A_{ijs}\delta y^s \qquad \textit{almost g-compatibility} \tag{2}$$

Chern Connection

- The *Chern connection* ∇ is the unique linear connection on $\pi^* TM$ whose connection 1-forms ω_j^i satisfy:

$$\mathrm{d}x^j \wedge \omega_j^i = 0 \qquad \textit{torsion free} \qquad (1)$$

$$\mathrm{d}g_{ij} - g_{kj}\omega_i^k - g_{ik}\omega_i^k = \frac{2}{F}A_{ijs}\delta y^s \qquad \textit{almost g-compatibility} \qquad (2)$$

where δy^s are the 1-forms on $\pi^* TM$ given as $\delta y^s := \mathrm{d}y^s + N_j^s \mathrm{d}x^j$, and

$$N_j^i(x,y) := \gamma_{jk}^i y^k - \frac{1}{F}A_{jk}^i \gamma_{rs}^k y^r y^s$$

are the coefficients of the so called *nonlinear connection* on $TM \setminus 0$, and

Chern Connection

- The *Chern connection* ∇ is the unique linear connection on $\pi^* TM$ whose connection 1-forms ω_j^i satisfy:

$$\mathrm{d}x^j \wedge \omega_j^i = 0 \qquad \textit{torsion free} \qquad (1)$$

$$\mathrm{d}g_{ij} - g_{kj}\omega_i^k - g_{ik}\omega_i^k = \frac{2}{F} A_{ijs}\delta y^s \qquad \textit{almost g-compatibility} \qquad (2)$$

where δy^s are the 1-forms on $\pi^* TM$ given as $\delta y^s := \mathrm{d}y^s + N_j^s \mathrm{d}x^j$, and

$$N_j^i(x, y) := \gamma_{jk}^i y^k - \frac{1}{F} A_{jk}^i \gamma_{rs}^k y^r y^s$$

are the coefficients of the so called nonlinear connection on $TM \setminus 0$, and

$$\gamma^i{}_{jk}(x, y) = \frac{1}{2} g^{is} \left(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j} \right), A_{ijk}(x, y) = \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k} = \frac{F}{4} \frac{\partial^3 (F^2)}{\partial y^i \partial y^j \partial y^k},$$

Covariant derivatives

Covariant derivatives

- The components of the Chern connection are given by:

$$\Gamma^i_{jk}(x,y) = \gamma^i_{jk} - \frac{g^{il}}{F}\left(A_{ljs}N^s_k - A_{jks}N^s_l + A_{kls}N^s_j\right).$$

that is, $\omega_j{}^i = \Gamma^i{}_{jk}dx^k$.

Covariant derivatives

- The components of the Chern connection are given by:

$$\Gamma^i_{jk}(x,y) = \gamma^i_{jk} - \frac{g^{il}}{F}\left(A_{ljs}N^s_k - A_{jks}N^s_l + A_{kls}N^s_j\right).$$

that is, $\omega_j{}^i = \Gamma^i{}_{jk} dx^k$.

- The Chern connection gives two different covariant derivatives:

$$D_T W = \left(\frac{\mathrm{d}W^i}{\mathrm{d}t} + W^j T^k \Gamma^i_{jk}(\gamma, T)\right) \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)} \quad \textit{with ref. vector } T,$$

$$D_T W = \left(\frac{\mathrm{d}W^i}{\mathrm{d}t} + W^j T^k \Gamma^i_{jk}(\gamma, W)\right) \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)} \quad \textit{with ref. vector } W.$$

Other connections

Other connections

- Cartan connection: metric compatible but has torsion

E. CARTAN (1861-1940)

Other connections

- Cartan connection: metric compatible but has torsion
- Hashiguchi connection

Masao Hashiguchi

E. Cartan (1861-1940)

Other connections

- Cartan connection: metric compatible but has torsion
- Hashiguchi connection
- Berwald connection: no torsion. Specially good to treat with Finsler spaces of constant flag curvature

Masao Hashiguchi

Ludwig Berwald 1883 (Prague)-1942

E. Cartan (1861-1940)

Other connections

- Cartan connection: metric compatible but has torsion
- Hashiguchi connection
- Berwald connection: no torsion. Specially good to treat with Finsler spaces of constant flag curvature
- Rund connection: coincides with Chern connection

MASAO HASHIGUCHI

LUDWIG BERWALD 1883 (PRAGUE)-1942

E. CARTAN (1861-1940)

HANNO RUND 1925-1993, SOUTH AFRICA

Curvature 2-forms of the Chern connection

The curvature 2-forms of the Chern connection are:

$$\Omega_j{}^i := d\omega_j{}^i - \omega_j{}^k \wedge \omega_k{}^i$$

Curvature 2-forms of the Chern connection

The curvature 2-forms of the Chern connection are:

$$\Omega_j{}^i := d\omega_j{}^i - \omega_j{}^k \wedge \omega_k{}^i$$

- It can be expanded as

$$\Omega_j{}^i := \frac{1}{2} R_j{}^i{}_{kl} dx^k \wedge dx^l + P_j{}^i{}_{kl} dx^k \wedge \frac{\delta y^l}{F} + \frac{1}{2} Q_j{}^i{}_{kl} \frac{\delta y^k}{F} \wedge \frac{\delta y^l}{F}$$

Curvature 2-forms of the Chern connection

The curvature 2-forms of the Chern connection are:

$$\Omega_j{}^i := d\omega_j{}^i - \omega_j{}^k \wedge \omega_k{}^i$$

- It can be expanded as

$$\Omega_j{}^i := \frac{1}{2} R_j{}^i{}_{kl} dx^k \wedge dx^l + P_j{}^i{}_{kl} dx^k \wedge \frac{\delta y^l}{F} + \frac{1}{2} Q_j{}^i{}_{kl} \frac{\delta y^k}{F} \wedge \frac{\delta y^l}{F}$$

- From free torsion of the Chern connection $Q_j{}^i{}_{kl} = 0$

Curvature 2-forms of the Chern connection

The curvature 2-forms of the Chern connection are:

$$\Omega_j{}^i := d\omega_j{}^i - \omega_j{}^k \wedge \omega_k{}^i$$

- It can be expanded as

$$\Omega_j{}^i := \frac{1}{2} R_j{}^i{}_{kl} dx^k \wedge dx^l + P_j{}^i{}_{kl} dx^k \wedge \frac{\delta y^l}{F}$$

- From free torsion of the Chern connection $Q_j{}^i{}_{kl} = 0$

Curvature 2-forms of the Chern connection

The curvature 2-forms of the Chern connection are:

$$\Omega_j{}^i := d\omega_j{}^i - \omega_j{}^k \wedge \omega_k{}^i$$

- It can be expanded as

$$\Omega_j{}^i := \frac{1}{2} R_j{}^i{}_{kl} dx^k \wedge dx^l + P_j{}^i{}_{kl} dx^k \wedge \frac{\delta y^l}{F}$$

- From free torsion of the Chern connection $Q_j{}^i{}_{kl} = 0$
- $R_j{}^i{}_{kl} = \frac{\delta \Gamma^i{}_{jl}}{\delta x^k} - \frac{\delta \Gamma^i{}_{jk}}{\delta x^k} + \Gamma^i{}_{hk} \Gamma^h{}_{jl} - \Gamma^i{}_{hl} \Gamma^h{}_{jk}$ $\left(\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - N^i \frac{\partial}{\partial y^i}\right)$

Curvature 2-forms of the Chern connection

The curvature 2-forms of the Chern connection are:

$$\Omega_j{}^i := d\omega_j{}^i - \omega_j{}^k \wedge \omega_k{}^i$$

- It can be expanded as

$$\Omega_j{}^i := \frac{1}{2} R_j{}^i{}_{kl} dx^k \wedge dx^l + P_j{}^i{}_{kl} dx^k \wedge \frac{\delta y^l}{F}$$

- From free torsion of the Chern connection $Q_j{}^i{}_{kl} = 0$
- $R_j{}^i{}_{kl} = \frac{\delta \Gamma^i{}_{jl}}{\delta x^k} - \frac{\delta \Gamma^i{}_{jk}}{\delta x^k} + \Gamma^i{}_{hk} \Gamma^h{}_{jl} - \Gamma^i{}_{hl} \Gamma^h{}_{jk}$ $\left(\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - N^i \frac{\partial}{\partial y^i}\right)$
- $P_j{}^i{}_{kl} = -F \frac{\partial \Gamma^i{}_{jk}}{\partial y^l}$

Bianchi Identities

First Bianchi Identity for R

LUIGI BIANCHI (1856-1928)

Bianchi Identities

First Bianchi Identity for R

- $R_j{}^i{}_{kl} + R_k{}^i{}_{lj} + R_l{}^i{}_{jk} = 0$

Luigi Bianchi (1856-1928)

Bianchi Identities

First Bianchi Identity for R

- $R_j{}^i{}_{kl} + R_k{}^i{}_{lj} + R_l{}^i{}_{jk} = 0$

Other identities:

- $P_k{}^i{}_{jl} = P_j{}^i{}_{kl}$

LUIGI BIANCHI (1856-1928)

Bianchi Identities

First Bianchi Identity for R

- $R_j{}^i{}_{kl} + R_k{}^i{}_{lj} + R_l{}^i{}_{jk} = 0$

Other identities:

- $P_k{}^i{}_{jl} = P_j{}^i{}_{kl}$
- $R_{ijkl} + R_{jikl} = 2B_{ijkl}$, where $B_{ijkl} := -A_{iju}R^u{}_{kl}$, $R^u{}_{kl} = \frac{y^j}{F}R_j{}^u{}_{kl}$ and $R_{ijkl} = g_{j\mu}R_i{}^{\mu}{}_{kl}$

LUIGI BIANCHI (1856-1928)

Bianchi Identities

First Bianchi Identity for R

- $R_{j\ kl}^{\ i} + R_{k\ lj}^{\ i} + R_{l\ jk}^{\ i} = 0$

Other identities:

- $P_{k\ jl}^{\ i} = P_{j\ kl}^{\ i}$
- $R_{ijkl} + R_{jikl} = 2B_{ijkl}$, where $B_{ijkl} := -A_{iju}R^{u}{}_{kl}$, $R^{u}{}_{kl} = \frac{y^j}{F} R_{j\ kl}^{\ u}$ and $R_{ijkl} = g_{j\mu} R_{i\ kl}^{\ \mu}$
- $R_{klji} - R_{jikl} = (B_{klji} - B_{jikl}) + (B_{kilj} + B_{ljki}) + (B_{ilji} + B_{jkil})$

LUIGI BIANCHI (1856-1928)

First Bianchi Identity for R

- $R_j{}^i{}_{kl} + R_k{}^i{}_{lj} + R_l{}^i{}_{jk} = 0$

Other identities:

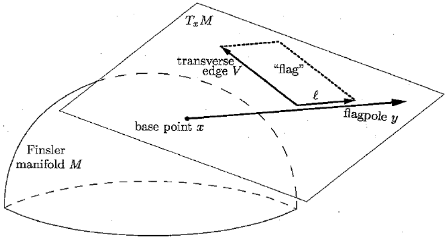
- $P_k{}^i{}_{jl} = P_j{}^i{}_{kl}$
- $R_{ijkl} + R_{jikl} = 2B_{ijkl}$, where $B_{ijkl} := -A_{iju}R^u{}_{kl}$, $R^u{}_{kl} = \frac{y^j}{F} R_j{}^u{}_{kl}$ and $R_{ijkl} = g_{j\mu} R_i{}^\mu{}_{kl}$
- $R_{klji} - R_{jikl} = (B_{klji} - B_{jikl}) + (B_{kilj} + B_{ljki}) + (B_{ilji} + B_{jkil})$

Second Bianchi identities: very complicated, mix terms in $R_j{}^i{}_{kl}$ and $P_j{}^i{}_{kl}$

LUIGI BIANCHI (1856-1928)

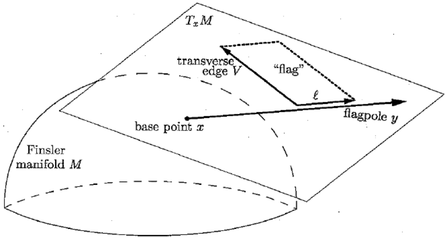
Flag Curvature

We must fix a flagpole y and then a transverse edge V



We must fix a flagpole y and then a transverse edge V

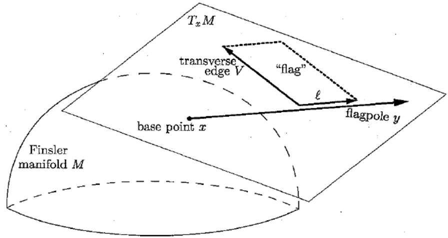
$$K(y,V) := \frac{V^i (y^j R_{jikl} y^l) V^k}{g(y,y)g(V,V) - g(y,V)^2}$$



We must fix a flagpole y and then a transverse edge V

$$K(y,V) := \frac{V^i (y^j R_{jikl} y^l) V^k}{g(y,y)g(V,V) - g(y,V)^2}$$

- We can change V by $W = \alpha V + \beta y$, that is, $K(y,W) = K(y,V)$.

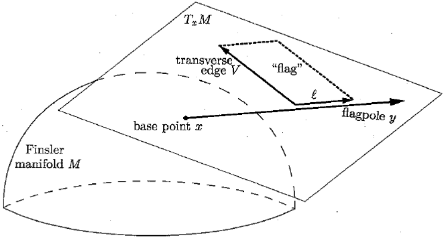


Flag Curvature

We must fix a flagpole y and then a transverse edge V

$$K(y,V) := \frac{V^i(y^j R_{jikl} y^l)V^k}{g(y,y)g(V,V) - g(y,V)^2}$$

- We can change V by $W = \alpha V + \beta y$, that is, $K(y,W) = K(y,V)$.
- We obtain the same quantity with the other connections (Cartan, Berwald, Hasiguchi...



Computing Flag curvature

Computing Flag curvature

- $G^i := \gamma^i{}_{jk} y^j y^k$ (spray coefficients)

Computing Flag curvature

- $G^i := \gamma^i{}_{jk} y^j y^k$ (spray coefficients)
- $2F^2 R^i{}_k = 2(G^i)_{x^k} - \frac{1}{2}(G^i)_{y^j}(G^j)_{y^k} - y^j (G^i)_{y^k x^j} + G^j (G^i)_{y^k y^j}$

Computing Flag curvature

- $G^i := \gamma^i{}_{jk} y^j y^k$ (spray coefficients)
- $2F^2 R^i{}_k = 2(G^i)_{x^k} - \frac{1}{2}(G^i)_{y^j}(G^j)_{y^k} - y^j (G^i)_{y^k x^j} + G^j (G^i)_{y^k y^j}$
- $K(y, V) = K(l, V) = \frac{V_i (R^i{}_k) V^k}{g(V,V) - g(l,V)^2}$, where $l = y/F$.

If we consider $F(x, y) = \sqrt{\langle y, y \rangle} + df[y]$, with $\langle \cdot, \cdot \rangle$ the Euclidean metric, then

Computing Flag curvature

- $G^i := \gamma^i{}_{jk} y^j y^k$ (spray coefficients)
- $2F^2 R^i{}_k = 2(G^i)_{x^k} - \frac{1}{2}(G^i)_{y^j}(G^j)_{y^k} - y^j (G^i)_{y^k x^j} + G^j (G^i)_{y^k y^j}$
- $K(y,V) = K(l,V) = \frac{V_i (R^i{}_k) V^k}{g(V,V) - g(l,V)^2}$, where $l = y/F$.

If we consider $F(x,y) = \sqrt{\langle y, y\rangle} + df[y]$, with $\langle \cdot, \cdot \rangle$ the Euclidean metric, then

- $G^i = \frac{1}{F} f_{x^j x^k} y^j y^k$, very simple!!!

Computing Flag curvature

- $G^i := \gamma^i{}_{jk} y^j y^k$ (spray coefficients)
- $2F^2 R^i{}_k = 2(G^i)_{x^k} - \frac{1}{2}(G^i)_{y^j}(G^j)_{y^k} - y^j (G^i)_{y^k x^j} + G^j (G^i)_{y^k y^j}$
- $K(y, V) = K(l, V) = \frac{V_i (R^i{}_k) V^k}{g(V,V) - g(l,V)^2}$, where $l = y/F$.

If we consider $F(x, y) = \sqrt{\langle y, y \rangle} + df[y]$, with $\langle \cdot, \cdot \rangle$ the Euclidean metric, then

- $G^i = \frac{1}{F} f_{x^j x^k} y^j y^k$, very simple!!!
- $K(y, V) = K(x, y) = \frac{3}{4F^4}(f_{x^i x^j} y^i y^j)^2 - \frac{1}{2F^3}(f_{x^i x^j x^k} y^i y^j y^k)$

Computing Flag curvature

- $G^i := \gamma^i{}_{jk} y^j y^k$ (spray coefficients)
- $2F^2 R^i{}_k = 2(G^i)_{x^k} - \frac{1}{2}(G^i)_{y^j}(G^j)_{y^k} - y^j (G^i)_{y^k x^j} + G^j (G^i)_{y^k y^j}$
- $K(y,V) = K(l,V) = \frac{V_i (R^i{}_k) V^k}{g(V,V) - g(l,V)^2}$, where $l = y/F$.

If we consider $F(x,y) = \sqrt{\langle y, y\rangle} + df[y]$, with $\langle \cdot, \cdot \rangle$ the Euclidean metric, then

- $G^i = \frac{1}{F} f_{x^j x^k} y^j y^k$, very simple!!!
- $K(y,V) = K(x,y) = \frac{3}{4F^4}(f_{x^i x^j} y^i y^j)^2 - \frac{1}{2F^3}(f_{x^i x^j x^k} y^i y^j y^k)$
- the flag curvature does not depend on the transverse edge!! it is scalar

Finsler metric with constant flag curvature

Finsler metric with constant flag curvature

- The complete classification is an open problem, no Hopf's theorem!!!

Finsler metric with constant flag curvature

- The complete classification is an open problem, no Hopf's theorem!!!
- In the class of Randers metrics there does exist a classification after a long story

Finsler metric with constant flag curvature

- The complete classification is an open problem, no Hopf's theorem!!!
- In the class of Randers metrics there does exist a classification after a long story
- In 1977 Yasuda and Shimada publishes a paper with a characterization of Randers metrics of scalar flag curvature

Hiroshi Yasuda (1925-1995)

Finsler metric with constant flag curvature

- The complete classification is an open problem, no Hopf's theorem!!!
- In the class of Randers metrics there does exist a classification after a long story
- In 1977 Yasuda and Shimada publishes a paper with a characterization of Randers metrics of scalar flag curvature
- As a particular case they obtain the Randers metrics of constant flag curvature

Hiroshi Yasuda (1925-1995)

Finsler metric with constant flag curvature

- The complete classification is an open problem, no Hopf's theorem!!!
- In the class of Randers metrics there does exist a classification after a long story
- In 1977 Yasuda and Shimada publishes a paper with a characterization of Randers metrics of scalar flag curvature
- As a particular case they obtain the Randers metrics of constant flag curvature
- Shibata-Kitayama in 1984 and Matsumoto in 1989 obtain alternate derivations of the Yasuda-Shimada theorem

Hiroshi Yasuda (1925-1995)

Makoto Matsumoto

Finsler metric with constant flag curvature

- The complete classification is an open problem, no Hopf's theorem!!!
- In the class of Randers metrics there does exist a classification after a long story
- In 1977 Yasuda and Shimada publishes a paper with a characterization of Randers metrics of scalar flag curvature
- As a particular case they obtain the Randers metrics of constant flag curvature
- Shibata-Kitayama in 1984 and Matsumoto in 1989 obtain alternate derivations of the Yasuda-Shimada theorem
- In summer 2000, P. Antonelli asks if Yasuda-Shimada theorem is indeed correct

HIROSHI YASUDA (1925-1995)

MAKOTO MATSUMOTO

Constant flag curvature and Zermelo metrics

Constant flag curvature and Zermelo metrics

- In the academic year of 2000-2001 Colleen Robles (a graduate student) and David Bao begins to work in a geometrical proof of Yasuda-Shimada theorem

Constant flag curvature and Zermelo metrics

- In the academic year of 2000-2001 Colleen Robles (a graduate student) and David Bao begins to work in a geometrical proof of Yasuda-Shimada theorem
- In 17th may 2001 Z. Shen phones D. Bao describing a counterexample to Yasuda-Shimada he found when working with Zermelo metrics

Constant flag curvature and Zermelo metrics

- In the academic year of 2000-2001 Colleen Robles (a graduate student) and David Bao begins to work in a geometrical proof of Yasuda-Shimada theorem
- In 17th may 2001 Z. Shen phones D. Bao describing a counterexample to Yasuda-Shimada he found when working with Zermelo metrics
- In the same year D. Bao-C. Robles and Matsumoto find independently the correct version of Yasuda-Shimada theorem.

Constant flag curvature and Zermelo metrics

- In the academic year of 2000-2001 Colleen Robles (a graduate student) and David Bao begins to work in a geometrical proof of Yasuda-Shimada theorem
- In 17th may 2001 Z. Shen phones D. Bao describing a counterexample to Yasuda-Shimada he found when working with Zermelo metrics
- In the same year D. Bao-C. Robles and Matsumoto find independently the correct version of Yasuda-Shimada theorem.
- Still no classification (solutions $\sqrt{h} + h(W, v)$ must have a h-Riemannian curvature related with the module of a h-Killing field W)

Constant flag curvature and Zermelo metrics

- In the academic year of 2000-2001 Colleen Robles (a graduate student) and David Bao begins to work in a geometrical proof of Yasuda-Shimada theorem
- In 17th may 2001 Z. Shen phones D. Bao describing a counterexample to Yasuda-Shimada he found when working with Zermelo metrics
- In the same year D. Bao-C. Robles and Matsumoto find independently the correct version of Yasuda-Shimada theorem.
- Still no classification (solutions $\sqrt{h} + h(W, v)$ must have a h-Riemannian curvature related with the module of a h-Killing field W)
- Finally they perceive that when considering Zermelo expression of Randers metrics the geometry comes out

Flag constant curvature and stationary spacetimes

Flag constant curvature and stationary spacetimes

- Zermelo metric:

$$\sqrt{\frac{1}{\alpha}g(v,v)+\frac{1}{\alpha^2}g(W,v)^2}-\frac{1}{\alpha}g(W,v),$$

where $\alpha = 1 - g(W,W)$.

Flag constant curvature and stationary spacetimes

- Zermelo metric:

$$\sqrt{\frac{1}{\alpha}g(v,v)+\frac{1}{\alpha^2}g(W,v)^2}-\frac{1}{\alpha}g(W,v),$$

  where $\alpha = 1 - g(W,W)$.

- Randers space forms are those Zermelo metrics having h of constant curvature and W a conformal Killing field

Flag constant curvature and stationary spacetimes

- Zermelo metric:

$$\sqrt{\frac{1}{\alpha}g(v,v)+\frac{1}{\alpha^2}g(W,v)^2}-\frac{1}{\alpha}g(W,v),$$

  where $\alpha = 1 - g(W,W)$.
- Randers space forms are those Zermelo metrics having h of constant curvature and W a conformal Killing field
- Katok metrics are Randers space forms

Flag constant curvature and stationary spacetimes

- Zermelo metric:

$$\sqrt{\frac{1}{\alpha}g(v,v)+\frac{1}{\alpha^2}g(W,v)^2}-\frac{1}{\alpha}g(W,v),$$

  where $\alpha = 1 - g(W,W)$.
- Randers space forms are those Zermelo metrics having h of constant curvature and W a conformal Killing field
- Katok metrics are Randers space forms
- When the Fermat metric associated to a stationary spacetime is of constant flag curvature, then the spacetime is locally conformally flat

Flag constant curvature and stationary spacetimes

- Zermelo metric:

$$\sqrt{\frac{1}{\alpha}g(v,v)+\frac{1}{\alpha^2}g(W,v)^2}-\frac{1}{\alpha}g(W,v),$$

  where $\alpha = 1 - g(W,W)$.
- Randers space forms are those Zermelo metrics having h of constant curvature and W a conformal Killing field
- Katok metrics are Randers space forms
- When the Fermat metric associated to a stationary spacetime is of constant flag curvature, then the spacetime is locally conformally flat
- Reciprocal is not true ($\sqrt{h}+df$)

Flag constant curvature and stationary spacetimes

- Zermelo metric:

$$\sqrt{\frac{1}{\alpha}g(v,v)+\frac{1}{\alpha^2}g(W,v)^2}-\frac{1}{\alpha}g(W,v),$$

  where $\alpha = 1 - g(W,W)$.
- Randers space forms are those Zermelo metrics having h of constant curvature and W a conformal Killing field
- Katok metrics are Randers space forms
- When the Fermat metric associated to a stationary spacetime is of constant flag curvature, then the spacetime is locally conformally flat
- Reciprocal is not true $(\sqrt{h}+df)$
- what about scalar flag curvature?

Schur's Lemma

Theorem

Let M be a Riemannian manifold with dimension ≥ 3. If for every point $x \in M$ the sectional curvature does not depend on the on the plain, then M has constant sectional curvature.

ISSAI SCHUR (1875-1941)

Schur's Lemma

Theorem

Let M be a Riemannian manifold with dimension ≥ 3. If for every point $x \in M$ the sectional curvature does not depend on the on the plain, then M has constant sectional curvature.

- It was established by Issai Schur (1875-1941)

Issai Schur (1875-1941)

Schur's Lemma

Theorem

Let M be a Riemannian manifold with dimension ≥ 3. If for every point $x \in M$ the sectional curvature does not depend on the on the plain, then M has constant sectional curvature.

Issai Schur (1875-1941)

- It was established by Issai Schur (1875-1941)
- Generalized to Finsler manifolds by Lilia del Riego in her Phd. Thesis in 1973.

Gauss-Bonnet Theorem

Theorem

Suppose M is a 2-dim compact Riemannian manifold with boundary ∂M. Then $\int_M K\ dA + \int_{\partial M} k_g\ ds = 2\pi\chi(M)$,

Gauss-Bonnet Theorem

Theorem

Suppose M is a 2-dim compact Riemannian manifold with boundary ∂M. Then $\int_M K\, dA + \int_{\partial M} k_g\, ds = 2\pi\chi(M)$,

- Gauss knew a version but never published it

Carl F. Gauss (1777-1855)

Theorem

Suppose M is a 2-dim compact Riemannian manifold with boundary ∂M. Then $\int_M K\, dA + \int_{\partial M} k_g\, ds = 2\pi\chi(M)$,

- Gauss knew a version but never published it
- Bonnet published a version in 1848

Pierre O. Bonnet (1819-1892)

Gauss-Bonnet Theorem

Theorem

Suppose M is a 2-dim compact Riemannian manifold with boundary ∂M. Then $\int_M K\, dA + \int_{\partial M} k_g\, ds = 2\pi\chi(M)$,

- Gauss knew a version but never published it
- Bonnet published a version in 1848
- Allendoerfer-Weil-Chern generalized Gauss-Bonnet to even dimensions using the Pfaffian in the mid-40's

S. S. Chern (1911-2004)

C. Allendoerfer (1911-1974)

André Weil (1906-1998)

Gauss-Bonnet Theorem

Theorem

Suppose M is a 2-dim compact Riemannian manifold with boundary ∂M. Then $\int_M K\, dA + \int_{\partial M} k_g\, ds = 2\pi\chi(M)$,

- Gauss knew a version but never published it
- Bonnet published a version in 1848
- Allendoerfer-Weil-Chern generalized Gauss-Bonnet to even dimensions using the Pfaffian in the mid-40's
- Lichnerowitz (Comm. Helv. Math. 1949) extends the theorem to the Finsler setting in some particular cases

André Lichnerowitz (1915-1998)

Gauss-Bonnet Theorem

Theorem

Suppose M is a 2-dim compact Riemannian manifold with boundary ∂M. Then
$\int_M K\, dA + \int_{\partial M} k_g\, ds = 2\pi\chi(M)$,

- Gauss knew a version but never published it
- Bonnet published a version in 1848
- Allendoerfer-Weil-Chern generalized Gauss-Bonnet to even dimensions using the Pfaffian in the mid-40's
- Lichnerowitz (Comm. Helv. Math. 1949) extends the theorem to the Finsler setting in some particular cases
- Bao-Chern (Ann. Math. 1996) extend it to a wider class of Finsler manifolds

David Bao and S. S. Chern

Bonnet-Myers Theorem

Theorem

If Ricci curvature of a complete Riemannian manifold M is bounded below by $(n-1)k > 0$, *then its diameter is at most* $\pi/\sqrt{k}$ *and the manifold is compact.*

Bonnet-Myers Theorem

Theorem

If Ricci curvature of a complete Riemannian manifold M is bounded below by $(n-1)k > 0$, *then its diameter is at most* $\pi/\sqrt{k}$ *and the manifold is compact.*

- Pierre Ossian Bonnet (1819-1892) obtained a version bounding from above the sectional curvatures with a positive constant

PIERRE O. BONNET (1819-1892)

Bonnet-Myers Theorem

Theorem

If Ricci curvature of a complete Riemannian manifold M is bounded below by $(n-1)k > 0$, then its diameter is at most $\pi/\sqrt{k}$ and the manifold is compact.

- Pierre Ossian Bonnet (1819-1892) obtained a version bounding from above the sectional curvatures with a positive constant
- Myers obtained the generalized version with *Ric* curvatures in 1941

Bonnet-Myers Theorem

Theorem

If Ricci curvature of a complete Riemannian manifold M is bounded below by $(n-1)k > 0$, then its diameter is at most $\pi/\sqrt{k}$ and the manifold is compact.

- Pierre Ossian Bonnet (1819-1892) obtained a version bounding from above the sectional curvatures with a positive constant
- Myers obtained the generalized version with *Ric* curvatures in 1941
- Louis Auslander extended the result to the Finsler setting in 1955 (Trans AMS)

Bonnet-Myers Theorem

Theorem

If Ricci curvature of a complete Riemannian manifold M is bounded below by $(n-1)k > 0$, *then its diameter is at most* $\pi/\sqrt{k}$ *and the manifold is compact.*

Bonnet-Myers Theorem

Theorem

If Ricci curvature of a complete Riemannian manifold M is bounded below by $(n-1)k > 0$, then its diameter is at most $\pi/\sqrt{k}$ and the manifold is compact.

D. Bao, S.S. Chern and Z. Shen

- Bao-Chern-Chen assume just forward completeness in their book "Introduction to Riemann-Finsler geometry"

Bonnet-Myers Theorem

Theorem

If Ricci curvature of a complete Riemannian manifold M is bounded below by $(n-1)k > 0$, then its diameter is at most $\pi/\sqrt{k}$ and the manifold is compact.

- Bao-Chern-Chen assume just forward completeness in their book "Introduction to Riemann-Finsler geometry"
- Causality reveals that completeness can be substituted by the condition

$$B^+(x,r) \cap B^-(x,r) \text{ compact for all } x \in M \text{ and } r > 0$$

(see Caponio-M.A.J.-Sánchez, preprint 09)

Synge's Theorem

Theorem

If M is an even-dimensional, oriented, complete and connected manifold, with all the sectional curvatures bounded by some positive constant, then M is simply connected.

Synge's Theorem

Theorem

If M is an even-dimensional, oriented, complete and connected manifold, with all the sectional curvatures bounded by some positive constant, then M is simply connected.

- John Lighton Synge (1897-1995) published this result in 1936 (Quaterly Journal of Mathematics).

JOHN SYNGE (1897-1995)

Synge's Theorem

Theorem

If M is an even-dimensional, oriented, complete and connected manifold, with all the sectional curvatures bounded by some positive constant, then M is simply connected.

- John Lighton Synge (1897-1995) published this result in 1936 (Quaterly Journal of Mathematics).
- Louis Auslander(1928-1997) extends the result for Finsler manifolds in 1955

JOHN SYNGE (1897-1995)

Synge's Theorem

Theorem

If M is an even-dimensional, oriented, complete and connected manifold, with all the sectional curvatures bounded by some positive constant, then M is simply connected.

- John Lighton Synge (1897-1995) published this result in 1936 (Quaterly Journal of Mathematics).
- Louis Auslander(1928-1997) extends the result for Finsler manifolds in 1955
- Again the completeness condition can be weakened.

JOHN SYNGE (1897-1995)

Cartan-Hadamard Theorem

Theorem

If M is a geodesically complete connected Riemannian manifold of non positive sectional curvature. Then

- *Geodesics do not have conjugate points*
- $\exp_p : T_pM \longrightarrow M$ *is globally defined and a local diffeorphism*
- *If M simply connected, then $\exp_p$ is a diffeomorphism*

Cartan-Hadamard Theorem

Theorem

If M is a geodesically complete connected Riemannian manifold of non positive sectional curvature. Then

- *Geodesics do not have conjugate points*
- $\exp_p : T_pM \longrightarrow M$ *is globally defined and a local diffeorphism*
- *If M simply connected, then $\exp_p$ is a diffeomorphism*

- Obtained for surfaces in 1898 by Hadamard

Jacques Hadamard (1865-1963)

Élie Cartan (1869-1951)

Cartan-Hadamard Theorem

Theorem

If M is a geodesically complete connected Riemannian manifold of non positive sectional curvature. Then

- *Geodesics do not have conjugate points*
- $\exp_p : T_pM \longrightarrow M$ *is globally defined and a local diffeorphism*
- *If M simply connected, then $\exp_p$ is a diffeomorphism*

Jacques Hadamard (1865-1963)

- Obtained for surfaces in 1898 by Hadamard
- Generalized for every dimension by Cartan

Élie Cartan (1869-1951)

Cartan-Hadamard Theorem

Theorem

If M is a geodesically complete connected Riemannian manifold of non positive sectional curvature. Then

- *Geodesics do not have conjugate points*
- $\exp_p : T_pM \longrightarrow M$ *is globally defined and a local diffeorphism*
- *If M simply connected, then $\exp_p$ is a diffeomorphism*

JACQUES HADAMARD (1865-1963)

ÉLIE CARTAN (1869-1951)

- Obtained for surfaces in 1898 by Hadamard
- Generalized for every dimension by Cartan
- Extended to Finsler manifolds in 1955 by L. Auslander

Rauch's Comparison Theorem

Theorem

For large curvature, geodesics tend to converge, while for small (or negative) curvature, geodesics tend to spread.

Rauch's Comparison Theorem

Theorem

For large curvature, geodesics tend to converge, while for small (or negative) curvature, geodesics tend to spread.

- Proved in the 40's by A. D. Aleksandrov for surfaces

A. D. ALEKSANDROV (1912-1999)

Rauch's Comparison Theorem

Theorem

For large curvature, geodesics tend to converge, while for small (or negative) curvature, geodesics tend to spread.

- Proved in the 40's by A. D. Aleksandrov for surfaces
- Generalized to Riemannian manifolds in 1951 by H. E. Rauch

A. D. ALEKSANDROV (1912-1999)

Rauch's Comparison Theorem

Theorem

For large curvature, geodesics tend to converge, while for small (or negative) curvature, geodesics tend to spread.

- Proved in the 40's by A. D. Aleksandrov for surfaces
- Generalized to Riemannian manifolds in 1951 by H. E. Rauch
- Probably P. Dazord was the first one in giving the generalized Rauch theorem in 1968

A. D. Aleksandrov (1912-1999)

Sphere Theorem

Theorem

A simply connected connected manifold with $\frac{1}{4} < K \leq 1$ *is homeomorphic to the sphere.*

Sphere Theorem

Theorem

A simply connected connected manifold with $\frac{1}{4} < K \leq 1$ is homeomorphic to the sphere.

- Conjecture by Rauch. First proof by M. Berger in 1960

Sphere Theorem

Theorem

A simply connected connected manifold with $\frac{1}{4} < K \leq 1$ is homeomorphic to the sphere.

- Conjecture by Rauch. First proof by M. Berger in 1960
- Alternative proof by Klingenberg in 1961 (obtaining homotopy equivalence rather than homeomorphism)

Sphere Theorem

Theorem

A simply connected connected manifold with $\frac{1}{4} < K \leq 1$ is homeomorphic to the sphere.

- Conjecture by Rauch. First proof by M. Berger in 1960
- Alternative proof by Klingenberg in 1961 (obtaining homotopy equivalence rather than homeomorphism)
- Dazord observes that Klingeberg proof works for reversible Finsler metrics in 1968

Sphere theorem

Theorem

A simply connected connected manifold with $\frac{1}{4} < K \leq 1$ *is homeomorphic to the sphere.*

Sphere theorem

Theorem

A simply connected connected manifold with $\frac{1}{4} < K \leq 1$ *is homeomorphic to the sphere.*

- In 2004 H. B. Rademacher (Math. Ann.) extends Klingenberg proof to non-reversible Finsler metrics using the hypothesis $\left(1 - \frac{1}{1+\lambda}\right)^2 < K \leq 1$), where $\lambda = \max\{F(-X) : F(X) = 1\}$

Sphere theorem

Theorem

A simply connected connected manifold with $\frac{1}{4} < K \leq 1$ is homeomorphic to the sphere.

- In 2004 H. B. Rademacher (Math. Ann.) extends Klingenberg proof to non-reversible Finsler metrics using the hypothesis $\left(1 - \frac{1}{1+\lambda}\right)^2 < K \leq 1$), where $\lambda = \max\{F(-X) : F(X) = 1\}$
- In 2007 S. Brendle and R. Schoen (J. Amer. Math. Soc 2009) prove by using Ricci-flow that there exists a diffeomorphism

Sphere theorem

Theorem

A simply connected connected manifold with $\frac{1}{4} < K \leq 1$ is homeomorphic to the sphere.

- In 2004 H. B. Rademacher (Math. Ann.) extends Klingenberg proof to non-reversible Finsler metrics using the hypothesis $\left(1 - \frac{1}{1+\lambda}\right)^2 < K \leq 1$), where $\lambda = \max\{F(-X) : F(X) = 1\}$
- In 2007 S. Brendle and R. Schoen (J. Amer. Math. Soc 2009) prove by using Ricci-flow that there exists a diffeomorphism
- To obtain Rademacher's result it is enough symmetrized compact balls and bounded reversivility index

Inextendible theorems

Inextendible theorems

- Toponogov theorem? Problems with angles

VICTOR A. TOPONOGOV (1930-2004)

Inextendible theorems

- Toponogov theorem? Problems with angles
- Submanifold theory (very difficult)

Victor A. Toponogov (1930-2004)

Inextendible theorems

- Toponogov theorem? Problems with angles
- Submanifold theory (very difficult)
- Laplacian theory

VICTOR A. TOPONOGOV (1930-2004)

Bibliography

D. Bao, S.-S. Chern, and Z. Shen, *An introduction to Riemann-Finsler geometry*, vol. 200 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.

Bibliography

D. Bao, S.-S. Chern, and Z. Shen, *An introduction to Riemann-Finsler geometry*, vol. 200 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.

Bibliography

D. Bao, S.-S. Chern, and Z. Shen, *An introduction to Riemann-Finsler geometry*, vol. 200 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.

Bibliography

D. Bao, S.-S. Chern, and Z. Shen, *An introduction to Riemann-Finsler geometry*, vol. 200 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.

Bibliography

D. Bao, S.-S. Chern, and Z. Shen, *An introduction to Riemann-Finsler geometry*, vol. 200 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.