Cohomology of Quaternionic Foliations and Orbifolds

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based on joint works with **R.A. Wolak**

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Outline

Section 1

Kähler case

Let (M, ω, J) be a Kähler manifold. The so-called Lefschetz operator is defined as follows:

$$L\colon H^k(M) \longrightarrow H^{k+2}(M), \; L([\alpha]) = [\omega \wedge \alpha].$$

Let (M, ω, J) be a Kähler manifold. The so-called Lefschetz operator is defined as follows:

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Theorem (Hard Lefschetz theorem)

Let (M^n, ω, J) be a compact Kähler manifold. The homomorphism

$$L^r\colon H^{n-r}(M) \longrightarrow H^{n+r}(M),\ \ L([\alpha]) = [\omega \wedge \alpha].$$

is an isomorphism for all $r \geq 0$.

Let $\Lambda\colon H^k(M) \longrightarrow H^{k-2}(M)$ be the formal adjoint of L. A k-form α is called **effective (or primitive)**, if $\Lambda\alpha = 0$. Let P^k be the space of all effective k-forms.

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The hard Lefschetz theorem then implies the following isomorphism, which is **the Lefschetz decomposition**:

$$H^k(M) = \bigoplus_{r\geq 0} L^r(P^{k-2r}).$$

Section 2

Quaternionic case

Let I, J, K be three almost complex structures on a $4n$-dimensional manifold M, such that they satisfy $I \circ J = K$ and its cyclic permutations, then the ordered triple $H = (I, J, K)$ on M is called an **almost hypercomplex structure**.

Let I, J, K be three almost complex structures on a $4n$-dimensional manifold M, such that they satisfy $I \circ J = K$ and its cyclic permutations, then the ordered triple $H = (I, J, K)$ on M is called an **almost hypercomplex structure**. An **almost quaternionic structure** on the manifold M is a rank 3 vector subbundle Q of the endomorphism bundle $End(TM)$ which locally is spanned by an almost hypercomplex structure $H = (I, J, K)$ which are transformed by $SO(3)$ on the their respective domains of existence.

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Quaternion Kähler manifolds are Riemannian manifolds (M, g) of real dimension 4n whose holonomy group can be reduced to $Sp(n).Sp(1)$. In dimension $4(n = 1)$ this condition means only that the manifold is Riemannian as $Sp(1).Sp(1) = SO(4)$. Therefore this condition is meaningful for $n \geq 2$.

Section 3

Foliation

Foliated manifolds

Let $(M, \mathcal{F})$ be a Riemannian foliation. Then it is defined by a cocycle $\mathcal{U} = \{U_i, f_i, k_{ij}\}_{i,j \in I}$ that is modeled on a Riemannian manifold $(N, \bar{g})$ such that

1. $f_i : U_i \to N$ is a submersion with connected fibers;
2. $k_{ij} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ are local isometries of $(N, \bar{g})$;
3. $f_i = k_{ij} f_j$ on $U_i \cap U_j$.

Definition

A foliation $\mathcal{F}$ is transversely quaternion Kähler if it is defined by a cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}_{i,j \in I}$ modeled on a quaternion Kähler manifold (N_0, g_0, Q_0) and the local diffeomorphisms g_{ij} are local automorphisms of the quaternion Kähler structure of (N_0, g_0, Q_0), i.e., the g_{ij} are local isometries and the induced mappings $\tilde{g}_{ij}$ of $End(TN_0)$ preserve the subbundle Q_0 of rank 3.

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Definition

A foliated quaternion Kähler structure on a foliated Riemannian manifold $(M, \mathcal{F})$ is given by the following data:

1. g is a foliated Riemannian metric in $N(M, \mathcal{F})$;
2. a 3-dimensional foliated subbundle Q of $End(N(M, \mathcal{F}))$ which is locally spanned by 3 almost complex foliated structures;
3. the metric g is Hermitian with respect to these local almost complex structures;
4. the subbundle Q is parallel with respect to the foliated Levi-Civita connection of g.

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$$\Omega_I(u, v) = g(Iu, v),\ \Omega_J(u, v) = g(Ju, v), \text{ and } \Omega_K(u, v) = g(Ku, v),$$

where $u, v \in N(M, \mathcal{F})$.

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The 4-form Ω

$$\Omega = \Omega_I \wedge \Omega_I + \Omega_J \wedge \Omega_J + \Omega_K \wedge \Omega_K$$

is well-defined, i.e., it is independent of the choice of the structures I, J, and K.

Section 4

Hodge theory for basic forms

On a foliated Riemannian manifold $(M, g, \mathcal{F})$ the set of all basic k-forms is

$$A^k(M, \mathcal{F}) = \{\alpha \in A^k(M) : i_X\alpha = 0,\ i_X d\alpha = 0, \text{ for all vectors } X \in T\mathcal{F}\},$$

which is a subcomplex of $A^k(M)$ and we denote its cohomology by $H^k(M, \mathcal{F})$.

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The restriction of the bundle-like metric to the normal bundle of the foliation of the Riemannian foliated manifold $(M, g, \mathcal{F})$ defines $\bar{*}$ operator,

$$\bar{*}\colon A^k(M, \mathcal{F}) \to A^{4n-k}(M, \mathcal{F}).$$

On the Riemannian manifold (M, g) we have the $*$-operator acting on the complex of smooth forms:

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On the subcomplex $A^k(M, \mathcal{F})$ of basic forms these two operators are related by the formula

$$\bar{*}\alpha = (-1)^{p(q-k)} * (\alpha \wedge \chi_{\mathcal{F}}),$$

for any $\alpha \in A^k(M, \mathcal{F})$, where $\chi_{\mathcal{F}}$ is the volume form of leaves.

In $A^k(M,\mathcal{F})$ we have the standard scalar product

$$\langle \alpha, \beta \rangle_b = \int_M \alpha \wedge \bar{*}\beta \wedge \chi_{\mathcal{F}},$$

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which is the restriction of the standard scalar product on $A^k(M)$. A Riemannian foliation on a compact manifold is said to be **taut** if there exists a Riemannian metric that makes all its leaves minimal submanifolds. Tautness is characterized by the nonvanishing of the top dimensional basic cohomology, i.e., $H^q(M,\mathcal{F}) \neq 0$. In this case we say that the foliation is **cohomologically taut**. In fact, this Riemannian metric can be chosen to be bundle-like.

The formal adjoint δ_b of d in the complex $A^k(M,\mathcal{F})$ with the scalar product $\langle .\,,.\rangle_b$ is the operator

$$\delta_b = (d - \kappa\wedge)^{\bar{*}} \colon A^k(M,\mathcal{F}) \to A^{k-1}(M,\mathcal{F}),$$

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$$\Delta_b = \delta_b d + d \delta_b$$

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A basic form α is called **harmonic** iff $\Delta_b \alpha = 0$. The basic Hodge theorem for compact Riemannian foliated manifolds asserts that α is harmonic iff $d\alpha = 0 = \delta_b \alpha$.

Using the 4-form Ω, we define L and Λ operators on the complex $A^*(M, \mathcal{F})$:

$$L\colon A^k(M,\mathcal{F}) \to A^{k+4}(M,\mathcal{F});\;\; L(\alpha) = \Omega \wedge \alpha$$
$$\Lambda\colon A^k(M,\mathcal{F}) \to A^{k-4}(M,\mathcal{F});\;\; \Lambda(\alpha) = \bar{*}(\Omega \wedge \bar{*}\alpha)$$

Basic forms that are annihilated by Λ are called **effective**.

On a compact manifold with a taut foliation one can define scalar products $\langle . , . \rangle$ and $\langle . , . \rangle_b$ on $A^k(M)$ and $A^k(M, \mathcal{F})$, respectively, as

1. $\langle \omega, \omega' \rangle = \int_M *(\omega \wedge *\omega') = \int_M \omega \wedge *\omega'$,
2. $\langle \omega, \omega' \rangle_b = \int_M \bar{*}(\omega \wedge \bar{*}\omega') = \int_M \omega \wedge \bar{*}\omega' \wedge \chi_{\mathcal{F}}$.

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Using this scalar product we have for any $\omega \in A^k(M, \mathcal{F})$ and $\omega' \in A^{k+4}(M, \mathcal{F})$

$$\langle L\omega, \omega' \rangle_b = \langle \omega, \Lambda\omega' \rangle_b.$$

Theorem (M., R. Wolak)

Let $(M, g, \mathcal{F})$ be a $(4n+p)$-dimensional Riemannian foliated manifold whose p-dimensional foliation $\mathcal{F}$ is transversely quaternion Kähler. Let ω be a basic differential form on $(M, \mathcal{F})$ of degree $p \leq n+1$. Then

$$\omega = \sum_{i=0}^{[p/4]} L^i \omega_e^{p-4i}$$

where ω_e^k is an effective basic k-form.

Let $(M, \mathcal{F})$ be a compact Riemannian foliated manifold. Assume that

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Let $(M,\mathcal{F})$ be a compact Riemannian foliated manifold. Assume that

1) its foliated normal bundle $(N(M,\mathcal{F}),\mathcal{F}_N)$ admits a reduction to a connected subgroup G of $O(q)$,
2) the corresponding foliated G-reduction $B((M,\mathcal{F}),G,\mathcal{F}_G)$ of the foliated frame bundle $L((M,\mathcal{F}),\mathcal{F}_L)$ admits a foliated connection without torsion.

The fiber bundle $\bigwedge^k N_x(M,\mathcal{F})^*$ can be understood as the associated bundle of $L((M,\mathcal{F}),\mathcal{F}_L)$ with the standard fiber $\bigwedge^k(R^{q*})$.The space of sections of this bundle we denote by $A^k(N)$. Since the normal frame bundle $L(M,\mathcal{F})$ is foliated, the foliation $\mathcal{F}_L$ induces a foliation $\mathcal{F}_L^k$ of the fiber bundle $\bigwedge^k N_x(M,\mathcal{F})^*$. The space of k-basic forms $A^k(M,\mathcal{F})$ is a subspace of $A^k(N)$. If the normal frame bundle $L(M,\mathcal{F})$ admits a foliated G-reduction $B((M,\mathcal{F}),G,\mathcal{F}_G)$, the bundle $\bigwedge^k N_x(M,\mathcal{F})^*$ can be understood as the associated fiber bundle of $B((M,\mathcal{F}),G,\mathcal{F}_G)$ with the standard fiber $\bigwedge^k(R^{q*})$. The natural induced foliations coincide. Let $W \subset \bigwedge^k(R^{q*})$ be an invariant subspace of $\bigwedge^k(R^{q*})$ under the standard action of G. There is the standard scalar product on $\bigwedge^k(R^{q*})$ for which the induced action of G is isometric.

The associated fiber bundle $\mathcal{W}$ of $B((M, \mathcal{F}), G, \mathcal{F}_G)$ with the standard fiber W can be understood as a foliated vector subbundle of the foliated vector bundle $(\bigwedge^k N_x(M, \mathcal{F})^*, \mathcal{F}_L^k)$. Therefore a k-differential form α which corresponds to a section of $\mathcal{W}$ is said to be of type W. The space of these $\mathcal{W}$-valued sections we denote also by $\mathcal{W}$. The projection $P_W \colon A^k(N) \to \mathcal{W}$ sends basic forms into basic forms as the operation is done point by point. Next we show that the result of S.S. Chern can be reformulated for the basic Laplacian Δ_b.

Proposition (M., R. Wolak)

Let $W \subset \bigwedge^k(R^{q*})$ be an invariant subspace of $\bigwedge^k(R^{q*})$ under the standard action of G, P_W be the projection $P_W\colon A^k(M,\mathcal{F}) \to \mathcal{W}$ and Δ_b be the basic Laplacian, then

$$P_W\Delta_b = \Delta_b P_W.$$

Moreover, let $W_1, \dots, W_s$ be irreducible invariant subspaces of $\bigwedge^k(R^{q*})$ for the action of the group G. Then if α is a harmonic basic k-form, the k-forms $P_{W_1}\alpha, \dots, P_{W_s}\alpha$ are basic and harmonic. Moreover, if α is a basic k-form of type W so is the form $\Delta_b\alpha$.

Theorem (M., R. Wolak)

Let $(M, \mathcal{F})$ be a compact Riemannian foliated manifold of codimension $4q$. If the foliation $\mathcal{F}$ is cohomologically taut and transversely quaternionic Kähler then the basic Betti numbers $B^i_{\mathcal{F}}$ of $(M, \mathcal{F})$ satisfy the inequalities:

$$B^i_{\mathcal{F}} \leq B^{i+4}_{\mathcal{F}} \leq \ldots \leq B^{i+4r}_{\mathcal{F}}$$

for $i + 4r \leq q + 1$, $i = 0, 1, 2$ or 3.

Theorem (M., R. Wolak)

Let $(M, g, Q, \mathcal{F})$ be a cohomologically taut quaternion Kähler foliated manifold of codimension $4q$. Then

1) *for any $k < q$ the linear map $L\colon H^k(M,\mathcal{F}) \to H^{k+4}(M,\mathcal{F})$ is injective,*
2) *and there is the direct sum decomposition* $H^k(M,\mathcal{F}) = \sum_{0\leq s\leq [k/4]} L^s H_e^{k-4s}(M,\mathcal{F})$, $k \leq q+3$.

Section 5

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Thank you.