

FIGUEIRA DA FOZ

1989



Reduction of time-dependent Hamiltonian systems in the presence of a Hamiltonian symmetry Lie group

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Symmetry and shape. Celebrating the 60 birthday of Prof.
E. García Río

23-27 September 2024

Santiago de Compostela, Spain

Work in progress (in collaboration with I Guhérrez,
D. Iglesias and E Padrón)

Aim of the talk: Cosymplectic geometry is not appropriate for the reduction of time-dependent Hamiltonian systems

Our idea: To replace cosymplectic structures by a special type of presymplectic structures: mechanical presymplectic structures

PLAN OF THE TALK

1. Cosymplectic structures and (unreduced) time-dependent Hamiltonian systems

2. Cosymplectic reduction and time-dependent Hamiltonian systems

3. Problems in the application of Albert's method to the reduction of time-dependent Hamiltonian systems

4. Reduction of mechanical presymplectic structures

5. Conclusions

1. Cosymplectic structures and (unreduced) time-dependent Hamiltonian systems

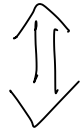
1.1 Cosymplectic structures

M a manifold, $\dim M = 2n+1$

$\omega \in \Omega^2(M)$, $\eta \in \Omega^1(M)$

(ω, η) a cosymplectic structure

$$\eta \wedge \omega \wedge \underbrace{\omega}_{n} \wedge \omega$$



$\eta \wedge \omega^n$ a volume form, $d\omega = 0$, $d\eta = 0$

(ω, η) a cosymplectic structure on M



$$\exists! R \in \mathfrak{X}(M) \quad / \quad i_R \omega = 0, \quad i_R \eta = 1$$

R the Reeb vector field of M

R preserves the cosymplectic structure



$$\mathcal{L}_R \omega = 0, \quad \mathcal{L}_R \eta = 0$$

Darboux coordinates for a coisotropic structure (ω, η)

$\exists (q^1, \dots, q^n, p_1, \dots, p_n, t)$ local coordinates on M /

$$\omega = dq^i \wedge dp_i, \quad \eta = dt, \quad \mathcal{R} = \frac{\partial}{\partial t}$$

Dynamics on cosymplectic manifolds

(ω, η) a cosymplectic structure on M

$$H: M \rightarrow \mathbb{R} \in C^\infty(M)$$

$$\exists! X_H^{(\omega, \eta)} \in \mathfrak{X}(M) \quad / \quad \begin{aligned} i_{X_H^{(\omega, \eta)}} \omega &= dH - \mathcal{R}(H)\eta \\ i_{X_H^{(\omega, \eta)}} \eta &= 0 \end{aligned}$$

$X_H^{(\omega, \eta)}$ \equiv the Hamiltonian vector field of H with respect to (ω, η)

The evolution vector field of H with respect to (ω, η)

$$E_H^{(\omega, \eta)} = R + X_H^{(\omega, \eta)}$$

$X_H^{(\omega, \eta)}$ and $E_H^{(\omega, \eta)}$ don't preserve the symplectic structure

$$\mathcal{L}_{X_H^{(\omega, \eta)}} \omega = \mathcal{L}_{E_H^{(\omega, \eta)}} \omega = -d(\Omega(H)) \wedge \eta$$

Local expressions

$$X_H^{(w,n)} = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i}$$

$$E_H^{(w,n)} = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} + \frac{\partial}{\partial t}$$

Integral curves of $E_H^{(w,n)}$

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \dot{t} = 1$$

Hamilton equations for $H = H(q^i, p_i, t)$

1.2 Time-dependent Hamiltonian systems

- The configuration space: Q a smooth manifold of dimension n
- The phase space of momenta: $\mathbb{R} \times T^*Q$
- The cosymplectic structure: on $\mathbb{R} \times T^*Q$:

$$(\omega, \eta) = (\omega_Q, dt)$$

$\omega_Q \equiv$ the canonical symplectic structure on T^*Q

- The Reeb vector field: $\partial/\partial t$

- The Hamiltonian function:

$$H: \mathbb{R} \times T^*Q \rightarrow \mathbb{R} \in C^\infty(\mathbb{R} \times T^*Q)$$

- The dynamical vector field:

$$E_H(w, dt) = \frac{\partial}{\partial t} + X_H(w, dt) \in \mathfrak{X}(\mathbb{R} \times T^*Q)$$

- Solutions of the Hamilton equations:

Integral curves of $E_H(w, dt)$

A common trick

We consider a new cosymplectic structure such that its Reeb vector field is just the evolution vector field of H

- H -cosymplectic structure (ω_H, η) on $\mathbb{R} \times T^*Q$

$$(\omega_H = \omega_Q + dH \wedge dt, dt)$$

R_H the Reeb vector field



$$\boxed{E_H = R_H}$$

Summarizing

- Unreduced time-dependent Hamiltonian systems



Geometry: ω -symplectic

Dynamics: Reeb ω -symplectic

2. Cosymplectic reduction and time-dependent Hamiltonian systems

Cosymplectic and Reeb ^{dynamics} reduction (Albert, 89)

(M, ω, η) a connected cosymplectic manifold

G a connected Lie group

$\phi: G \times M \rightarrow M$ an action of G on M

ϕ a cosymplectic action if

$$\phi_g^* \omega = \omega, \quad \phi_g^* \eta = \eta, \quad \forall g \in G$$

- ϕ cosymplectic



$$0 = \mathcal{L}_{\xi_{\mathfrak{g}}} \omega = d(i_{\xi_{\mathfrak{g}}} \omega), \quad \langle \eta, \xi_{\mathfrak{g}} \rangle = c_{\eta} \xi \in \mathbb{R}, \quad \forall \xi \in \mathfrak{g}$$

- ϕ cosymplectic \Rightarrow

$$\mathcal{R} \text{ is } G\text{-invariant}$$

$$\Downarrow$$

$$[\mathcal{R}, \xi_{\mathfrak{g}}] = 0, \quad \forall \xi \in \mathfrak{g}$$

Hamiltonian symplectic action

$$\exists J: M \rightarrow \mathfrak{g}^* \text{ a smooth map} / i_{\xi} \omega = dJ\xi, \forall \xi \in \mathfrak{g}$$

$$J: M \rightarrow \mathfrak{g}^* \text{ a momentum map for } \phi$$

Noether theorem

$$J: M \rightarrow \mathfrak{g}^* \text{ a momentum map for } \phi$$



J is a first integral of R

Assumption: $J: M \rightarrow \mathfrak{g}^*$ is G -equivariant

Theorem (Albert, 89)

$\phi: G \times M \rightarrow M$ a Hamiltonian co-symplectic action

$J: M \rightarrow \mathfrak{g}^*$ a momentum map for ϕ

$\mu \in \mathfrak{g}^*$ a regular value of J

1) G_μ acts \Downarrow on the regular submanifold $J^{-1}(\mu)$

If $M_\mu = \frac{J^{-1}(\mu)}{G_\mu}$ is a quotient manifold and

$$\langle \Omega, \xi \rangle = 0, \forall \xi \in \mathfrak{g}$$



cosymplectic reduction

ii) $\exists!$ cosymplectic structure (ω_μ, Ω_μ) on M_μ such that

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega, \quad \pi_\mu^* \Omega_\mu = i_\mu^* \Omega$$

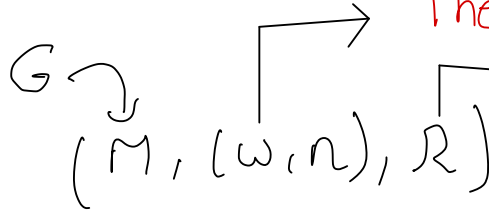
$$\pi_\mu: J^{-1}(\mu) \longrightarrow M_\mu = \frac{J^{-1}(\mu)}{G_\mu} \quad \text{canonical projection}$$

$$i_\mu: J^{-1}(\mu) \hookrightarrow M \quad \text{canonical inclusion}$$

Reeb reduction

iii) $R|_{J^{-1}(M)}$ is tangent to $J^{-1}(M)$ and TU -projectable.
Its projection R_M is the Reeb vector field of the
cosymplectic manifold (M, ω_M, η_M)

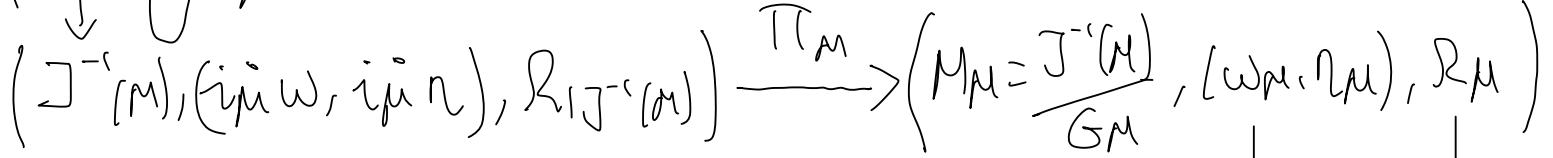
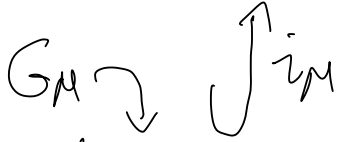
Summarizing:



The unreduced cosymplectic structure
 The unreduced Reeb vector field

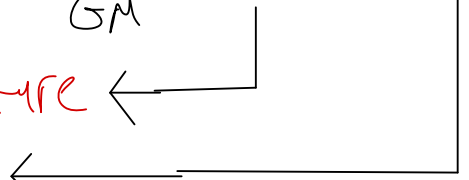
$$\langle \Omega, \xi_M \rangle = 0, \forall \xi \in \mathfrak{g}$$

(Albert's condition)



The reduced cosymplectic structure

The reduced Reeb vector field



3. Problems in the application of Albert's method to the reduction of time-dependent Hamiltonian systems

Problems are related with Albert's condition

• The typical momentum maps in Albert's reduction don't depend on time \Rightarrow First integrals don't depend on time! (however, we deal with time-dependent Hamiltonian systems!)

• $\langle dt, \mathbb{R} \times T^*Q \rangle = 0 \Rightarrow$ Action ϕ leaves invariant the time! (however, we deal with time-dependent Hamiltonian systems!)

An example

- Particle of mass m moving in a harmonic oscillator potential with frequency Ω in N dimensions
- The system is described by an inertial observer that moves with constant velocity $\mathbf{v} = (v, 0, \dots, 0)$

Hamiltonian of the system $H: \mathbb{R} \times T^* \mathbb{R}^N \rightarrow \mathbb{R}$

$$H(t, q^i, p_i) = \frac{1}{2m} \sum_{i=1}^N p_i^2 + \frac{1}{2} m \Omega^2 \left((q^1 + vt)^2 + \sum_{\alpha=2}^N (q^\alpha)^2 \right)$$

- ϕ cosymplectic action of \mathbb{R} for the cosymplectic structure $(\omega_H, d\theta)$ on $\mathbb{R} \times T^*\mathbb{R}^N$

$$\phi(s, (t, q^i, p_i)) = (t+s, q^i - vs, q^i, p_i), \quad s \in \mathbb{R}, (t, q^i, p_i) \in \mathbb{R} \times T^*\mathbb{R}^N$$

- A \mathbb{R} -equivariant momentum map $J: \mathbb{R} \times T^*\mathbb{R}^N \rightarrow \mathbb{R}$

$$J(t, q^i, p_i) = -v p_1 + H(t, q^i, p_i)$$

It seems that we could make reduction...

• Ho never

$$\langle 2, 1 \rangle_{\mathbb{R} \times T^* \mathbb{R}^N} = 1 \neq 0$$



Albert's reduction doesn't work!

Albert's condition

$$\langle \Omega, \xi_{112} \times T^*Q \rangle = 0, \quad \forall \xi \in \mathfrak{g}$$

is the origin of our problems

We need this condition in order to obtain a reduced
cosymplectic structure in Albert's method

Conclusion: Cosymplectic geometry is not appropriate for
reduced time-dependent Hamiltonian systems

We must remove the cosymplectic 1-form η of the picture



We maintain the cosymplectic 2-form ω and the Reeb vector field R



We have a presymplectic structure of corank 1 (the 2-form ω) with parallelizable characteristic foliation (which is generated by R)

4. Reduction of mechanical presymplectic structures

4.1 Mechanical presymplectic structures

M a smooth manifold, $\dim M = 2n+1$

A mechanical presymplectic structure on M

$$(\omega, \mathcal{R}) \in \Omega^2(M) \times \mathcal{X}(M) \quad /$$

i) ω is a presymplectic structure of corank 1

$$\boxed{d\omega = 0, \omega \text{ of rank } 2n} \quad \left[\omega^n(x) = \omega(x) \wedge^{n-1} \omega(x) \neq 0 \right. \\ \left. \forall x \in M \right]$$

ii) \mathcal{R} is the Reeb vector field and $\boxed{\text{Ker } \omega = \langle \mathcal{R} \rangle}$

Examples:

i) (ω, η) a cosymplectic structure with Reeb vector field $\mathcal{R} \Rightarrow (\omega, \mathcal{R})$ a mechanical presymplectic structure

ii) The canonical contact structure on S^{2n+1} ($n \geq 1$) induces a mechanical presymplectic structure on S^{2n+1} .
However, S^{2n+1} doesn't admit cosymplectic structures by topological obstructions.

4.2 Symmetry Lie group for a mechanical presymplectic structure and reduction

(ω, \mathcal{R}) a mechanical presymplectic structure on M

G a connected Lie group

$\phi: G \times M \rightarrow M$ an action of G on M

ϕ a presymplectic mechanical action

$$i) \phi_g^* \omega = \omega, \quad (T\phi_g) \circ \mathcal{R} = \mathcal{R} \circ \phi_g, \quad \forall g \in G$$

$$ii) \mathcal{R}(x) \notin T_x(G \cdot x), \quad \forall x \in M$$

Remark:

$$i) \Leftrightarrow \mathcal{L}_{\xi_M} \omega = d(i_{\xi_M} \omega) = 0, \quad [\mathcal{R}, \xi_M] = 0, \quad \forall \xi \in \mathfrak{g}$$

Hamiltonian mechanical presymplectic action

$\phi: G \times M \rightarrow M$ a mechanical presymplectic action /

$\exists J: M \rightarrow \mathfrak{g}^*$ a smooth map and

$$i_{\xi_M} \omega = dJ_\xi, \quad \forall \xi \in \mathfrak{g}$$

$J: M \rightarrow \mathfrak{g}^*$ a momentum map for ϕ

Noether theorem

$J: \mathcal{M} \rightarrow \mathfrak{g}^*$ a momentum map for ϕ



J is a first integral of R

Assumption: $J: \mathcal{M} \rightarrow \mathfrak{g}^*$ is G -equivariant

Theorem

$\phi: G \times \mathcal{M} \rightarrow \mathcal{M}$ a Hamiltonian presymplectic mechanical action

$J: \mathcal{M} \rightarrow \mathfrak{g}^*$ a momentum map for ϕ

$\mu \in \mathfrak{g}^*$ a regular value of J

1) G_μ acts \Downarrow on the regular submanifold $J^{-1}(\mu)$

If $M_\mu = \frac{J^{-1}M}{G_\mu}$ is a quotient manifold then

Presymplectic reduction

(ii) $\exists!$ a closed 2-form ω_μ on M_μ

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega,$$
$$\pi_\mu: J^{-1}(M) \longrightarrow M_\mu = \frac{J^{-1}(M)}{G_\mu} \text{ canonical projection}$$
$$i_\mu: J^{-1}(M) \hookrightarrow M \text{ canonical inclusion}$$

Reeb reduction {
iii) $R|_{J^{-1}(M)}$ is tangent to $J^{-1}(M)$ and $T\mu$ -projectable.
Its projection R_M satisfies
 $\text{Ker } \omega_M = \langle R_M \rangle$

Thus, (ω_M, R_M) is a (reduced) presymplectic mechanical structure on $M_M = \frac{J^{-1}(M)}{GM}$

4.3 Application to the reduction of time-dependent Hamiltonian systems

The unreduced time-dependent Hamiltonian system

A cosymplectic structure (ω, η) on M

$\phi: G \times M \rightarrow M$ a Hamiltonian cosymplectic action

$J: M \rightarrow \mathfrak{g}^*$ an equivariant momentum map

$H: M \rightarrow \mathbb{R}$ a G -invariant Hamiltonian function

$E_H^{(\omega, \eta)}$ the evolution vector field of H

$E_H^{(\omega, \eta)}(x) \in T_x(G \cdot x), \forall x \in M$

First step: Modification of the cosymplectic structure and the momentum map

$(\omega_H = \omega + dH \lrcorner \eta, \eta)$ new cosymplectic structure on M

$E_H^{(\omega, \eta)}$ the Reeb vector field of (ω_H, η)

$J_H: M \rightarrow \mathfrak{g}^*$ new equivariant momentum map

$$J_H(x) = J(x) - H(x) \eta_x, \quad \forall x \in M$$



• $(\omega_H, E_H^{(\omega, \eta)})$ a mechanical presymplectic structure

• $\phi: GX(M, \omega_H, E_H^{(\omega, \eta)}) \rightarrow (M, \omega_H, E_H^{(\omega, \eta)})$ Hamiltonian presymplectic action

Second step: Reduction of the mechanical presymplectic structure (ω_H, E_H)

$\mu \in \mathfrak{g}^*$ a regular value of J_H

Reduced mechanical presymplectic structure on $\frac{J_H^{-1}(\mu)}{G\mu}$
 $((\omega_H)_\mu, (E_H)_\mu)$

Our example (revisited)

- Particle of mass m moving in a harmonic oscillator potential with frequency Ω in N dimensions
- The system is described by an inertial observer that moves with constant velocity $\mathbf{V} = (v, 0, \dots, 0)$

We can apply our method and we obtain a **geometric reduction** (a reduced mechanical presymplectic structure) and a **dynamical reduction** (a reduced Reeb vector field)

5. Conclusions

- Albert's reduction method doesn't work satisfactorily in the reduction of time-dependent Hamiltonian systems and, therefore, **cosymplectic geometry is not appropriate for this reduction.**
- We replace cosymplectic structures by **mechanical presymplectic structures**
- We develop reduction of mechanical presymplectic structures and we apply this process to interesting examples of time-dependent Hamiltonian systems for which Albert's reduction method doesn't work.

THANKS!

CONGRATULATIONS EDUARDO!