FIGUEIRA DA FOZ

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Reduction of time-dependent Hamiltonian systems in the presence of a Hamiltonian symmetry Lie group

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Aim of the talk: Cosymplectic geometry is not appropriate for the reduction of time-dependent Hamiltonian systems

Our idea: To replace cosymplectic structures by a special type of presymplectic structures: mechanical presymplectic structures

PLAN OF THE TALK

1. Cosymplectic structures and (unreduced) time-dependent Hamiltonian systems
2. Cosymplectic reduction and time-dependent Hamiltonian systems
3. Problems in the application of Albert's method to the reduction of time-dependent Hamiltonian systems
4. Reduction of mechanical presymplectic structures
5. Conclusions

1. Cosymplectic structures and (unreduced) time-dependent Hamiltonian systems

1.1 Cosymplectic structures

M a manifold, $\dim M = 2n+1$

$\omega \in \Omega^2(M)$, $\eta \in \Omega^1(M)$

(ω, η) a cosymplectic structure

$\eta \wedge \underbrace{\omega \wedge \cdots \wedge \omega}_{n}$ $\Longleftrightarrow$ $d\omega = 0$, $d\eta = 0$

$\quad \parallel$

$\eta \wedge \omega^n$ a volume form

(ω, η) a cosymplectic structure on M

$$\Downarrow$$

$$\exists!\ R \in \mathfrak{X}(M) \ / \ i_R \omega = 0, \ i_R \eta = 1$$

R the Reeb vector field of M

R preserves the cosymplectic structure

$$\Updownarrow$$

$$\boxed{\mathcal{L}_R \omega = 0, \ \mathcal{L}_R \eta = 0}$$

Darboux coordinates for a cosymplectic structure (ω, η)

$$\boxed{\begin{array}{l} \exists\, (q^1, \dots, q^n, p_1, \dots, p_n, t) \text{ local coordinates on } M \;/ \\ \omega = dq^i \wedge dp_i, \quad \eta = dt, \quad R = \partial/\partial t \end{array}}$$

Dynamics on cosymplectic manifolds

(ω, η) a cosymplectic structure on M

$H : M \to \mathbb{R} \in C^\infty(M)$

$\Downarrow$

$$\boxed{\exists!\, X_H^{(\omega,\eta)} \in \mathfrak{X}(M) \;/\; \begin{array}{l} i_{X_H^{(\omega,\eta)}}\, \omega = dH - \mathcal{R}(H)\, \eta \\ i_{X_H^{(\omega,\eta)}}\, \eta = 0 \end{array}}$$

$X_H^{(\omega,\eta)} \equiv$ the Hamiltonian vector field of H with respect to (ω, η)

The evolution vector field of H with respect to (ω, η)

$$\boxed{E_H^{(\omega,\eta)} = R + X_H^{(\omega,\eta)}}$$

$X_H^{(\omega,\eta)}$ and $E_H^{(\omega,\eta)}$ don't preserve the cosymplectic structure

$$\boxed{\mathcal{L}_{X_H^{(\omega,\eta)}} \omega = \mathcal{L}_{E_H^{(\omega,\eta)}} \omega = -d(R(H)) \wedge \eta}$$

Local expressions

$$\boxed{\begin{aligned} X_H^{(\omega,\eta)} &= \frac{\partial H}{\partial p_i}\partial/\partial q^i - \frac{\partial H}{\partial q^i}\partial/\partial p_i \\ E_H^{(\omega,\eta)} &= \frac{\partial H}{\partial p_i}\partial/\partial q^i - \frac{\partial H}{\partial q^i}\partial/\partial p_i + \partial/\partial t \end{aligned}}$$

Integral curves of $E_H^{(\omega,\eta)}$

$$\boxed{\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \dot{t} = 1}$$

Hamilton equations for $H = H(q^i, p_i, t)$

1.2 Time-dependent Hamiltonian systems

- The configuration space: Q a smooth manifold of dimension n
- The phase space of momenta: $\mathbb{R} \times T^*Q$
- The cosymplectic structure on $\mathbb{R} \times T^*Q$:

$$(\omega, \eta) = (\omega_Q, dt)$$

$\omega_Q \equiv$ the canonical symplectic structure on T^*Q

- The Reeb vector field: $\partial/\partial t$

- The Hamiltonian function:

$$H : \mathbb{R} \times T^*Q \to \mathbb{R} \in C^\infty(\mathbb{R} \times T^*Q)$$

- The dynamical vector field:

$$E_H^{(\omega_Q, dt)} = \partial/\partial t + X_H^{(\omega_Q, dt)} \in \mathfrak{X}(\mathbb{R} \times T^*Q)$$

- Solutions of the Hamilton equations:

Integral curves of $E_H^{(\omega_Q, dt)}$

A common trick

We consider a new cosymplectic structure such that its Reeb vector field is just the evolution vector field of H

- H-cosymplectic structure (ω_H, η) on $\mathbb{R}\times T^*Q$

$$(\omega_H = \omega_Q + dH\wedge dt, dt)$$

R_H the Reeb vector field

$$\Downarrow$$

$$\boxed{E_H^{(\omega_Q, dt)} = R_H}$$

Summarizing

· Unreduced time-dependent Hamiltonian systems

$\Updownarrow$

Geometry: Cosymplectic

Dynamics: Reeb cosymplectic

2. Cosymplectic reduction and time-dependent Hamiltonian systems

Cosymplectic and $\underbrace{\text{Reeb}}_{}$ reduction (Albert, 89)

dynamics

(M, ω, η) a connected cosymplectic manifold

G a connected Lie group

$\phi : G \times M \longrightarrow M$ an action of G on M

ϕ a cosymplectic action if

$$\boxed{\phi_g^* \omega = \omega, \quad \phi_g^* \eta = \eta, \quad \forall g \in G}$$

- ϕ cosymplectic $\Updownarrow$

$$\boxed{0 = \mathcal{L}_{\xi_M}\omega = d(i_{\xi_M}\omega), \quad \langle \eta, \xi_M \rangle = c_\eta \xi \in \mathbb{R}, \ \forall \xi \in \mathfrak{g}}$$

- ϕ cosymplectic $\Rightarrow$ $\boxed{\begin{array}{c} R \text{ is } G\text{-invariant} \\ \Updownarrow \\ [R, \xi_M] = 0, \ \forall \xi \in \mathfrak{g} \end{array}}$

Hamiltonian symplectic action

$\boxed{\exists\, J : M \to \mathfrak{g}^* \text{ a smooth map } / \ i_{\xi_M}\omega = dJ_\xi,\ \forall \xi \in \mathfrak{g}}$

$J : M \to \mathfrak{g}^*$ a momentum map for ϕ

Noether theorem

$J : M \to \mathfrak{g}^*$ a momentum map for ϕ

$\Downarrow$

J is a first integral of R

Assumption : $J : M \to \mathfrak{g}^*$ is G-equivariant

Theorem (Albert, 89)

$\phi : G \times M \to M$ a Hamiltonian cosymplectic action

$J : M \to \mathfrak{g}^*$ a momentum map for ϕ

$\mu \in \mathfrak{g}^*$ a regular value of J

$\Downarrow$

1) G_μ acts on the regular submanifold $J^{-1}(\mu)$

If $M_\mu = \dfrac{J^{-1}(\mu)}{G_\mu}$ is a quotient manifold and $\boxed{\langle \eta, \xi_M \rangle = 0, \ \forall \xi \in \mathfrak{g}}$

$$\Downarrow$$

Cosymplectic reduction

ii) $\exists!$ cosymplectic structure (ω_μ, η_μ) on M_μ such that

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega, \quad \pi_\mu^* \eta_\mu = i_\mu^* \eta$$

$\pi_\mu : J^{-1}(\mu) \longrightarrow M_\mu = \dfrac{J^{-1}(\mu)}{G_\mu}$ canonical projection

$i_\mu : J^{-1}(\mu) \hookrightarrow M$ canonical inclusion

Reeb reduction

iii) $R_{|J^{-1}(\mu)}$ is tangent to $J^{-1}(\mu)$ and π_μ-projectable.
Its projection R_μ is the Reeb vector field of the cosymplectic manifold $(M_\mu, \omega_\mu, \eta_\mu)$

Summarizing:

The unreduced cosymplectic structure

The unreduced Reeb vector field

$$G \curvearrowright \left(M, (\omega, \eta), R\right) + \boxed{\langle \eta, \xi_M \rangle = 0, \ \forall \xi \in \mathfrak{g}}$$

(Albert's condition)

$$\Big\uparrow i_\mu$$

$$G_\mu \curvearrowright \left(J^{-1}(\mu), (i_\mu^* \omega, i_\mu^* \eta), R|_{J^{-1}(\mu)}\right) \xrightarrow{\pi_\mu} \left(M_\mu = \frac{J^{-1}(\mu)}{G_\mu}, (\omega_\mu, \eta_\mu), R_\mu\right)$$

The reduced cosymplectic structure

The reduced Reeb vector field

3. Problems in the application of Albert's method to the reduction of time-dependent Hamiltonian systems

Problems are related with Albert's condition

- The typical momentum maps in Albert's reduction don't depend on time $\Rightarrow$ First integrals don't depend on time!. (however, we deal with time-dependent Hamiltonian systems!)

- $\langle dt, \xi_{\mathbb{R}\times T^*Q} \rangle = 0 \Rightarrow$ Action Φ leaves invariant the time! (however, we deal with time-dependent Hamiltonian systems!)

An example

- Particle of mass m moving in a harmonic oscillator potential with frequency Ω in N dimensions
- The system is described by an inertial observer that moves with constant velocity $\mathbf{V}=(v,0,\dots,0)$

Hamiltonian of the system $\quad H: \mathbb{R}\times T^*\mathbb{R}^N \longrightarrow \mathbb{R}$

$$H(t,q^i,p_i)=\frac{1}{2m}\sum_{i=1}^{N}p_i^2+\frac{1}{2}m\Omega^2\left((q^1+vt)^2+\sum_{\alpha=2}^{N}(q^\alpha)^2\right)$$

- Φ cosymplectic action of $\mathbb{R}$ for the cosymplectic structure (ω_H, dt) on $\mathbb{R}\times T^*\mathbb{R}^N$

$$\boxed{\Phi(s,(t,q^i,p_i)) = (t+s,\ q^1 - vs,\ q^\alpha,\ p_i),\ s\in\mathbb{R},\ (t,q^i,p_i)\in\mathbb{R}\times T^*\mathbb{R}^N}$$

- A $\mathbb{R}$-equivariant momentum map $J:\mathbb{R}\times T^*\mathbb{R}^N \longrightarrow \mathbb{R}$

$$\boxed{J(t,q^i,p_i) = -v\,p_1 + H(t,q^i,p_i)}$$

It seems that we could make reduction...

- However

$$\boxed{\langle 2 \cdot 1_{\mathbb{R}\times T^{\circ}\mathbb{R}^N} \rangle = 1 \neq 0}$$

$$\Downarrow$$

$\boxed{\text{Albert's reduction doesn't work!}}$

Albert's condition

$$\boxed{\langle \Omega, \xi_{\mathbb{R}\times TQ} \rangle = 0, \quad \forall \xi \in \mathfrak{g}}$$

is the origin of our problems

We need this condition in order to obtain a reduced cosymplectic structure in Albert's method

Conclusion: Cosymplectic geometry is not appropriate for reduced time-dependent Hamiltonian systems

We must remove the cosymplectic 1-form η of the picture

$$\Downarrow$$

We maintain the cosymplectic 2-form ω and the Reeb vector field R

$$\Downarrow$$

$$\boxed{\text{We have a presymplectic structure of corank 1 (the 2-form } \omega\text{) with parallelizable characteristic foliation (which is generated by } R\text{)}}$$

4. Reduction of mechanical presymplectic structures

4.1 Mechanical presymplectic structures

M a smooth manifold, $\dim M = 2n+1$

A mechanical presymplectic structure on M

$(\omega, R) \in \Omega^2(M) \times \mathfrak{X}(M)$ /

i) ω is a presymplectic structure of corank 1

$\boxed{d\omega = 0, \quad \omega \text{ of rank } 2n}$ $\left(\omega^n(x) = \omega(x) \wedge \overset{(n}{\ldots} \wedge \omega(x) \neq 0 \right.$
$\left. \forall x \in M \right)$

ii) R is the Reeb vector field and $\boxed{\operatorname{Ker} \omega = \langle R \rangle}$

Examples:

i) (ω, η) a cosymplectic structure with Reeb vector field $\mathcal{R}$ $\Rightarrow$ $(\omega, \mathcal{R})$ a mechanical presymplectic structure

ii) The canonical contact structure on S^{2n+1} $(n \geq 1)$ induces a mechanical presymplectic structure on S^{2n+1}. However, S^{2n+1} doesn't admit cosymplectic structures by topological obstructions.

4.2 Symmetry Lie group for a mechanical presymplectic structure and reduction

(ω, Ω) a mechanical presymplectic structure on M

G a connected Lie group

$\Phi : G \times M \longrightarrow M$ an action of G on M

Φ a presymplectic mechanical action

$$\boxed{\begin{array}{l} i) \quad \Phi_g^* \omega = \omega, \quad (T\Phi_g) \circ \Omega = \Omega \circ \Phi_g, \quad \forall g \in G \\[1em] ii) \quad \Omega(x) \notin T_x(G \cdot x), \quad \forall x \in M \end{array}}$$

Remark:

$$i) \iff \boxed{\mathcal{L}_{\xi_M} \omega = d(i_{\xi_M} \omega) = 0, \quad [\Omega, \xi_M] = 0, \quad \forall \xi \in \mathfrak{g}}$$

Hamiltonian mechanical presymplectic action

$\boxed{\begin{array}{l}\phi : G \times M \longrightarrow M \text{ a mechanical presymplectic action } / \\ \exists\, J : M \to \mathfrak{g}^* \text{ a smooth map and} \\ \qquad i_{\xi_M}\omega = dJ_\xi, \ \forall \xi \in \mathfrak{g}\end{array}}$

$J : M \to \mathfrak{g}^*$ a momentum map for ϕ

Noether theorem

$$\boxed{\begin{array}{c} J : M \to \mathfrak{g}^* \text{ a momentum map for } \phi \\ \Downarrow \\ J \text{ is a first integral of } R \end{array}}$$

Assumption : $J : M \to \mathfrak{g}^*$ is G-equivariant

Theorem

$\phi : G \times M \to M$ a Hamiltonian presymplectic mechanical action

$J : M \to \mathfrak{g}^*$ a momentum map for ϕ

$\mu \in \mathfrak{g}^*$ a regular value of J

$\Downarrow$

1) G_μ acts on the regular submanifold $J^{-1}(\mu)$

If $M_\mu = \dfrac{J^{-1}\mu}{G_\mu}$ is a quotient manifold then

Presymplectic reduction

ii) $\exists!$ a closed 2-form ω_μ on M_μ

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega ,$$

$\pi_\mu : J^{-1}(\mu) \longrightarrow M_\mu = \dfrac{J^{-1}(\mu)}{G_\mu}$ canonical projection

$i_\mu : J^{-1}(\mu) \hookrightarrow M$ canonical inclusion

Reeb reduction

iii) $R_{|J^{-1}(\mu)}$ is tangent to $J^{-1}(\mu)$ and π_μ-projectable.

Its projection R_μ satisfies

$$\ker \omega_\mu = \langle R_\mu \rangle$$

Thus, (ω_μ, R_μ) is a (reduced) presymplectic mechanical structure on $M_\mu = \frac{J^{-1}(\mu)}{G_\mu}$

4.3 Application to the reduction of time-dependent Hamiltonian systems

The unreduced time-dependent Hamiltonian system

A cosymplectic structure (ω, η) on M

$\Phi : G \times M \longrightarrow M$ a Hamiltonian cosymplectic action

$J : M \longrightarrow \mathfrak{g}^*$ an equivariant momentum map

$H : M \longrightarrow \mathbb{R}$ a G-invariant Hamiltonian function

$E_H^{(\omega,\eta)}$ the evolution vector field of H

$E_H^{(\omega,\eta)}(x) \notin T_x(G \cdot x), \ \forall x \in M$

First step: Modification of the cosymplectic structure and the momentum map

$\boxed{(\omega_H = \omega + dH \wedge \eta, \eta) \text{ new cosymplectic structure on } M}$

$E_H^{(\omega,\eta)}$ the Reeb vector field of (ω_H, η)

$$\boxed{\begin{array}{l} J_H : M \longrightarrow \mathfrak{g}^* \text{ new equivariant momentum map} \\ J_H(x) = J(x) - H(x)c\eta, \quad \forall x \in M \end{array}}$$

$$\Downarrow$$

- $\boxed{(\omega_H, E_H^{(\omega,\eta)}) \text{ a mechanical presymplectic structure}}$

- $\boxed{\phi : G \times (M, \omega_H, E_H^{(\omega,\eta)}) \longrightarrow (M, \omega_H, E_H^{(\omega,\eta)}) \text{ Hamiltonian presymplectic action}}$

Second step: Reduction of the mechanical presymplectic structure (ω_H, E_H)

$\mu \in \mathfrak{g}^*_H$ a regular value of J_H

$$\boxed{\begin{array}{l}\text{Reduced mechanical presymplectic structure on } \dfrac{J_H^{-1}(\mu)}{G_\mu} \\ \left((\omega_H)_\mu, \left(E_H^{(\omega, \eta)}\right)_\mu \right)\end{array}}$$

Our example (revisited)

- Particle of mass m moving in a harmonic oscillator potential with frequency Ω in N dimensions
- The system is described by an inertial observer that moves with constant velocity $\mathbf{V}=(v,0,\dots,0)$

We can apply our method and we obtain a geometric reduction (a reduced mechanical presymplectic structure) and a dynamical reduction (a reduced Reeb vector field)

5. Conclusions

- Albert's reduction method doesn't work satisfactorily in the reduction of time-dependent Hamiltonian systems and, therefore, cosymplectic geometry is not appropriate for this reduction.
- We replace cosymplectic structures by mechanical presymplectic structures
- We develop reduction of mechanical presymplectic structures and we apply this process to interesting examples of time-dependent Hamiltonian systems for which Albert's reduction method doesn't work.

THANKS!

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