Conformal structures with an infinitesimal symmetry

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Motivation and objective

Geometric structures equipped with (an infinitesimal) symmetry form one of the richest and most interesting cases to study in various areas of physics and geometry.

E.g. Einstein vacuum spacetimes with a Killing vector, Kähler metrics with (torus) symmetry, and CR structures with symmetry.

Geometrically, an infinitesimal symmetry is a redundancy in the description of the geometry.

Objective: A (local) characterization of (certain) geometric structures with a choice of infinitesimal symmetry via a 1-1 correspondence which recovers all such structures from another geometric structure on a lower dimensional manifold.

As a result, such a symmetry reduction removes the redundancy.

An illustrating case of redundancy

Example : A real hypersur $\phi^{-1}(0) =: M^{2n-1} \subset \mathbb{C}^{2n}$ has a CR str

$$(\mathscr{H}, J), \quad \mathscr{H}^{2n-2} \subset TM, \quad J\colon \mathscr{H} \to \mathscr{H}.$$

Assume $\partial\bar{\partial}\phi$ is non-degenerate and locally describe M as

$$\Im(z_n) = F(z, \bar{z}, \Re(z_n)), \quad z = (z_1, \cdots, z_{n-1}).$$

An infinitesimal symmetry of a CR non-degenerate hypersurface is a vector field $\zeta \in \Gamma(TM)$ such that its flow preserves $\mathscr{H}$ and J.

{CR non-deg hypersurfaces with nonvanishing inf. symmetry}

$$\overset{1-1}{\longleftrightarrow}$$

{ Kähler metrics (g, J) in $\dim_{\mathbb{C}}\ n-1$ up to homothety} i.e. on $N := M/\langle\zeta\rangle$.

Let $K(z, \bar{z})$ be a **Kähler potential**, i.e. $g_{ij} = \partial_{z_i}\partial_{\bar{z}_j}K$.
Rectifying ζ, $F(z, \bar{z})$ gives $K(z, \bar{z})$ (no dependency on $\Re(z_n)$.)

[Cahen-Schwachhöfer 2009]: **special symplectic connections** (e.g. **Bochner-flat Kähler metrics**) $\overset{1-1}{\leftrightarrow}$ **Flat parabolic contact strs** (e.g. **Flat CR strs**) + an infinitesimal symmetry.

Related results and structures

[Caþ-Salač 2014,2018,2018,2019] treated curved ***parabolic contact structures*** with an infin symm.

Key ingredient: Identifying the *contact distribution* $\mathscr{H} \subset TM$ with the tangent space of the quotient $N = M/\langle\zeta\rangle$ whenever $\zeta \neq 0$.

In [M-Sagerschnig 2023] first 3 articles of Čap-Salač are extended to

- (2,3,5)-strs + infin symm $\overset{1-1}{\leftrightarrow}$ var'l scalar 4th order ODEs
- (3,6)-strs + null infin symm $\overset{1-1}{\leftrightarrow}$ var'l pairs of 3rd order ODEs
- Pseudo-conf and causal strs + infin symm $\overset{1-1}{\leftrightarrow}$ var'l orthopath strs

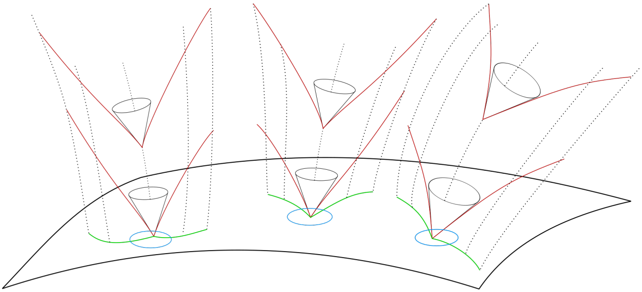
Example : If (M, g) is Riem, $(M \times \mathbb{R}, [\tilde{g}])$, where $\tilde{g} = g - (\mathrm{d}t)^2$, has a conformal Killing field $\zeta = \frac{\partial}{\partial t}$ i.e.

$$\mathscr{L}_\zeta \tilde{g} \in [\tilde{g}] \quad \text{where} \quad [\tilde{g}] = \{e^{2\lambda}\tilde{g} \,\big|\, \lambda \in C^\infty(M)\}.$$

If (M, g) has constant sectional curvature then $[\tilde{g}]$ is conformally flat.

Using a different technique, [Caponio-Javaloyes-Sanchez 2011] obtained stationary spacetimes from certain Finsler metrics.

Main steps in the case of Lorentzian conformal strs



- Sky bdle $\mathscr{C}^{2n} \to M^{n+1}$ has a filtered str with a quasi-contact dist

$$\text{Sky at } x \in M: \quad \mathscr{C}^{n-1}_x := \{[v] \in \mathbb{P}T_xM \mid g_x(v,v) = 0\}.$$

- Any conf Kill field $\zeta \in \Gamma(TM)$ lifts to a transverse v.f on $\mathscr{C}$ a.e.
- The leaf space of ζ has a natural str (augmented path geometry)
- Augmented path geometries are variational + extra conditions
- Via quasi-contactification one obtains a 1-1 correspondence

Pseudo-conformal structures via their sky bundle

Let $(M^{n+1},[g])$ be a **strictly** pseudo-conf str of sign $(p+1,q+1), p,q\geq 0$:

$$g=2\omega^0\omega^n+\varepsilon_{ab}\omega^a\omega^b, \qquad [\varepsilon_{ab}]=\mathrm{diag}(\underbrace{1,\cdots,1}_{p},\underbrace{-1,\cdots,-1}_{q}).$$

Sky bundle of $[g]$ is the *bdle of projectivized null cones* $\mathscr{C}\subset\mathbb{P}TM$:

$$\mathscr{C}_x^{n-1}\hookrightarrow\mathscr{C}^{2n}\xrightarrow{\pi}M^{n+1}, \quad \text{where} \quad \mathscr{C}_x:=\{[v]\in\mathbb{P}T_xM\,|\,g_x(v,v)=0\}.$$

Lift of *null geodesics* and the fibers $\mathscr{C}_x^{n-1}$ foliate $\mathscr{C}$ and induce filtration

$$\begin{array}{ccccc} T^{-1}\mathscr{C} & \subset & T^{-2}\mathscr{C} & \subset & T^{-3}\mathscr{C}, \\ \| & & \| & & \| \\ \ell^1\oplus\mathcal{V}^{n-1} & \subset & \mathscr{H}^{2n-1}:=\langle[\ell,\mathcal{V}],\ell,\mathcal{V}\rangle & \subset & T\mathscr{C}, \end{array}$$

with growth vector $(n,2n-1,2n)$. $\mathscr{H}$ is tautologically defined:

$$\mathscr{H}=\ker\pi^*g_x(y,.) \quad \text{at} \quad (x,[y])\in\mathscr{C}.$$

$\mathscr{H}$ is max'ly non-integrable and defines a **quasi-contact** str on $\mathscr{C}^{2n}$

$$\alpha=\mathrm{Ann}\mathscr{H}, \quad \alpha\wedge(\mathrm{d}\alpha)^{n-1}\neq 0, \quad (\mathrm{d}\alpha)^n=0, \quad \ell=\mathrm{Char}(\alpha).$$

$\forall v,w\in\mathcal{V}:\ [v,[w,\ell]]/\mathscr{H}\in T\mathscr{C}/\mathscr{H}\rightsquigarrow[\mathrm{h}]\subset\mathrm{Sym}^2(\mathcal{V}^*)$ of sign (p,q).

Causal structures with symmetry

A Pseudo-Riemannian conformal manifold $(M^{n+1},[g])$ is encoded by

$$\mathscr{C}^{2n} := \{[v] \in \mathbb{P}TM \mid g(v,v)=0\}.$$

Relaxing the quadratic assumption one obtains a causal structure i.e.

$$\mathscr{C}^{2n} := \{[v] \in \mathbb{P}TM \mid G(v)=0\}.$$

Viewing $\mathscr{C} \subset \mathbb{P}TM$ as a graph using affine coordinates $p^0=1$ for $\mathbb{P}TM$

$$p^n = K(x^0,\cdots,x^n,p^1,\cdots,p^{n-1}), \quad \det(\partial^2_{p^a p^b}K) \neq 0$$

Theorem [M-Sagerschnig 2023]:
{Causal structures with an inf symmetry} $\overset{1-1}{\leftrightarrow}$ {variational orthopath strs}

Variational orthopath geom is the study of Lagrangians under div equiv.
1st order problem of variational calculus for function(s) of one variable:
Find $u\colon (p,q) \to \mathbb{R}^{n-1}, n \geq 2$, that extremizes

$$I[u] = \int_p^q L(x,u^1(x),\cdots,u^{n-1}(x),(u^1(x))',\cdots,(u^{n-1}(x))')\mathrm{d}x.$$

allowing $L\mathrm{d}x \to cL\mathrm{d}x + \mathrm{d}_H f$. Its extremals are curves given by

$$E_a(L) = \frac{\partial L}{\partial u^a} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial (u^a)'} = 0 \Rightarrow (u^a)'' = F^a(x,u,u').$$

Variational orthopath geometry and Finsler structures

Thus, variational orthopath geometry is equivalent to (pseudo-)Finsler structures under divergence equivalence:

- geodesics are **unparameterized**
- **2nd fundamental form of indicatrices** are defined up to a scale.

Riem metrics can be described by their unit sphere bdle $\Sigma^{2n-1} \to M^n$

$$\Sigma^{2n-1} := \{v \in TM \,\big|\, G(v) = 1\}, \quad \Sigma_x^{n-1} \subset T_x M, \quad G(v) = \sqrt{g(v,v)}.$$

In Finsler strs the norm $G\colon TM \to \mathbb{R}$ doesn't arise from an inner product.
On an open set $U \subset \Sigma$ of $\Sigma \subset TM$ one can describe $J^1(\mathbb{R}, \mathbb{R}^{n-1}) \cong \Sigma$ as

$$(x, y^a, p^a) \to \frac{1}{L(x, y^a, p^a)} \left(\frac{\partial}{\partial x} + p^a \frac{\partial}{\partial y^a} \right), \quad \det\left(\left[\frac{\partial^2 L}{\partial p^a \partial p^b} \right] \right) \neq 0$$

for a nonvanishing function $L(x, y^a, p^a)$.
Addressing redundancy: $K(x^0, x^a, x^n, p^a)$ is given by $L(x^0, x^1, \cdots, x^{n-1}, p^a)$ (no dependency on x^n after rectifying ζ.)

Pseudo-conformal structures with symmetry

The fundamental invariants of var orthopath strs are $[\mathbf{A}], [\mathbf{T}], [\mathbf{N}], [\mathbf{q}]$.

Corollary [M-Sagerschnig]
Variational orthopath geoms $+\mathbf{A}=0 \leftrightarrow [g]+$ an infin symmetry and $\mathbf{W}=\tau^*\mathbf{T}$ using the quotient map $\tau : \mathscr{C}^{2n} \to \Sigma^{2n-1}$ where $\mathbf{W}$ is the generator of the pull-back of Weyl curv on $\mathscr{C}$.

- $\mathbf{q}, \mathbf{A} = 0 \iff$ The conformal Killing field is null.
- $\mathbf{q}, \mathbf{A}, \mathbf{N} = 0 \Leftrightarrow$ The pseudo-conf str has null symm with orth foliation.
- $\mathbf{q}, \mathbf{A}, \mathbf{N} = 0$ + 1st order condition on $\mathbf{T} \Rightarrow$ conf hol of $[g]$ is $\subsetneq P_2$.
- $\mathbf{A}, \mathbf{T} = 0$ (finite local moduli) $\iff$ Flat pseudo-conf str + infin symm

Example Quasi-contactification of the orthopath str of a *Riem metric* g on M gives the pseudo-conf str $[g-(\mathrm{d}t)^2]$ on $M \times \mathbb{R}$.
Example *Fefferman conformal metrics* for a CR str are a class of pseudo-conformal structures with a null infinitesimal symmetry
$\Rightarrow$ Theorem : *Chains of CR structures are variational*.
Our proof is Cartan geometric unlike the proof in [Cheng et al. 2019].

Thank you for your attention