Nearly Kähler geometry and totally geodesic submanifolds

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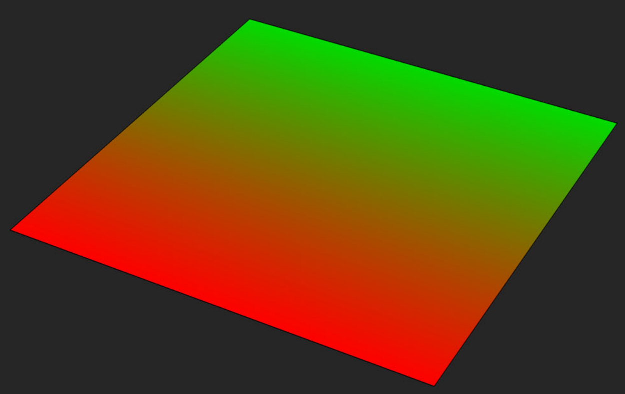
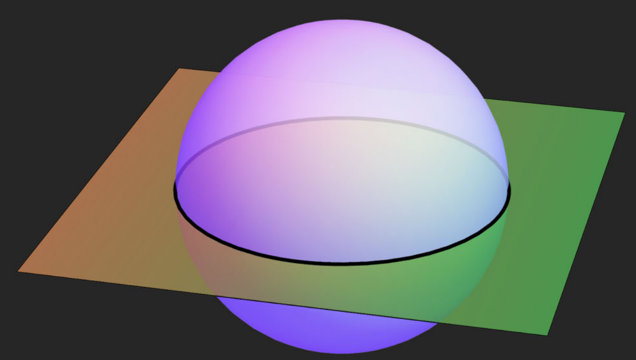
- f is *totally geodesic* if:

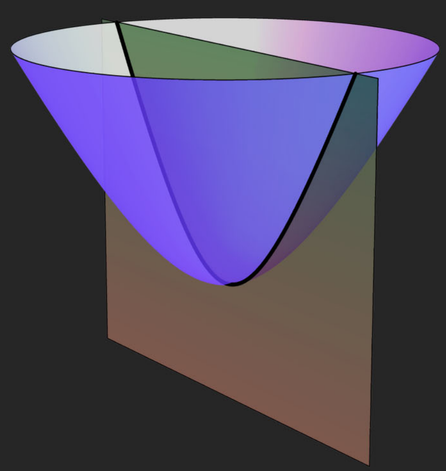
$$\gamma\colon I \to \Sigma \text{ geodesic} \Rightarrow f \circ \gamma\colon I \to M \text{ geodesic.}$$



$\mathbb{R}^k \subseteq \mathbb{R}^n$



$\mathsf{S}^k \subseteq \mathsf{S}^n$



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General problem

- $f\colon \Sigma^k \to M^n$ is *compatible* if $\widetilde{f}\colon \Sigma \to \mathsf{Gr}_k(TM)$ given by

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Given M, classify (equivalence classes of) inextendable compatible totally geodesic immersions to M up to congruence.

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- Butruille ('06): Classification of homogeneous nearly Kähler manifolds G/K in dimension six:

M	G	K
S^6	G_2	$\mathsf{SU}(3)$
$\mathsf{F}(\mathbb{C}^3)$	$\mathsf{SU}(3)$	T^2
$\mathbb{C}\mathsf{P}^3$	$\mathsf{Sp}(2)$	$\mathsf{U}(1) \times \mathsf{Sp}(1)$
$\mathsf{S}^3 \times \mathsf{S}^3$	$\mathsf{SU}(2)^3$	$\Delta\mathsf{SU}(2)$

Previously known results

- $\mathbb{C}\mathrm{P}^3$:
  - Totally geodesic + Lagrangian (Aslan '23, Liefsoens '22).
  - Totally geodesic + J-holomorphic curve (Cwiklinski, Vrancken '22).
- $\mathrm{F}(\mathbb{C}^3)$:
  - Totally geodesic + Lagrangian (Storm '20).
- $\mathrm{S}^3 \times \mathrm{S}^3$:
  - Totally geodesic + Lagrangian (Zhang, Dioos, Hu, Vrancken, Wang '16).
  - Totally geodesic + J-holomorphic curve (Bolton, Dillen, Dioos, Vrancken '22).

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$$(\nabla^c_{X^*} Y^*)_o = -[X, Y]_{\mathfrak{p}}.$$

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- Difference tensor $\alpha = \nabla - \nabla^c \colon \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$.

$$\alpha(X, Y) = \frac{1}{2}[X, Y]_{\mathfrak{p}}.$$

Totally geodesic subspaces

- $f\colon \Sigma \to M$ totally geodesic. Then f is determined by any tangent subspace $V \in \widetilde{f}(\Sigma)$:

$$f_i\colon \Sigma_i \to M, \quad \widetilde{f_1}(\Sigma_1) \cap \widetilde{f_2}(\Sigma_2) \neq \varnothing \Rightarrow f_1 \simeq f_2.$$

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Tojo's criterion. M naturally reductive, $\mathfrak{v} \subseteq \mathfrak{p}$. The following are equivalent:

1. $\mathfrak{v}$ is a totally geodesic subspace.
2. The subspace $e^{\nabla X^*}\mathfrak{v}$ is R-invariant for all $X \in \mathfrak{v}$.

α-invariant totally geodesic submanifolds

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- Σ is α-*invariant* if $T_x\Sigma$ is α-invariant for all $x \in \Sigma$.

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Theorem. $\mathfrak{v} \subseteq \mathfrak{p}$ invariant under R and α. Then:

1. $\mathfrak{s} = [\mathfrak{v}, \mathfrak{v}] + \mathfrak{v} = [\mathfrak{v}, \mathfrak{v}]_{\mathfrak{k}} \oplus \mathfrak{v}$ is a subalgebra.
2. $\Sigma = \mathsf{S} \cdot o$ is totally geodesic and α-invariant with $T_o\Sigma = \mathfrak{v}$.

$\mathbb{C}\mathsf{P}^3 = \mathsf{Sp}(2)/(\mathsf{U}(1) \times \mathsf{Sp}(1))$

Submanifold	Orbit of	Relationship with J
$\mathbb{R}\mathsf{P}^3_{\mathbb{C},1/2}(\sqrt{2})$	$\mathsf{SU}(2)$	Lagrangian
$\mathsf{S}^2(1/\sqrt{2})$	$\mathsf{Sp}(1)_f$	J-holomorphic
$\mathsf{S}^2(1)$	$\mathsf{SU}(2)$	J-holomorphic
$\mathsf{S}^2(\sqrt{5})$	$\mathsf{SU}(2)_{\Lambda_3}$	J-holomorphic

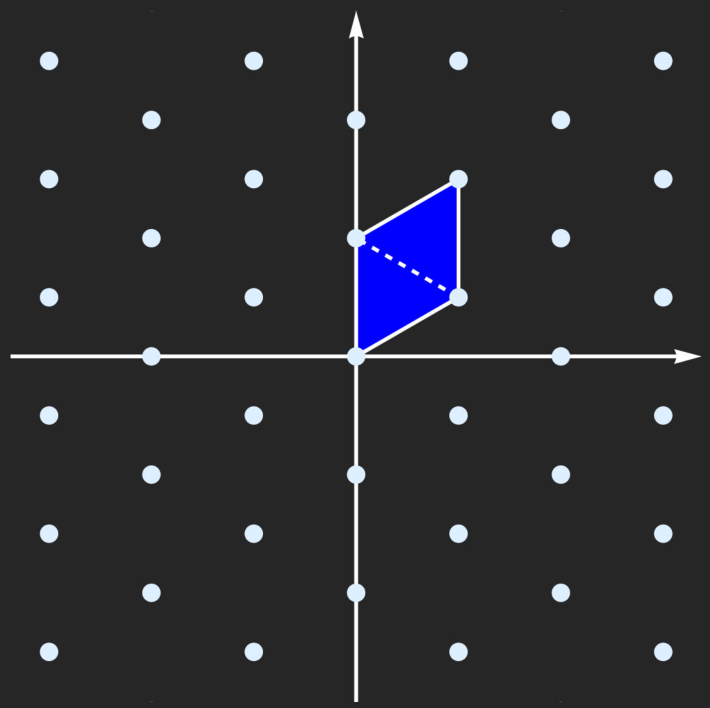
$\Lambda_3 = \mathsf{S}^3(\mathbb{C}^2)$ is the four-dimensional irrep. of $\mathsf{SU}(2)$.

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mapsto \begin{pmatrix} a(|a|^2 - 2|b|^2) & -\sqrt{3}a^2\bar{b} \\ \sqrt{3}a^2\bar{b} & a^3 \end{pmatrix} + \mathbf{j} \begin{pmatrix} b(2|a|^2 - |b|^2) & -\sqrt{3}ab^2 \\ \sqrt{3}\bar{a}b^2 & -b^3 \end{pmatrix}$$

$\mathsf{F}(\mathbb{C}^3) = \mathsf{SU}(3)/\mathsf{T}^2$

Submanifold	Orbit of	Relationship with J
$\mathsf{F}(\mathbb{R}^3) = \mathsf{S}^3(2\sqrt{2})/\mathsf{Q}_8$	$\mathsf{SO}(3)$	Lagrangian
$\mathsf{S}^3_{\mathbb{C},1/4}(2)$	$\mathsf{SU}(2)$	Lagrangian
T_Λ	T^2	J-holomorphic
$\mathsf{S}^2(1/\sqrt{2})$	$\mathsf{U}(2)$	J-holomorphic
$\mathsf{S}^2(\sqrt{2})$	$\mathsf{SO}(3)$	J-holomorphic
$\mathbb{R}\mathsf{P}^2(2\sqrt{2})$	Inhomogeneous	Totally real

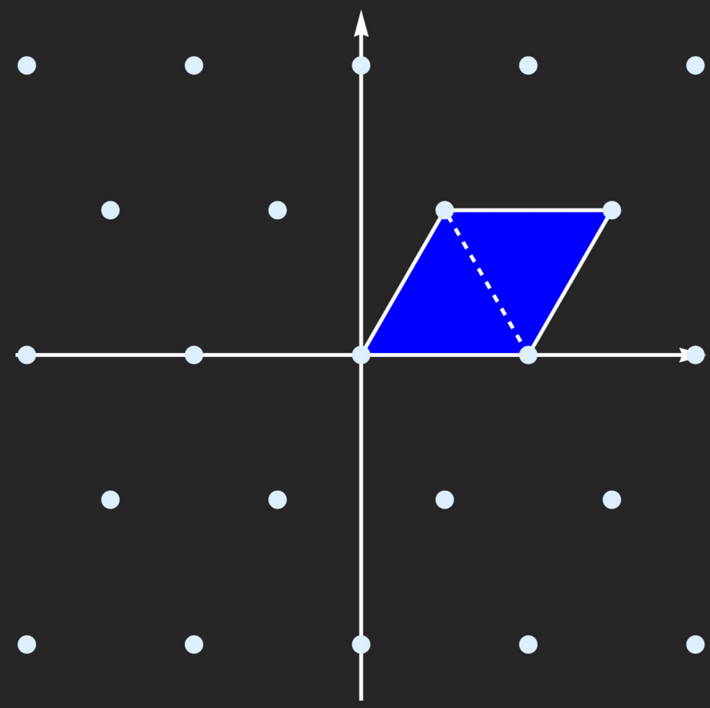
$$\Lambda = \left\langle \sqrt{\tfrac{2}{3}}\pi\,(0,2), \sqrt{2}\pi\left(1, \tfrac{1}{\sqrt{3}}\right) \right\rangle$$



$S^3 \times S^3 = SU(2)^3/\Delta SU(2)$

Submanifold	Orbit of	Relationship with J
$S^3(2/\sqrt{3})$	$SU(2)_2$	Lagrangian
$S^3_{\mathbb{C},1/3}(2)$	$SU(2)_{13,2}$	Lagrangian
T_Γ	$T \subseteq U(1)^3$	J-holomorphic
$S^2(\sqrt{3/2})$	$\Delta SU(2)$	J-holomorphic
$S^2(2/\sqrt{3})$	$H \subseteq SU(2)_{13,2}$	Totally real

$$\Gamma = \left\langle \frac{4\pi}{\sqrt{3}}(1,0), \frac{2\pi}{\sqrt{3}}(1,\sqrt{3}) \right\rangle$$



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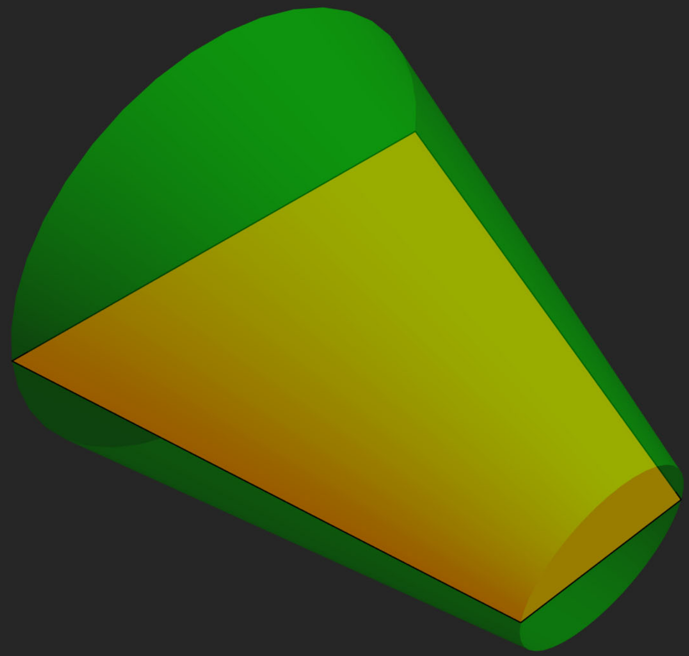
- If $M \neq \mathsf{S}^6$ and $\pi_1(M) = 0$, then $\mathrm{Hol}(\widehat{M}) = \mathsf{G}_2$.

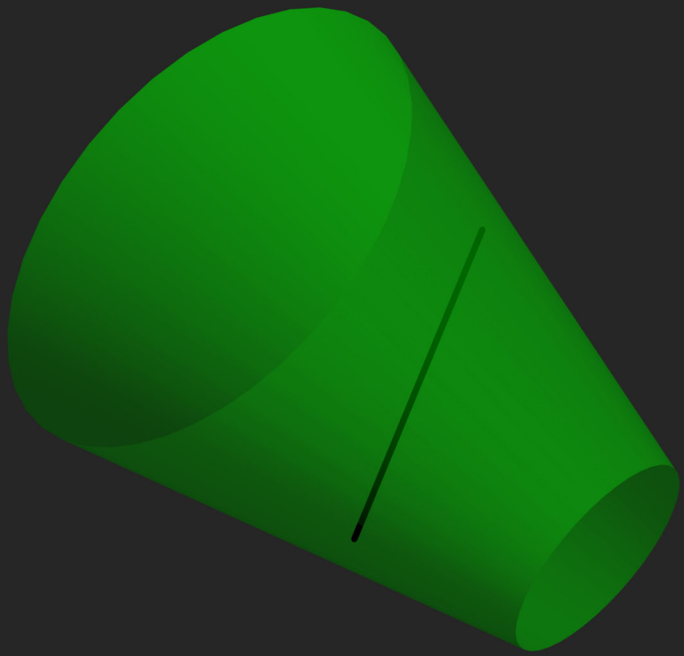
Theorem (LN, Rodríguez-Vázquez). Σ totally geodesic in $\widehat{M}$. Then one of the two holds:

i. $\Sigma = \widehat{S}$ for $S \to M$ totally geodesic.

ii. Σ is (up to surjective local isometry) a totally geodesic hypersurface in $\widehat{S}$ for $S \to M$ totally geodesic.



$\Sigma = \widehat{S}$



$\Sigma \subseteq \widehat{S}$

Corollary. $\Sigma \to \widehat{M}$ maximal totally geodesic submanifold. Then one of the two holds:

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Theorem (LN, Rodríguez-Vázquez). Let $\widehat{M}$ be a cohomogeneity one G_2-cone. Every maximal totally geodesic submanifold of $\widehat{M}$ is

i. Associative (i.e. calibrated by the G_2-structure ϕ) if $\dim \Sigma = 3$.

ii. Coassociative (i.e. calibrated by $\star\phi$) if $\dim \Sigma = 4$.

Ambient	Submanifold	Orbit of	Relationship with J
$\mathbb{C}P^3$	$\mathbb{R}P^3_{\mathbb{C},1/2}(\sqrt{2})$	$SU(2)^j$	Lagrangian
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$F(\mathbb{C}^3)$	$F(\mathbb{R}^3)$	$SO(3)$	Lagrangian
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$S^3 \times S^3$	$S^3(2/\sqrt{3})$	$SU(2)_2$	Lagrangian
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