Polygons & G_2-symmetry

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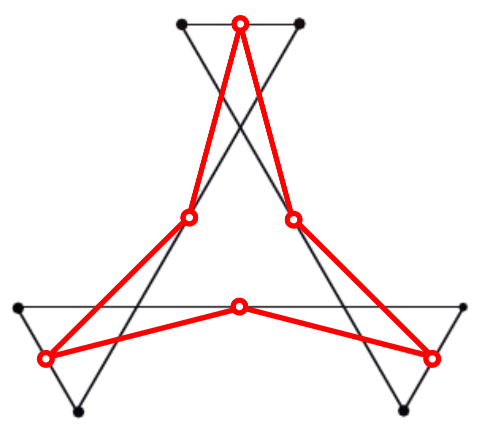
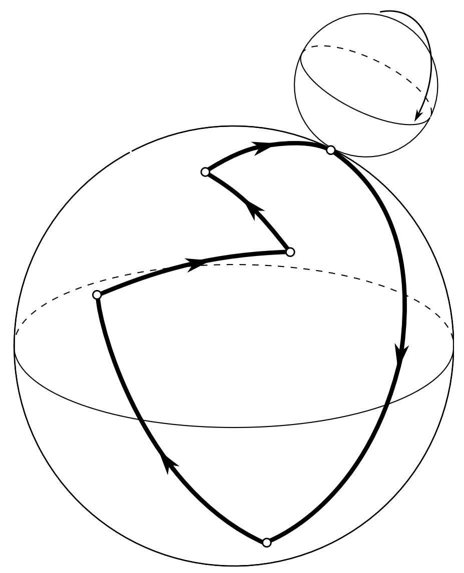
Symmetry & Shape, September 23, 2024
Santiago de Compostela, Galicia

Theorem (main): there is a 1:1 correspondence

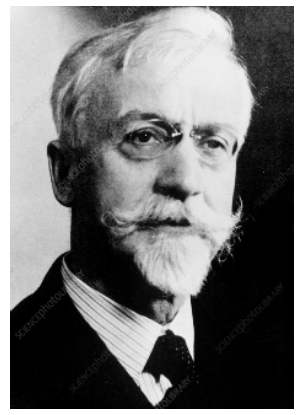
{ Dancing pairs of planar polygons } ⟷ { Spherical polygons with trivial rolling monodromy }





Proof: ⤡ ⤢

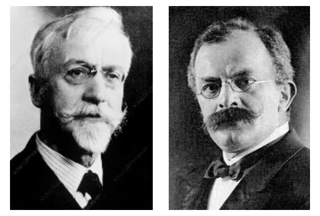
{ Closed piece-wise rigid Horizontal curves of the Cartan-Engel distribution }



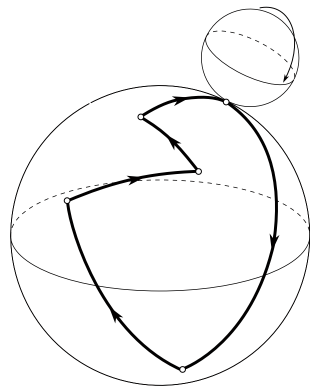
$$\widetilde{Q}^{\texttt{oct}}$$

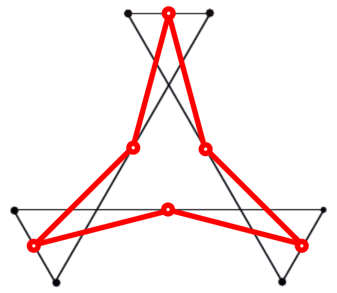
2:1   2:1

$$Q^{\texttt{dance}} \subset Q^{\texttt{oct}}$$



$$Q^{\texttt{roll}}$$



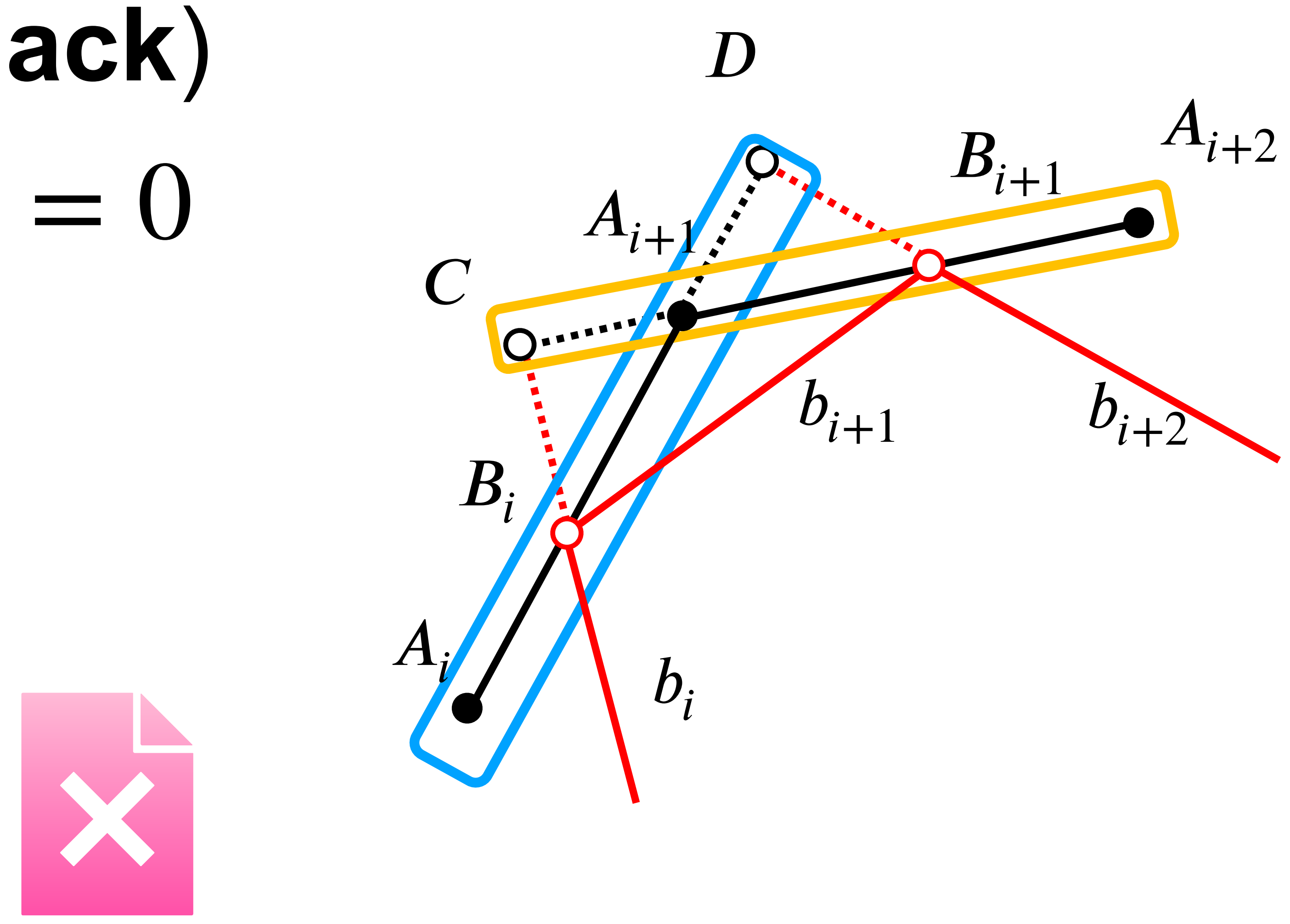


$$\mathbb{RP}^2$$

Dancing pair: is a pair of polygons in $\mathbb{RP}^2$, with vertices $A_1, A_2, \ldots, A_n$ and edges $b_1, b_2, \ldots, b_n$, such that for all i

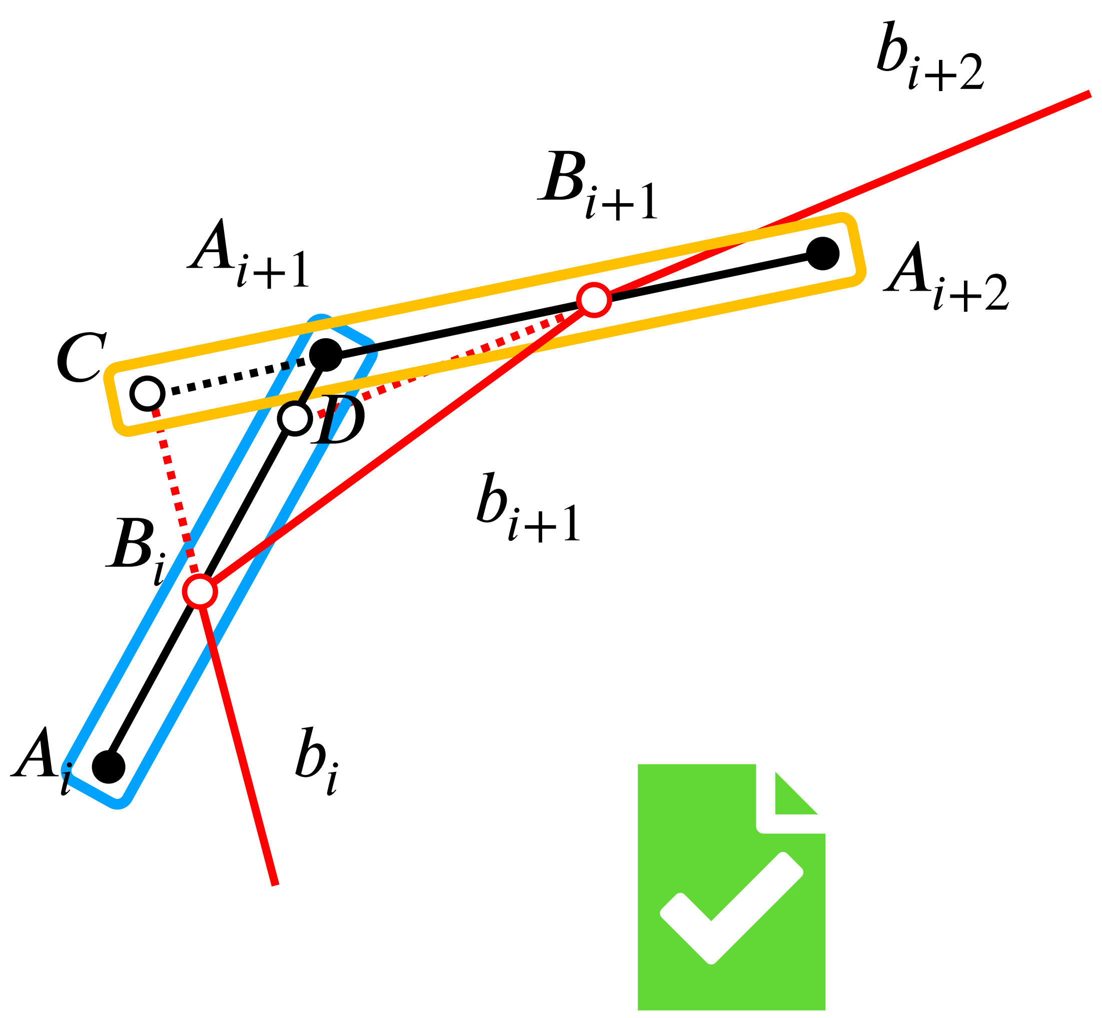
(1) $b_i b_{i+1} \in A_i A_{i+1}$ (**red** is inscribed in **black**)

(2) $[A_{i+1}, B_i, A_i, D] + [A_{i+1}, B_{i+1}, A_{i+2}, C] = 0$



Cross-ratio

$$[x_1, x_2, x_3, x_4] := \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)}$$



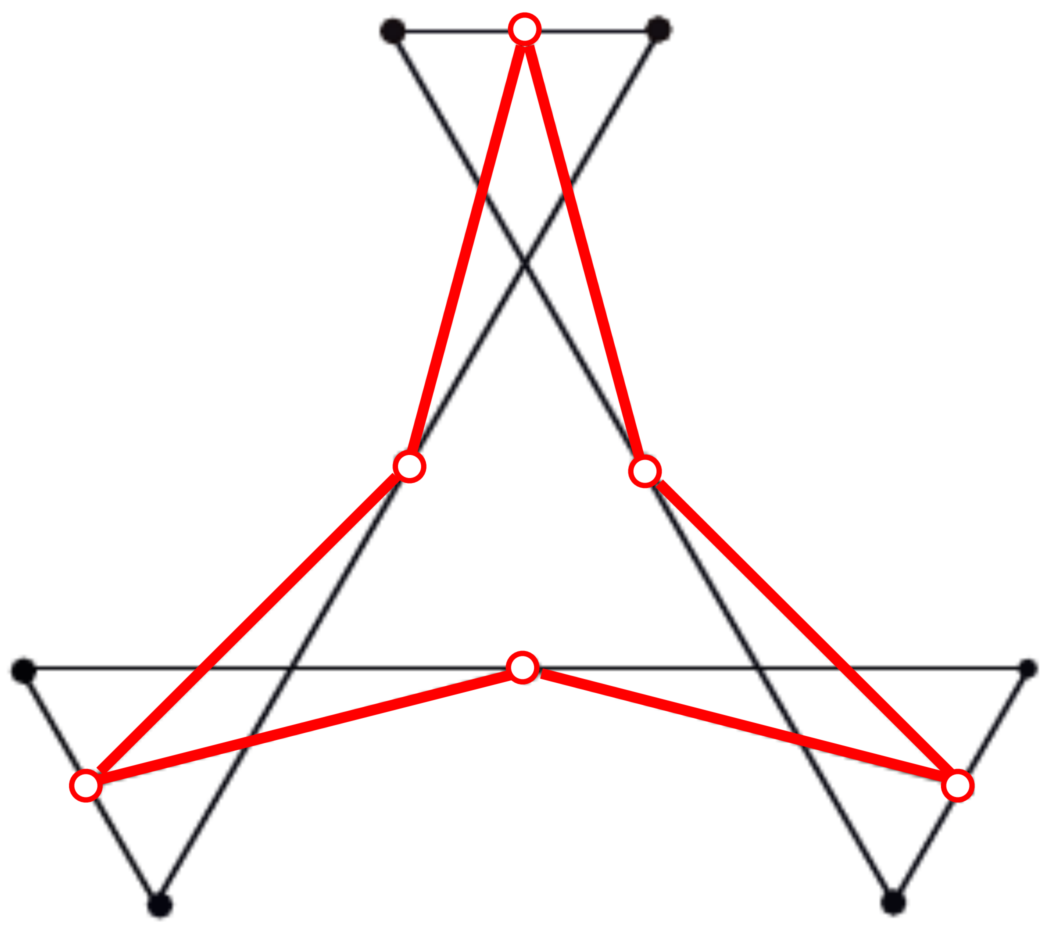
Non degeneracy conditions:

(1) No 3 consecutive A's are colinear.

(2) No 3 consecutive b's are concurrent.

(3) $A_i \notin b_i$, for all i.

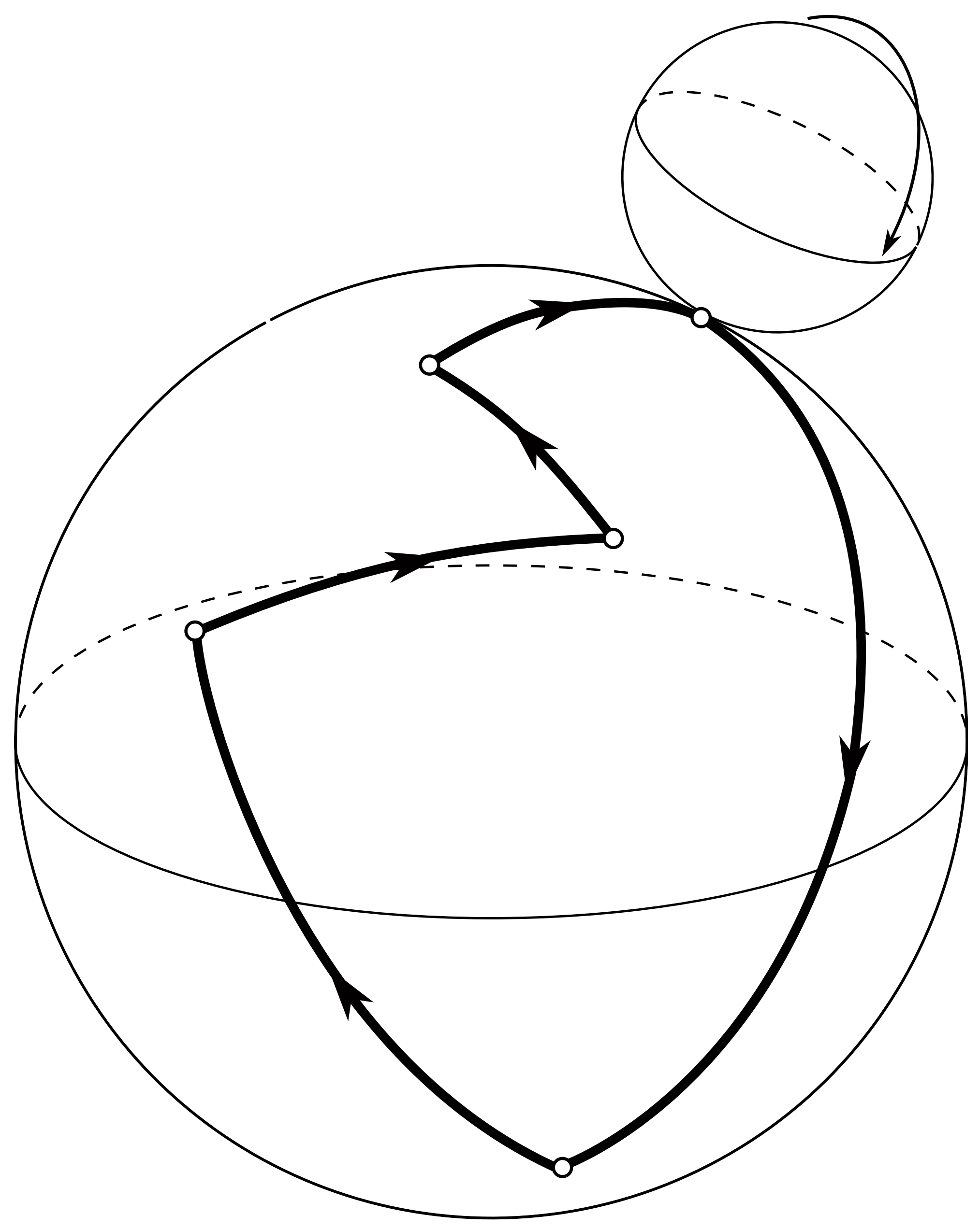
Theorem: there are dancing pairs of closed n gons iff $n \geq 6$.



$n = 6$

Proof: via rolling balls

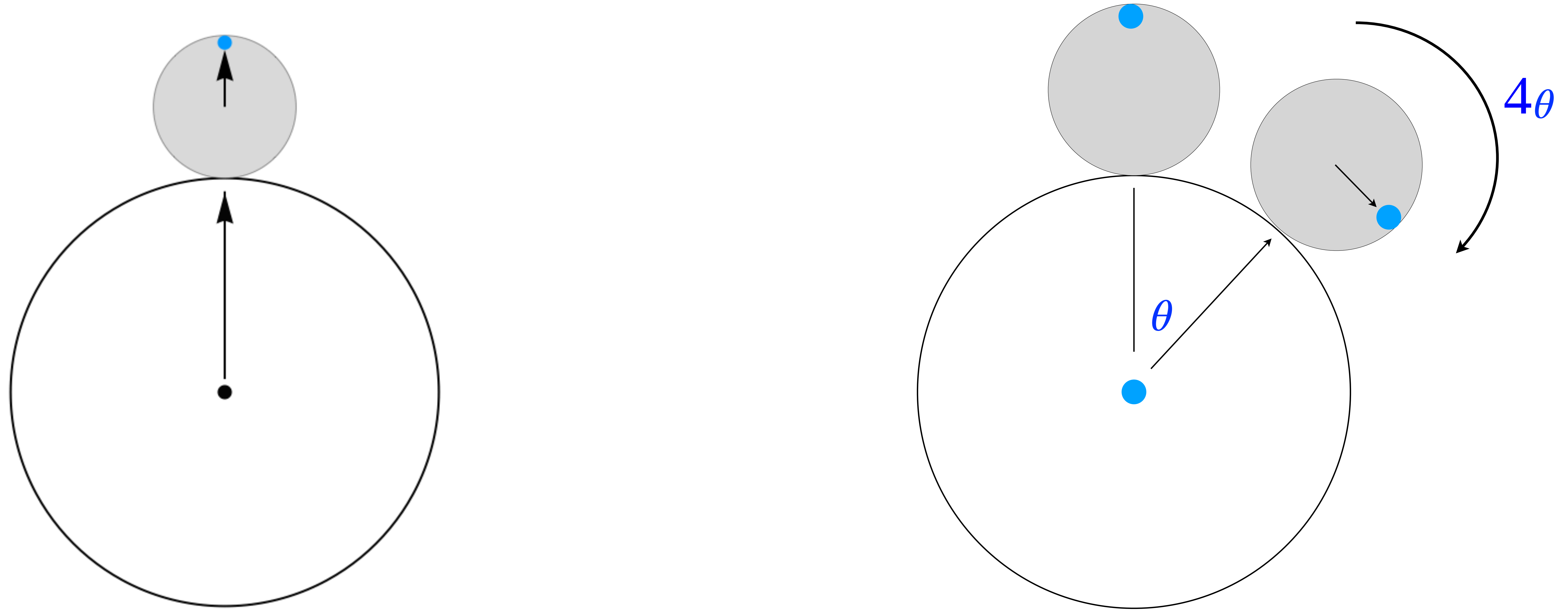
Rolling balls: a sphere of radius 1 is rolling without sliding and twisting along a closed polygon on a sphere of radius 3



Non-degeneracy condition:
No 3 consecutive vertices
are collinear.

The rolling ball defines a path in SO_3, starting at **I,** whose endpoint is ``the **rolling monodromy"**

Rolling monodromy (3:1 ratio)



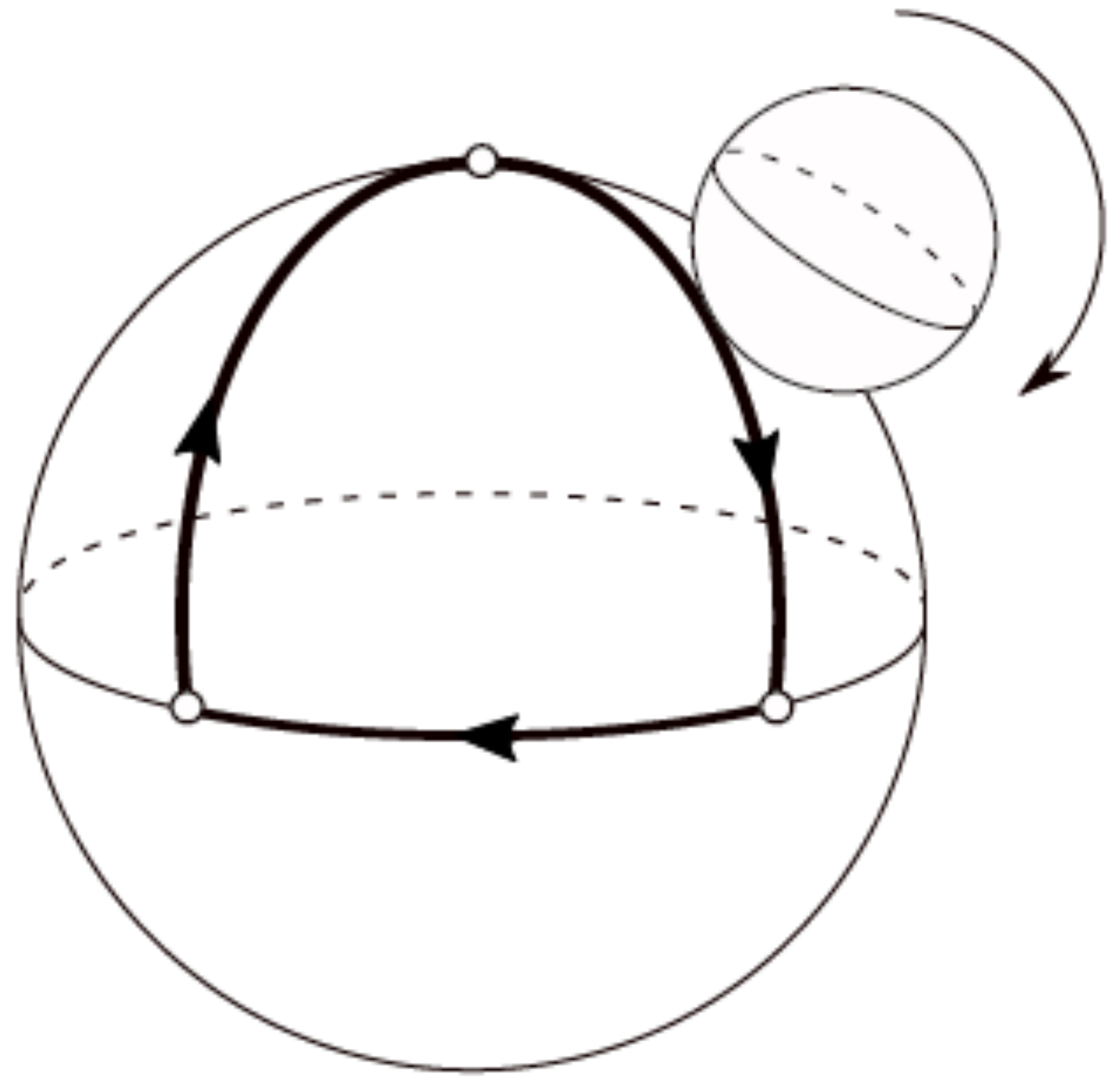
Definition: the rolling monodromy is **trivial** if the corresponding path in SO_3 is **closed** and **contractible**.

Equivalently: the lifted path in S^3 is **closed**.

Recall:

$$\begin{array}{ccc} S^3 & \ni & \mathbf{q} \\ {\scriptstyle 2:1}\downarrow & & \downarrow \\ SO_3 & \ni & [\mathbf{v} \mapsto \mathbf{q}\mathbf{v}\bar{\mathbf{q}}] \end{array}$$

Example: a triangular ‘octant’.



Each edge is ¼ of a great circle

$\Longrightarrow$ small sphere makes 1 full turn going along each edge

$\Longrightarrow$ lifted monodromy for each edge is -1

$\Longrightarrow$ lifted monodromy of the triangle is $(-1)^3 = -1$

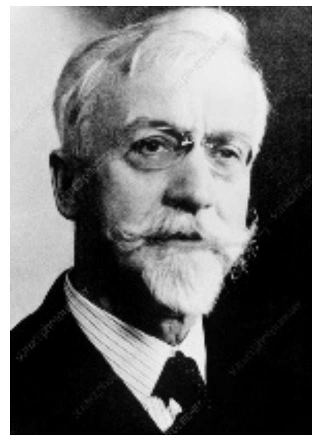
$\Longrightarrow$ lifted monodromy of going **twice** around the triangle is trivial.

Cartan-Engel distribution (1893)

max symm

Is a “flat” 2-plane dist $D \subset TQ$, non-integrable, on a 5-mfld Q

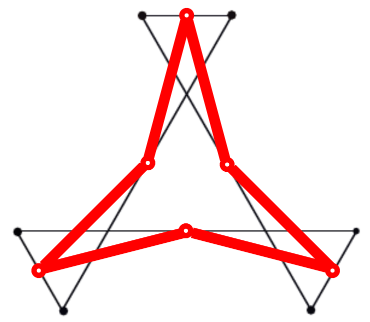
Theorem (Cartan, 1910):



1. It is a “flat” 235 dist: The (local) symmetry gp is G_2 (a 14-dim non-cpct simple Lie gp), max-dim possible for a 235 dist.

2. All “flat” 235 dist are loc diffeo.

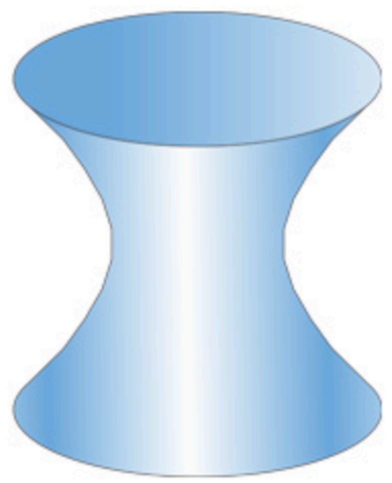
3. Submax symmetry for 235 dist: dim Aut $\geq 8 \Rightarrow$ Aut = G_2 .



$$\left\{ \begin{array}{c}\text{Dancing pairs of}\\ \text{planar polygons}\end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c}\text{Closed piece-wise rigid}\\ \text{Horizontal curves of the}\\ \text{Cartan-Engel distribution}\end{array} \right\}$$

$$\begin{array}{ccc} \mathbb{R}^3 \times \left(\mathbb{R}^3\right)^* & \longrightarrow & \mathbb{R} \\ (\mathbf{A}, \mathbf{b}) & \mapsto & \mathbf{bA} \end{array}$$

is a (3,3) metric



$$Q^{\texttt{dance}} = \left\{ (\mathbf{A}, \mathbf{b}) \,|\, \mathbf{bA} = 1 \right\} \subset \mathbb{R}^3 \times (\mathbb{R}^3)^*$$

$\mathbf{A}_1 \times \mathbf{A}_2 := \det(\mathbf{A}_1, \mathbf{A}_2, \cdot\,)$

$D_{\mathbf{A},\mathbf{b}} = \left\{ (\dot{\mathbf{A}}, \dot{\mathbf{b}}) \in TQ^{dance}_{\mathbf{A},\mathbf{b}} \,\middle|\, \dot{\mathbf{b}} = \mathbf{A} \times \dot{\mathbf{A}} \right\}$, the **dancing distribution,** is a Cartan-Engel 235-distribution:

Proof: 8-dimensional $SL_3(\mathbb{R})$ acts trans. on Q^{dance} preserving $D \implies$ 14-dim symmetry ∎

G Bor, LH, P Nurowski (2018), *The dancing metric, G_2-symmetry and projective rolling*, Trans. Amer. Math. Soc. **370(6)**

Q^{dance} fibers over the space M^4 of non-incident pairs of point-line

$$\begin{array}{ccc} (\mathbf{A}, \mathbf{b}) & \in & Q^{\texttt{dance}} \subset \mathbb{R}^3 \times (\mathbb{R}^3)^* \\ \downarrow & & \downarrow \\ (A, b) & \in & M^4 = \left\{ (A, b) \,\middle|\, A \notin b \right\} \subset \mathbb{RP}^2 \times (\mathbb{RP}^2)^* \end{array}$$

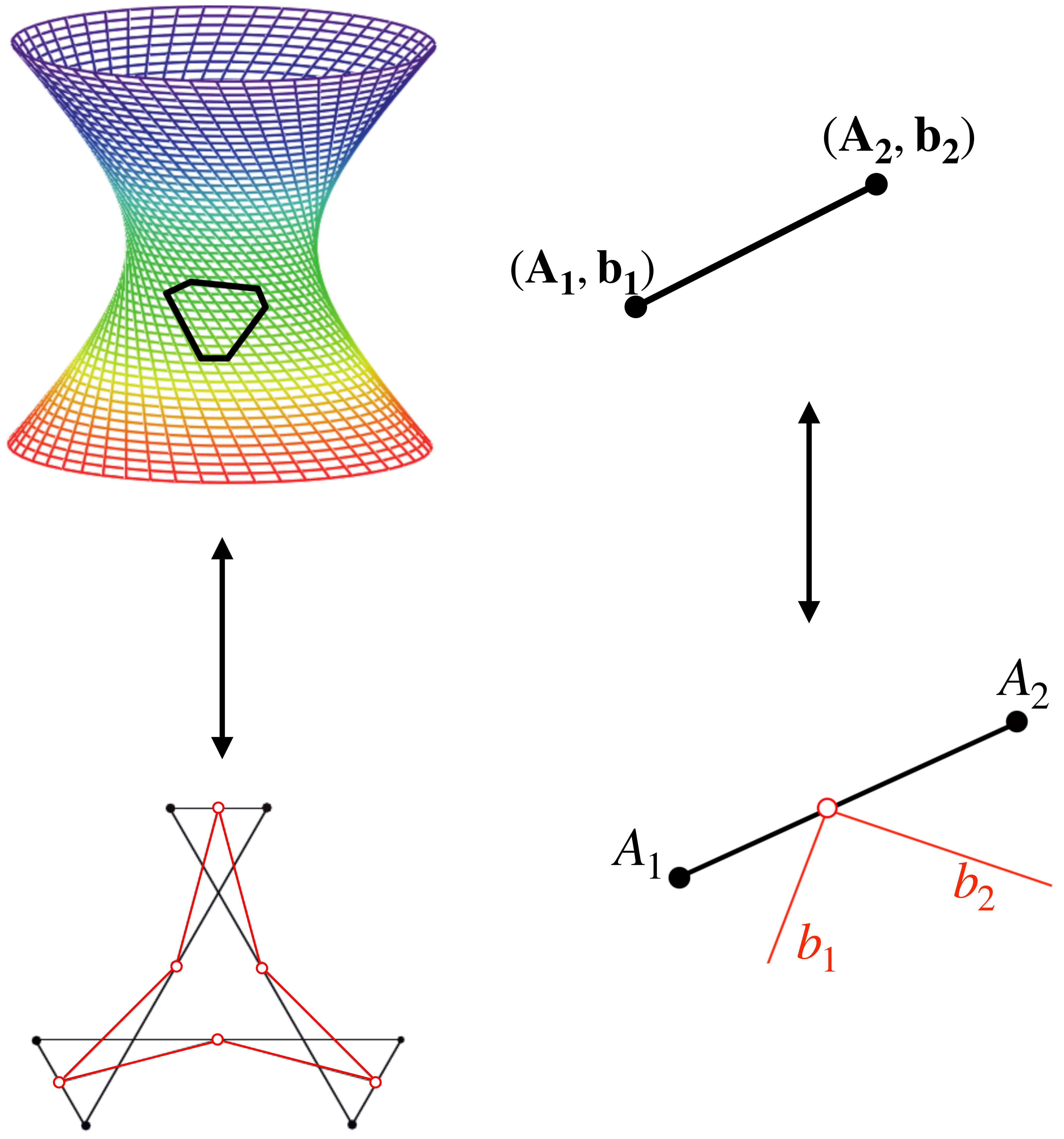
$\mathbb{R}^*$ acts on Q^{dance}: $r \cdot (\mathbf{A}, \mathbf{b}) = (r\mathbf{A}, r^{-1}\mathbf{b})$

Q^{dance} includes many horizontal lines: in fact,

$$\forall (\mathbf{A}, \mathbf{b}) \in Q^{dance}, (\mathbf{A}, \mathbf{b}) + D_{\mathbf{A},\mathbf{b}} \subset Q^{dance}$$

These horizontal lines are precisely the rigid curves (a la Bryant-Hsu) of the distribution.

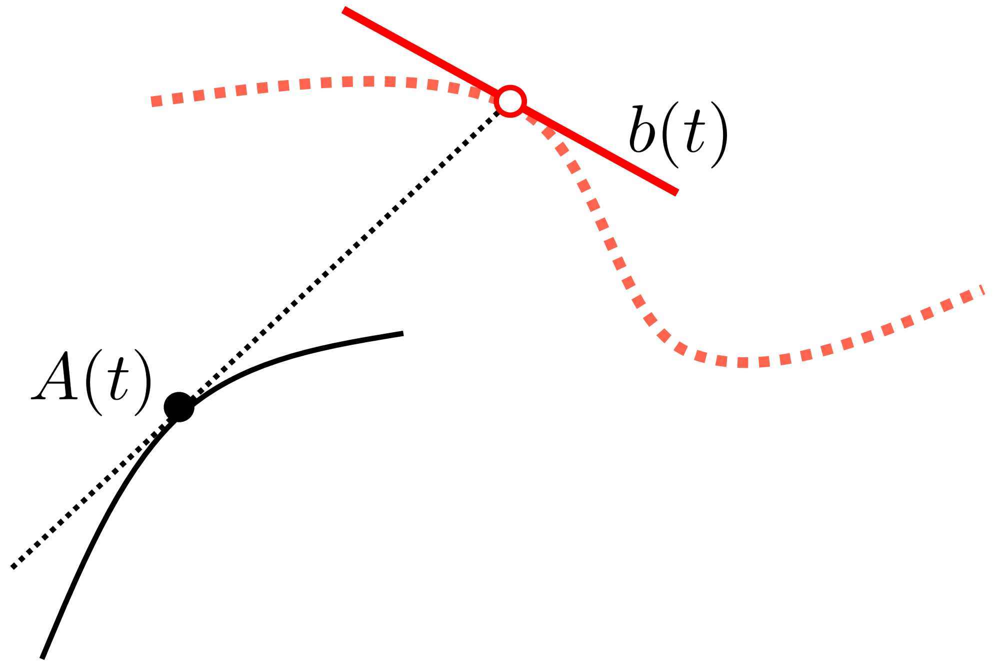
Theorem: a pair of polygons, the 1st with vertices $A_1, \ldots, A_n \in \mathbb{RP}^2$, the 2nd with edges $b_1, b_2, \ldots, b_n \in (\mathbb{RP}^2)^*$, is ***dancing*** iff there are homogeneous coordinates $\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathbb{R}^3, \mathbf{b}_1, \ldots, \mathbf{b}_n \in (\mathbb{R}^3)^*$, such that $(\mathbf{A}_1, \mathbf{b}_1), \ldots, (\mathbf{A}_n, \mathbf{b}_n)$ are the vertices of a **horizontal** polygon in Q^{dance}.



Remark.

dancing pairs of polygons = discrete version of ‘dancing’ point-line pairs in M^4:

(1) $A(t)$ always moves towards the “turning pt” of $b(t)$ (‘ice skate’)



$\Leftrightarrow$ null curves of a **metric of signature (2,2)** on

Ψ-riem symm space → $SL_3(\mathbb{R})/GL_2(\mathbb{R})$

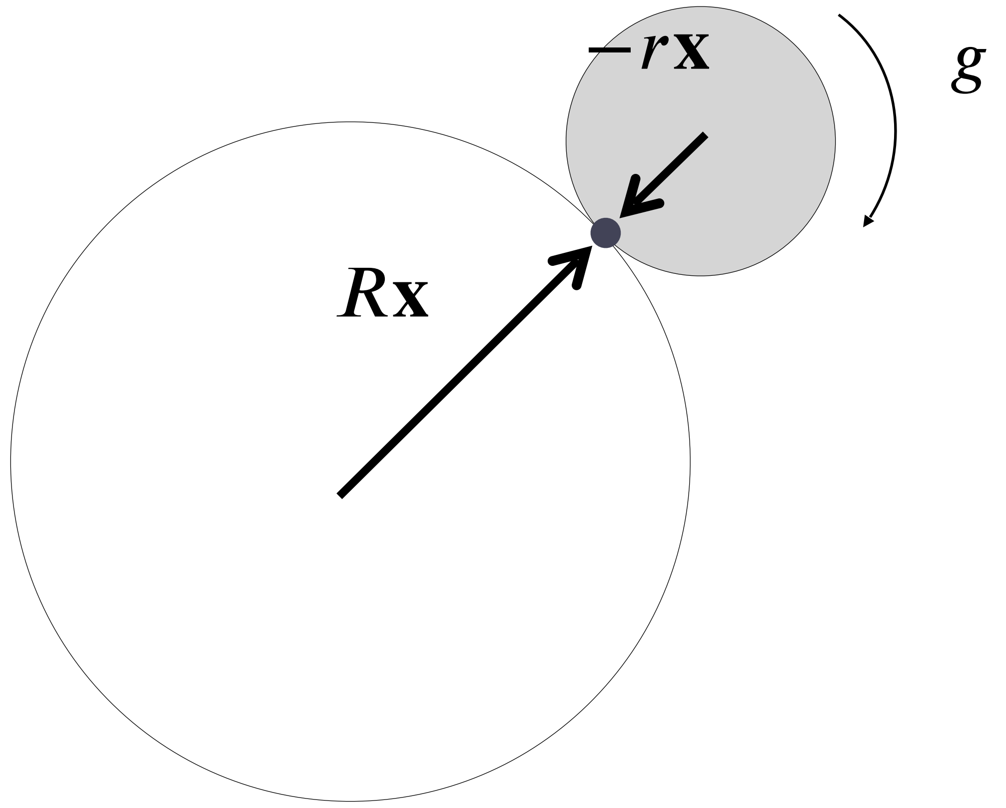
$$\begin{gathered} SL_3(\mathbb{R})/GL_2(\mathbb{R}) \\ \wr\wr \\ M^4 = \left\{(A,b)\,|\,A \notin b\right\} \subset \mathbb{RP}^2 \times (\mathbb{RP}^2)^* \end{gathered}$$

(2) The tangent SD 2-plane field along the curve $\big(A(t), b(t)\big)$ in M^4 is **parallel**

(a ‘half-geodesic’).

$$\left\{ \begin{array}{c} \text{Rolling balls} \\ \text{trajectories} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Horizontal curves of the} \\ \textbf{Rolling distribution} \end{array} \right\}$$

$Q^{\mathtt{roll}} = S^2 \times \mathrm{SO}_3$= configuration space for rolling balls



$$\begin{array}{ccc} \mathrm{SO}_3 & \longrightarrow & Q^{\mathtt{roll}} \\ & & \downarrow \\ & & S^2 \end{array}$$

The **rolling distribution:** $D \subset TQ^{\mathtt{roll}}$, 235-dist if $\rho = R/r \neq 1$,

No slip & no twist conditions

$$\begin{cases} (\rho + 1)\mathbf{x}' = \boldsymbol{\omega} \times \mathbf{x}, \\ \boldsymbol{\omega} \cdot \mathbf{x} = 0 \end{cases}$$

$\mathbf{x} \in S^2$, $\boldsymbol{\omega} = g^{-1}g' \in \mathbb{R}^3 \simeq \mathfrak{so}_3$

Theorem (R Bryant ~2000):

The rolling dist for a pair of balls with $R/r \neq 1$

is *flat* (ie a Cartan-Engel dist) $\Leftrightarrow R/r = \frac{1}{3}$ or 3

(for $R/r \neq 1,3,1/3,$ sym gp is 6-dim)

"Rigid" curves of rolling distributions: rolling along **geodesics** (great circles)

R Bryant, L Hsu (1993), *Rigidity of integral curves of rank 2 distributions*, Invent. Math. 114.

G Bor, R Montgomery (2009), *G_2 and the 'rolling distribution'*, Enseign. Math. 55.

JC Baez, J Huerta (2014), *G_2 and the rolling ball*, Trans. AMS 366.

The “octonions” model of the Cartan-Engel distribution

$$\mathbb{O} = \mathbb{R}1 \oplus \mathrm{Im}\mathbb{O} \simeq \mathbb{R}^8$$ the split octonions, $G_2 = \mathrm{Aut}(\mathbb{O})$

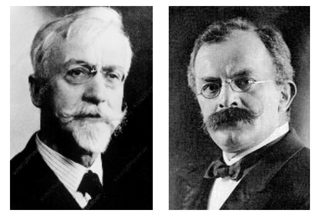
$$Q^{oct} \subset \mathbb{RP}^6 = \mathbb{P}(\mathrm{Im}\mathbb{O})$$

Projectivized null cone

$$Q^{oct} = \left\{ [\zeta] \,|\, \zeta \in \mathrm{Im}\mathbb{O}, \langle \zeta, \zeta \rangle = 0 \right\}$$

$$T_{[\zeta]}Q^{oct} \supset D^{oct}_{[\zeta]} = T_{[\zeta]}\left(\mathbb{P}\left[\underbrace{\{\eta \in \mathrm{Im}\mathbb{O} \,|\, \zeta\eta = 0\}}_{\zeta^\circ} \right] \right)$$

ζ°, the 3-dim annihilator of $\zeta \supset \mathbb{R}\zeta$

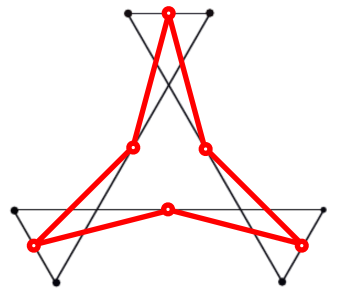


G_2 acts

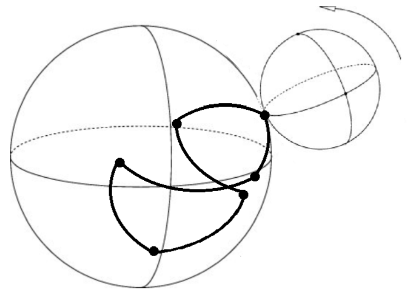
$\widetilde{Q}^{\texttt{oct}} \xleftarrow{\simeq} \widetilde{Q}^{\texttt{roll}}$

2:1   2:1

$\mathfrak{g}_2$ acts



$Q^{\texttt{dance}} \subset Q^{\texttt{oct}}$ $Q^{\texttt{roll}}$



$(\mathbf{A}, \mathbf{b}) \mapsto \left[\begin{pmatrix} 1 & \mathbf{A} \\ \mathbf{b} & -1 \end{pmatrix}\right]$

Local $\mathfrak{g}_2$ action only

$S^2 \times S^3 \rightleftarrows S^2 \times S^3$

$(\pm 1,\ 1)$ $(1,\ \pm 1)$

$\mathbb{RP}^2 \times S^3$ $S^2 \times \mathrm{SO}_3$

Spherical **regular** n-gons with trivial rolling monodromy

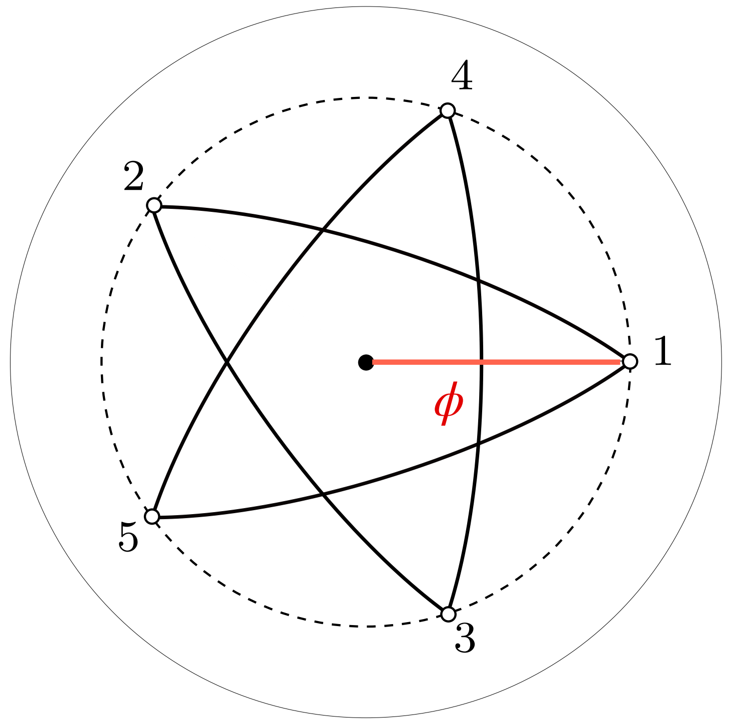
Proposition

(a) A regular spherical polygon (n, w, ϕ) has trivial 3:1 rolling monodromy iff there exists an integer w' such that

- $$\cos\left(\frac{\pi w'}{n}\right) = \cos\left(\frac{\pi w}{n}\right)\left[1 - 4\sin^2\left(\frac{\pi w}{n}\right)\sin^2\phi\right].$$

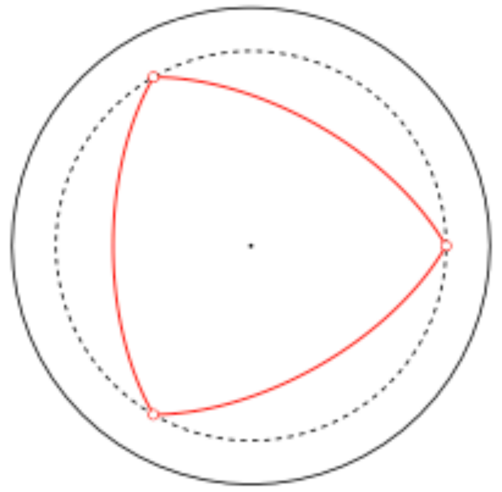
- $$w \equiv w' \pmod 2$$

w' is the winding number of the curve traced on the small sphere

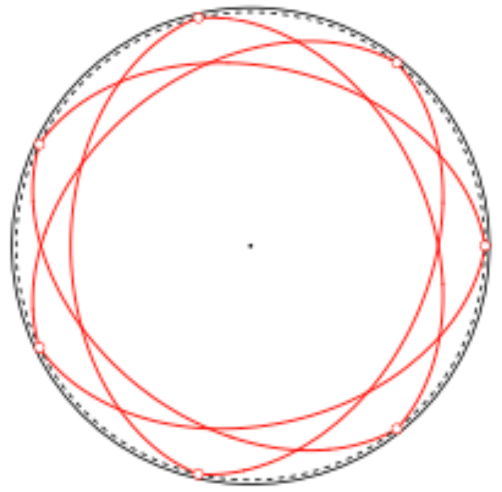


$n = 5,\ w = 2$

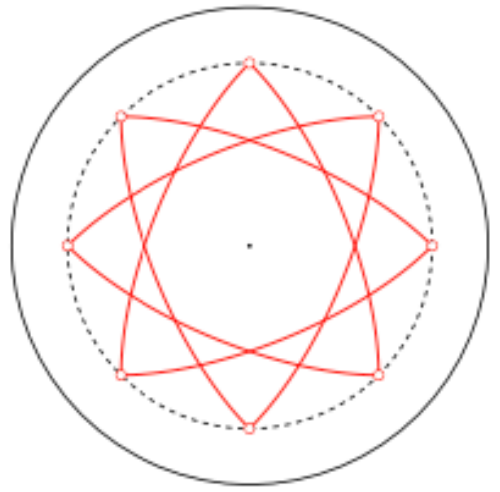
(b) There are solutions iff $n \geq 6$



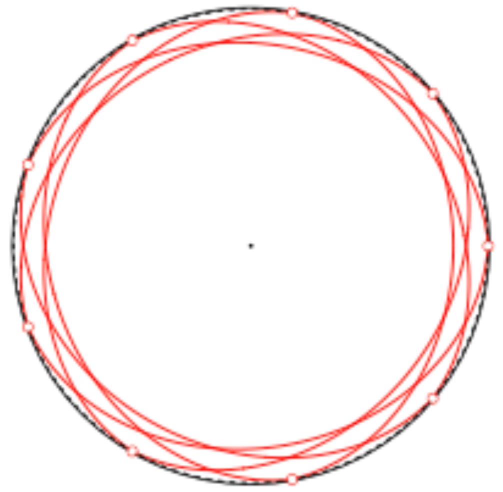
$(6, 2, 4)$



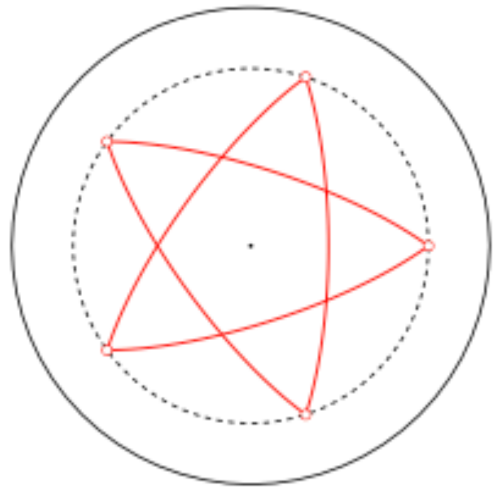
$(7, 2, 4)$



$(8, 3, 5)$



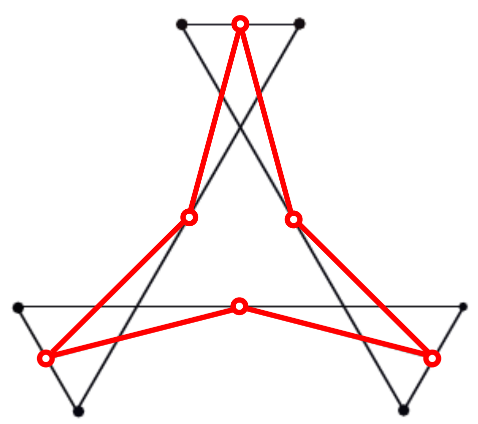
$(9, 2, 4)$

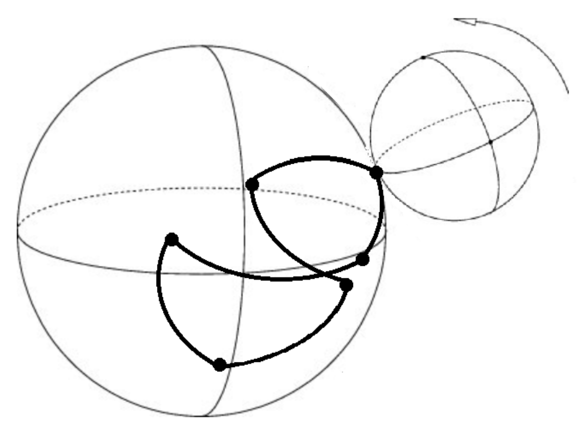


$(10, 4, 6)$

Theorem (main): there is a 1:1 correspondence

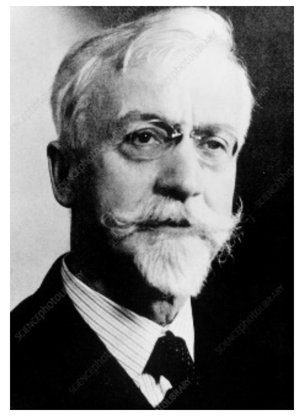
{ Dancing pairs of planar polygons } $\longleftrightarrow$ { Spherical polygons with trivial rolling monodromy }





Proof: ↖↘ ↗↙

{ Closed piece-wise rigid Horizontal curves of the Cartan-Engel distribution }



Thank you!