Classification of cohomogeneity-one actions on noncompact symmetric spaces

Ivan Solonenko (joint work with Víctor Sanmartín-Lopez)

University of Stuttgart

Symmetry and Shape, 25 Sep 2024

Definition and motivation

Definition and motivation

Definition

Let M be a Riemannian manifold and H a connected Lie group. A proper isometric action $H \curvearrowright M$ is said to be of **cohomogeneity one** (**C1**-) if it has an orbit of codimension one.

Definition and motivation

Definition

Let M be a Riemannian manifold and H a connected Lie group. A proper isometric action $H \curvearrowright M$ is said to be of **cohomogeneity one** (**C1**-) if it has an orbit of codimension one.

Some motivation:

Definition and motivation

Definition

Let M be a Riemannian manifold and H a connected Lie group. A proper isometric action $H \curvearrowright M$ is said to be of **cohomogeneity one** (**C1**-) if it has an orbit of codimension one.

Some motivation:

- C1-actions can be used to construct *metrics with special properties* (*Einstein*, *special holonomy*, *positive sectional curvature*, etc.). C1-symmetry allows to *reduce PDEs to ODEs* (can be applied, e.g., to the Ricci flow).

Definition and motivation

Definition

Let M be a Riemannian manifold and H a connected Lie group. A proper isometric action $H \curvearrowright M$ is said to be of **cohomogeneity one** (**C1**-) if it has an orbit of codimension one.

Some motivation:

- C1-actions can be used to construct *metrics with special properties* (*Einstein*, *special holonomy*, *positive sectional curvature*, etc.). C1-symmetry allows to *reduce PDEs to ODEs* (can be applied, e.g., to the Ricci flow).
- The principal orbits of a C1-action are *homogeneous hypersurfaces* and thus are *isoparametric* and have *constant principal curvatures*.

Definition and motivation

Definition

Let M be a Riemannian manifold and H a connected Lie group. A proper isometric action $H \curvearrowright M$ is said to be of **cohomogeneity one** (**C1**-) if it has an orbit of codimension one.

Some motivation:

- C1-actions can be used to construct *metrics with special properties* (*Einstein*, *special holonomy*, *positive sectional curvature*, etc.). C1-symmetry allows to *reduce PDEs to ODEs* (can be applied, e.g., to the Ricci flow).
- The principal orbits of a C1-action are *homogeneous hypersurfaces* and thus are *isoparametric* and have *constant principal curvatures*.
- C1-actions form one of the chief examples of *hyperpolar* (and thus *polar*) *actions*.

Classification problem

Classification problem

The existence of a C1-action $\rightsquigarrow$ a large degree of symmetry

Classification problem

The existence of a C1-action $\rightsquigarrow$ a large degree of symmetry

Definition

Two isometric action $H_1 \curvearrowright M$ and $H_2 \curvearrowright M$ are called **orbit-equivalent** if there exists an isometry $\varphi \in I(M)$ identifying their orbits.

Classification problem

The existence of a C1-action $\rightsquigarrow$ a large degree of symmetry

Definition

Two isometric action $H_1 \curvearrowright M$ and $H_2 \curvearrowright M$ are called **orbit-equivalent** if there exists an isometry $\varphi \in I(M)$ identifying their orbits.

Problem

Classify C1-actions on symmetric spaces up to orbit-equivalence.

Classification problem

The existence of a C1-action $\rightsquigarrow$ a large degree of symmetry

Definition

Two isometric action $H_1 \curvearrowright M$ and $H_2 \curvearrowright M$ are called **orbit-equivalent** if there exists an isometry $\varphi \in I(M)$ identifying their orbits.

Problem

Classify C1-actions on symmetric spaces up to orbit-equivalence.

This problem is essentially equivalent to:

Problem

Classify homogeneous hypersurfaces in symmetric spaces up to isometric congruence.

History

History

- In $\mathbb{R}^n$, homogeneous = isoparametric $\rightsquigarrow$ classified by Somigliana (1919), Levi-Civita ('37), and Segre ('38).

History

- In $\mathbb{R}^n$, homogeneous = isoparametric $\rightsquigarrow$ classified by Somigliana (1919), Levi-Civita ('37), and Segre ('38).
- In $\mathbb{R}H^n$, homogeneous = isoparametric $\rightsquigarrow$ classified by Cartan ('38).

History

- In $\mathbb{R}^n$, homogeneous $=$ isoparametric $\rightsquigarrow$ classified by Somigliana (1919), Levi-Civita ('37), and Segre ('38).
- In $\mathbb{R}H^n$, homogeneous $=$ isoparametric $\rightsquigarrow$ classified by Cartan ('38).
- In $\mathbb{S}^n$ homogeneous $\subsetneq$ isoparametric. Classified by Hsiang & Lawson ('71). Every C1-action on $\mathbb{S}^n$ is orbit equivalent to the action induced by the isotropy representation of a symmetric space of rank 2 and dimension $n+1$.

History

- In $\mathbb{R}^n$, homogeneous = isoparametric $\rightsquigarrow$ classified by Somigliana (1919), Levi-Civita ('37), and Segre ('38).
- In $\mathbb{R}H^n$, homogeneous = isoparametric $\rightsquigarrow$ classified by Cartan ('38).
- In $\mathbb{S}^n$ homogeneous $\subsetneq$ isoparametric. Classified by Hsiang & Lawson ('71). Every C1-action on $\mathbb{S}^n$ is orbit equivalent to the action induced by the isotropy representation of a symmetric space of rank 2 and dimension $n+1$.
- On $\mathbb{C}P^n$: Takagi ('73).

History

- In $\mathbb{R}^n$, homogeneous $=$ isoparametric $\rightsquigarrow$ classified by Somigliana (1919), Levi-Civita ('37), and Segre ('38).
- In $\mathbb{R}H^n$, homogeneous $=$ isoparametric $\rightsquigarrow$ classified by Cartan ('38).
- In $\mathbb{S}^n$ homogeneous $\subsetneq$ isoparametric. Classified by Hsiang & Lawson ('71). Every C1-action on $\mathbb{S}^n$ is orbit equivalent to the action induced by the isotropy representation of a symmetric space of rank 2 and dimension $n+1$.
- On $\mathbb{C}P^n$: Takagi ('73).
- On $\mathbb{H}P^n$: D'Atri ('79) and Iwata ('78).

History

- In $\mathbb{R}^n$, homogeneous $=$ isoparametric $\rightsquigarrow$ classified by Somigliana (1919), Levi-Civita ('37), and Segre ('38).
- In $\mathbb{R}H^n$, homogeneous $=$ isoparametric $\rightsquigarrow$ classified by Cartan ('38).
- In $\mathbb{S}^n$ homogeneous $\subsetneq$ isoparametric. Classified by Hsiang & Lawson ('71). Every C1-action on $\mathbb{S}^n$ is orbit equivalent to the action induced by the isotropy representation of a symmetric space of rank 2 and dimension $n+1$.
- On $\mathbb{C}P^n$: Takagi ('73).
- On $\mathbb{H}P^n$: D'Atri ('79) and Iwata ('78).
- On $\mathbb{O}P^2$: Iwata ('81).

History

- In $\mathbb{R}^n$, homogeneous $=$ isoparametric $\rightsquigarrow$ classified by Somigliana (1919), Levi-Civita ('37), and Segre ('38).
- In $\mathbb{R}H^n$, homogeneous $=$ isoparametric $\rightsquigarrow$ classified by Cartan ('38).
- In $\mathbb{S}^n$ homogeneous $\subsetneq$ isoparametric. Classified by Hsiang & Lawson ('71). Every C1-action on $\mathbb{S}^n$ is orbit equivalent to the action induced by the isotropy representation of a symmetric space of rank 2 and dimension $n+1$.
- On $\mathbb{C}P^n$: Takagi ('73).
- On $\mathbb{H}P^n$: D'Atri ('79) and Iwata ('78).
- On $\mathbb{O}P^2$: Iwata ('81).
- On irreducible symmetric spaces of compact type: Kollross ('02).

Symmetric spaces of noncompact type

Symmetric spaces of noncompact type

- $M = G/K$ a symmetric space of noncompact type, $G = I^0(M)$, $K =$ the isotropy of some $o \in M$.

Symmetric spaces of noncompact type

- $M = G/K$ a symmetric space of noncompact type, $G = I^0(M)$, $K =$ the isotropy of some $o \in M$.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition with $\mathfrak{k} = \mathrm{Lie}(K)$, $\theta =$ the corresponding Cartan involution.

Symmetric spaces of noncompact type

- $M = G/K$ a symmetric space of noncompact type, $G = I^0(M)$, $K =$ the isotropy of some $o \in M$.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition with $\mathfrak{k} = \mathrm{Lie}(K)$, $\theta =$ the corresponding Cartan involution.
- $B =$ the Killing form of $\mathfrak{g}$, $B_\theta(X, Y) = -B(X, \theta Y)$ a K-invariant inner product on $\mathfrak{g}$.

Symmetric spaces of noncompact type

- $M = G/K$ a symmetric space of noncompact type, $G = I^0(M)$, $K =$ the isotropy of some $o \in M$.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition with $\mathfrak{k} = \mathrm{Lie}(K)$, $\theta =$ the corresponding Cartan involution.
- $B =$ the Killing form of $\mathfrak{g}$, $B_\theta(X, Y) = -B(X, \theta Y)$ a K-invariant inner product on $\mathfrak{g}$.
- Pick a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p} \rightsquigarrow$ induces the restricted root system $\Sigma \subset \mathfrak{a}^*$ and the restricted root space decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$. Here $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, where $\mathfrak{k}_0 = Z_{\mathfrak{k}}(\mathfrak{a})$.

Symmetric spaces of noncompact type

- $M = G/K$ a symmetric space of noncompact type, $G = I^0(M)$, $K =$ the isotropy of some $o \in M$.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition with $\mathfrak{k} = \mathrm{Lie}(K)$, $\theta =$ the corresponding Cartan involution.
- $B =$ the Killing form of $\mathfrak{g}$, $B_\theta(X, Y) = -B(X, \theta Y)$ a K-invariant inner product on $\mathfrak{g}$.
- Pick a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p} \rightsquigarrow$ induces the restricted root system $\Sigma \subset \mathfrak{a}^*$ and the restricted root space decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$. Here $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, where $\mathfrak{k}_0 = Z_\mathfrak{k}(\mathfrak{a})$.
- Make a choice of simple roots $\Lambda = \{\alpha_1, \ldots, \alpha_r\} \subseteq \Sigma^+ \subset \Sigma \rightsquigarrow$ induces a nilpotent subalgebra $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$.

Symmetric spaces of noncompact type

- $M = G/K$ a symmetric space of noncompact type, $G = I^0(M)$, $K =$ the isotropy of some $o \in M$.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition with $\mathfrak{k} = \mathrm{Lie}(K)$, $\theta =$ the corresponding Cartan involution.
- $B =$ the Killing form of $\mathfrak{g}$, $B_\theta(X, Y) = -B(X, \theta Y)$ a K-invariant inner product on $\mathfrak{g}$.
- Pick a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p} \rightsquigarrow$ induces the restricted root system $\Sigma \subset \mathfrak{a}^*$ and the restricted root space decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$. Here $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, where $\mathfrak{k}_0 = Z_{\mathfrak{k}}(\mathfrak{a})$.
- Make a choice of simple roots $\Lambda = \{\alpha_1, \ldots, \alpha_r\} \subseteq \Sigma^+ \subset \Sigma \rightsquigarrow$ induces a nilpotent subalgebra $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$.
- Iwasawa decomposition: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $G = KAN$.

Homogeneous codimension-one foliations

Homogeneous codimension-one foliations

For a C1-action $H \curvearrowright M$ on a noncompact symmetric space, there are two possibilities:

Homogeneous codimension-one foliations

For a C1-action $H \curvearrowright M$ on a noncompact symmetric space, there are two possibilities:

1. There are no singular orbits, all orbits are diffeomorphic to $\mathbb{R}^{n-1}$ and form a Riemannian foliation.

Homogeneous codimension-one foliations

For a C1-action $H \curvearrowright M$ on a noncompact symmetric space, there are two possibilities:

1. There are no singular orbits, all orbits are diffeomorphic to $\mathbb{R}^{n-1}$ and form a Riemannian foliation.
2. There is one singular orbit diffeomorphic to $\mathbb{R}^k$, and the other orbits are equidistant tubes around it diffeomorphic to $\mathbb{R}^k \times \mathbb{S}^{n-k-1}$.

Homogeneous codimension-one foliations

For a C1-action $H \curvearrowright M$ on a noncompact symmetric space, there are two possibilities:

1. There are no singular orbits, all orbits are diffeomorphic to $\mathbb{R}^{n-1}$ and form a Riemannian foliation.
2. There is one singular orbit diffeomorphic to $\mathbb{R}^k$, and the other orbits are equidistant tubes around it diffeomorphic to $\mathbb{R}^k \times \mathbb{S}^{n-k-1}$.

Actions of the first type were classified by Berndt, Tamaru, and Díaz-Ramos ('03, '10):

Homogeneous codimension-one foliations

For a C1-action $H \curvearrowright M$ on a noncompact symmetric space, there are two possibilities:

1. There are no singular orbits, all orbits are diffeomorphic to $\mathbb{R}^{n-1}$ and form a Riemannian foliation.
2. There is one singular orbit diffeomorphic to $\mathbb{R}^k$, and the other orbits are equidistant tubes around it diffeomorphic to $\mathbb{R}^k \times \mathbb{S}^{n-k-1}$.

Actions of the first type were classified by Berndt, Tamaru, and Díaz-Ramos ('03, '10):

- $\ell \subseteq \mathfrak{a}$ a line $\rightsquigarrow \mathfrak{h}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$ a Lie subalgebra, whose corresponding Lie subgroup $\boldsymbol{H_\ell}$ induces a C1-foliation on M.

Homogeneous codimension-one foliations

For a C1-action $H \curvearrowright M$ on a noncompact symmetric space, there are two possibilities:

1. There are no singular orbits, all orbits are diffeomorphic to $\mathbb{R}^{n-1}$ and form a Riemannian foliation.
2. There is one singular orbit diffeomorphic to $\mathbb{R}^k$, and the other orbits are equidistant tubes around it diffeomorphic to $\mathbb{R}^k \times \mathbb{S}^{n-k-1}$.

Actions of the first type were classified by Berndt, Tamaru, and Díaz-Ramos ('03, '10):

- $\ell \subseteq \mathfrak{a}$ a line $\rightsquigarrow \mathfrak{h}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$ a Lie subalgebra, whose corresponding Lie subgroup $\boldsymbol{H_\ell}$ induces a C1-foliation on M.
- $\alpha_i \in \Lambda$ a simple root and $\ell_{\alpha_i} \subseteq \mathfrak{g}_{\alpha_i}$ a line $\rightsquigarrow \mathfrak{h}_{\alpha_i} = \mathfrak{a} \oplus (\mathfrak{n} \ominus \ell_{\alpha_i})$ is a Lie subalgebra, whose corresponding Lie subgroup $\boldsymbol{H_{\alpha_i}}$ induces a C1-foliation on M.

Homogeneous codimension-one foliations

For a C1-action $H \curvearrowright M$ on a noncompact symmetric space, there are two possibilities:

1. There are no singular orbits, all orbits are diffeomorphic to $\mathbb{R}^{n-1}$ and form a Riemannian foliation.
2. There is one singular orbit diffeomorphic to $\mathbb{R}^k$, and the other orbits are equidistant tubes around it diffeomorphic to $\mathbb{R}^k \times \mathbb{S}^{n-k-1}$.

Actions of the first type were classified by Berndt, Tamaru, and Díaz-Ramos ('03, '10):

- $\ell \subseteq \mathfrak{a}$ a line $\rightsquigarrow \mathfrak{h}_{\boldsymbol{\ell}} = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$ a Lie subalgebra, whose corresponding Lie subgroup $\boldsymbol{H_\ell}$ induces a C1-foliation on M.
- $\alpha_i \in \Lambda$ a simple root and $\ell_{\alpha_i} \subseteq \mathfrak{g}_{\alpha_i}$ a line $\rightsquigarrow \mathfrak{h}_{\boldsymbol{\alpha_i}} = \mathfrak{a} \oplus (\mathfrak{n} \ominus \ell_{\alpha_i})$ is a Lie subalgebra, whose corresponding Lie subgroup $\boldsymbol{H_{\alpha_i}}$ induces a C1-foliation on M. A different choice of ℓ_{α_i} leads to an orbit-equivalent action.

C1-actions with a totally geodesic singular orbit

C1-actions with a totally geodesic singular orbit

Definition

A submanifold $S \subseteq M$ is called **reflective** if it is a connected component of M^σ for some involutive isometry $\sigma \in I(M)$.

C1-actions with a totally geodesic singular orbit

Definition

A submanifold $S \subseteq M$ is called **reflective** if it is a connected component of M^σ for some involutive isometry $\sigma \in I(M)$.

- Leung ('75, '79) classified reflective submanifolds in irreducible compact symmetric spaces.

C1-actions with a totally geodesic singular orbit

Definition

A submanifold $S \subseteq M$ is called **reflective** if it is a connected component of M^σ for some involutive isometry $\sigma \in I(M)$.

- Leung ('75, '79) classified reflective submanifolds in irreducible compact symmetric spaces.
- Berndt and Tamaru ('04) classified C1-actions with a *totally geodesic singular orbit* for M irreducible:

C1-actions with a totally geodesic singular orbit

Definition

A submanifold $S \subseteq M$ is called **reflective** if it is a connected component of M^σ for some involutive isometry $\sigma \in I(M)$.

- Leung ('75, '79) classified reflective submanifolds in irreducible compact symmetric spaces.
- Berndt and Tamaru ('04) classified C1-actions with a *totally geodesic singular orbit* for M irreducible:
  - either the orbit is reflective $\rightsquigarrow$ these were classified using Leung's result,

C1-actions with a totally geodesic singular orbit

Definition

A submanifold $S \subseteq M$ is called **reflective** if it is a connected component of M^σ for some involutive isometry $\sigma \in I(M)$.

- Leung ('75, '79) classified reflective submanifolds in irreducible compact symmetric spaces.
- Berndt and Tamaru ('04) classified C1-actions with a *totally geodesic singular orbit* for M irreducible:
  - either the orbit is reflective $\rightsquigarrow$ these were classified using Leung's result,
  - or non-reflective $\rightsquigarrow$ 5 examples, all related to G_2.

C1-actions with a totally geodesic singular orbit

Definition

A submanifold $S \subseteq M$ is called **reflective** if it is a connected component of M^σ for some involutive isometry $\sigma \in I(M)$.

- Leung ('75, '79) classified reflective submanifolds in irreducible compact symmetric spaces.
- Berndt and Tamaru ('04) classified C1-actions with a *totally geodesic singular orbit* for M irreducible:
  - either the orbit is reflective $\rightsquigarrow$ these were classified using Leung's result,
  - or non-reflective $\rightsquigarrow$ 5 examples, all related to G_2.
- Díaz-Ramos, Domínguez-Vázquez, and Otero ('23) found another example for M reducible. Let M_1 and M_2 be homothetic spaces of rank one, pick an isomorphism $\varphi\colon \mathfrak{g}_1 \xrightarrow{\sim} \mathfrak{g}_2$, and consider $\mathfrak{g}_\varphi = \{X + \varphi(X) \mid X \in \mathfrak{g}_1\} \subseteq \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Then the corresponding subgroup $G_\varphi \subseteq G_1 \times G_2$ acts on $M_1 \times M_2$ with cohomogeneity one. This is called a **diagonal C1-action**.

Canonical extension

Canonical extension

- Pick $\Phi \subseteq \Lambda$, define $\mathfrak{a}_\Phi = \bigcap_{\alpha \in \Phi} \mathrm{Ker}(\alpha) \subseteq \mathfrak{a}$, $\mathfrak{l}_\Phi = Z_{\mathfrak{g}}(\mathfrak{a}_\Phi), \mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$, and $M_\Phi \subseteq G$ the subgroup corresponding to $\mathfrak{m}_\Phi$.

Canonical extension

- Pick $\Phi \subseteq \Lambda$, define $\mathfrak{a}_\Phi = \bigcap_{\alpha \in \Phi} \mathrm{Ker}(\alpha) \subseteq \mathfrak{a}$, $\mathfrak{l}_\Phi = Z_{\mathfrak{g}}(\mathfrak{a}_\Phi), \mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$, and $M_\Phi \subseteq G$ the subgroup corresponding to $\mathfrak{m}_\Phi$.
- $M_\Phi \cdot o = \boldsymbol{B_\Phi}$ is a **boundary component** of M: itself a noncompact symmetric space of rank $|\Phi|$.

Canonical extension

- Pick $\Phi \subseteq \Lambda$, define $\mathfrak{a}_\Phi = \bigcap_{\alpha \in \Phi} \mathrm{Ker}(\alpha) \subseteq \mathfrak{a}$, $\mathfrak{l}_\Phi = Z_{\mathfrak{g}}(\mathfrak{a}_\Phi), \mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$, and $M_\Phi \subseteq G$ the subgroup corresponding to $\mathfrak{m}_\Phi$.
- $M_\Phi \cdot o = \boldsymbol{B_\Phi}$ is a **boundary component** of M: itself a noncompact symmetric space of rank $|\Phi|$.
- $\Sigma_\Phi =$ the root subsystem generated by Φ, $\Sigma_\Phi^+ = \Sigma_\Phi \cap \Sigma^+$. Then Φ is its set of simple roots.

Canonical extension

- Pick $\Phi \subseteq \Lambda$, define $\mathfrak{a}_\Phi = \bigcap_{\alpha \in \Phi} \mathrm{Ker}(\alpha) \subseteq \mathfrak{a}$, $\mathfrak{l}_\Phi = Z_{\mathfrak{g}}(\mathfrak{a}_\Phi), \mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$, and $M_\Phi \subseteq G$ the subgroup corresponding to $\mathfrak{m}_\Phi$.
- $M_\Phi \cdot o = \boldsymbol{B_\Phi}$ is a **boundary component** of M: itself a noncompact symmetric space of rank $|\Phi|$.
- $\Sigma_\Phi =$ the root subsystem generated by Φ, $\Sigma_\Phi^+ = \Sigma_\Phi \cap \Sigma^+$. Then Φ is its set of simple roots.
- Σ_Φ is the restricted root system of B_Φ, and the subalgebra $\mathfrak{g}'_\Phi$ generated by $\mathfrak{g}_\alpha$, $\alpha \in \Sigma_\Phi$, is the isometry Lie algebra of B_Φ.

Canonical extension

- Pick $\Phi \subseteq \Lambda$, define $\mathfrak{a}_\Phi = \bigcap_{\alpha \in \Phi} \mathrm{Ker}(\alpha) \subseteq \mathfrak{a}$, $\mathfrak{l}_\Phi = Z_\mathfrak{g}(\mathfrak{a}_\Phi), \mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$, and $M_\Phi \subseteq G$ the subgroup corresponding to $\mathfrak{m}_\Phi$.
- $M_\Phi \cdot o = \boldsymbol{B_\Phi}$ is a **boundary component** of M: itself a noncompact symmetric space of rank $|\Phi|$.
- $\Sigma_\Phi =$ the root subsystem generated by Φ, $\Sigma_\Phi^+ = \Sigma_\Phi \cap \Sigma^+$. Then Φ is its set of simple roots.
- Σ_Φ is the restricted root system of B_Φ, and the subalgebra $\mathfrak{g}_\Phi'$ generated by $\mathfrak{g}_\alpha$, $\alpha \in \Sigma_\Phi$, is the isometry Lie algebra of B_Φ.
- Define $\Delta_\Phi = \Sigma^+ \setminus \Sigma_\Phi^+$ and $\mathfrak{n}_\Phi = \bigoplus_{\lambda \in \Delta_\Phi} \mathfrak{g}_\lambda \subseteq \mathfrak{n}$.

Canonical extension

- Pick $\Phi \subseteq \Lambda$, define $\mathfrak{a}_\Phi = \bigcap_{\alpha \in \Phi} \mathrm{Ker}(\alpha) \subseteq \mathfrak{a}$, $\mathfrak{l}_\Phi = Z_{\mathfrak{g}}(\mathfrak{a}_\Phi), \mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$, and $M_\Phi \subseteq G$ the subgroup corresponding to $\mathfrak{m}_\Phi$.
- $M_\Phi \cdot o = \boldsymbol{B_\Phi}$ is a **boundary component** of M: itself a noncompact symmetric space of rank $|\Phi|$.
- $\Sigma_\Phi =$ the root subsystem generated by Φ, $\Sigma_\Phi^+ = \Sigma_\Phi \cap \Sigma^+$. Then Φ is its set of simple roots.
- Σ_Φ is the restricted root system of B_Φ, and the subalgebra $\mathfrak{g}'_\Phi$ generated by $\mathfrak{g}_\alpha$, $\alpha \in \Sigma_\Phi$, is the isometry Lie algebra of B_Φ.
- Define $\Delta_\Phi = \Sigma^+ \setminus \Sigma_\Phi^+$ and $\mathfrak{n}_\Phi = \bigoplus_{\lambda \in \Delta_\Phi} \mathfrak{g}_\lambda \subseteq \mathfrak{n}$.
- The **canonical extension**:

Canonical extension

- Pick $\Phi \subseteq \Lambda$, define $\mathfrak{a}_\Phi = \bigcap_{\alpha \in \Phi} \mathrm{Ker}(\alpha) \subseteq \mathfrak{a}$, $\mathfrak{l}_\Phi = Z_\mathfrak{g}(\mathfrak{a}_\Phi), \mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$, and $M_\Phi \subseteq G$ the subgroup corresponding to $\mathfrak{m}_\Phi$.
- $M_\Phi \cdot o = \boldsymbol{B_\Phi}$ is a **boundary component** of M: itself a noncompact symmetric space of rank $|\Phi|$.
- $\Sigma_\Phi =$ the root subsystem generated by Φ, $\Sigma_\Phi^+ = \Sigma_\Phi \cap \Sigma^+$. Then Φ is its set of simple roots.
- Σ_Φ is the restricted root system of B_Φ, and the subalgebra $\mathfrak{g}'_\Phi$ generated by $\mathfrak{g}_\alpha$, $\alpha \in \Sigma_\Phi$, is the isometry Lie algebra of B_Φ.
- Define $\Delta_\Phi = \Sigma^+ \setminus \Sigma_\Phi^+$ and $\mathfrak{n}_\Phi = \bigoplus_{\lambda \in \Delta_\Phi} \mathfrak{g}_\lambda \subseteq \mathfrak{n}$.
- The **canonical extension**:

$$H_\Phi \subseteq I^0(B_\Phi) \rightsquigarrow \mathfrak{h}_\Phi \subseteq \mathfrak{g}'_\Phi \rightsquigarrow \boldsymbol{\mathfrak{h}_\Phi^\Lambda} = \mathfrak{h}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi \subseteq \mathfrak{g} \rightsquigarrow \boldsymbol{H_\Phi^\Lambda} \subseteq G$$
$$H_\Phi \curvearrowright B_\Phi \text{ is C1} \quad \Rightarrow \quad H_\Phi^\Lambda \curvearrowright M \text{ is C1.}$$

Nilpotent construction

Nilpotent construction

- Suppose $\Phi = \Lambda \setminus \{\alpha_j\}$. In this case, we write j instead of Φ in subscripts.

Nilpotent construction

- Suppose $\Phi = \Lambda \setminus \{\alpha_j\}$. In this case, we write j instead of Φ in subscripts.
- $\Delta_j = \sqcup_{\nu \geq 1} \Delta_j^\nu$, where $\Delta_j^\nu = \left\{ \nu\alpha_j + \sum_{i \neq j} m_i \alpha_i \right\} \rightsquigarrow \mathfrak{n}_j^\nu = \bigoplus_{\lambda \in \Delta_j^\nu} \mathfrak{g}_\lambda$, $\mathfrak{n}_j = \bigoplus_{\nu \geq 1} \mathfrak{n}_j^\nu$.

Nilpotent construction

- Suppose $\Phi = \Lambda \setminus \{\alpha_j\}$. In this case, we write j instead of Φ in subscripts.
- $\Delta_j = \sqcup_{\nu \geq 1} \Delta_j^\nu$, where $\Delta_j^\nu = \left\{ \nu\alpha_j + \sum_{i \neq j} m_i \alpha_i \right\} \rightsquigarrow \mathfrak{n}_j^\nu = \bigoplus_{\lambda \in \Delta_j^\nu} \mathfrak{g}_\lambda$, $\mathfrak{n}_j = \bigoplus_{\nu \geq 1} \mathfrak{n}_j^\nu$.
- Write $K_j = M_j \cap K$. Given a subspace $\mathfrak{v} \subseteq \mathfrak{n}_j^1$, define a subalgebra $\mathfrak{n}_{j,\mathfrak{v}} = \mathfrak{n}_j \ominus \mathfrak{v}$ and $N_{j,\mathfrak{v}} \subseteq N$ the corresponding subgroup. We call $\mathfrak{v}$

Nilpotent construction

- Suppose $\Phi = \Lambda \setminus \{\alpha_j\}$. In this case, we write j instead of Φ in subscripts.
- $\Delta_j = \sqcup_{\nu \geq 1} \Delta_j^\nu$, where $\Delta_j^\nu = \left\{ \nu \alpha_j + \sum_{i \neq j} m_i \alpha_i \right\} \rightsquigarrow \mathfrak{n}_j^\nu = \bigoplus_{\lambda \in \Delta_j^\nu} \mathfrak{g}_\lambda$, $\mathfrak{n}_j = \bigoplus_{\nu \geq 1} \mathfrak{n}_j^\nu$.
- Write $K_j = M_j \cap K$. Given a subspace $\mathfrak{v} \subseteq \mathfrak{n}_j^1$, define a subalgebra $\mathfrak{n}_{j,\mathfrak{v}} = \mathfrak{n}_j \ominus \mathfrak{v}$ and $N_{j,\mathfrak{v}} \subseteq N$ the corresponding subgroup. We call $\mathfrak{v}$
  1. **admissible** if $N_{M_j}(\mathfrak{n}_{j,\mathfrak{v}})$ acts transitively on B_j.

Nilpotent construction

- Suppose $\Phi = \Lambda \setminus \{\alpha_j\}$. In this case, we write j instead of Φ in subscripts.
- $\Delta_j = \sqcup_{\nu \geq 1} \Delta_j^\nu$, where $\Delta_j^\nu = \left\{ \nu \alpha_j + \sum_{i \neq j} m_i \alpha_i \right\} \rightsquigarrow \mathfrak{n}_j^\nu = \bigoplus_{\lambda \in \Delta_j^\nu} \mathfrak{g}_\lambda$, $\mathfrak{n}_j = \bigoplus_{\nu \geq 1} \mathfrak{n}_j^\nu$.
- Write $K_j = M_j \cap K$. Given a subspace $\mathfrak{v} \subseteq \mathfrak{n}_j^1$, define a subalgebra $\mathfrak{n}_{j,\mathfrak{v}} = \mathfrak{n}_j \ominus \mathfrak{v}$ and $N_{j,\mathfrak{v}} \subseteq N$ the corresponding subgroup. We call $\mathfrak{v}$
  1. **admissible** if $N_{M_j}(\mathfrak{n}_{j,\mathfrak{v}})$ acts transitively on B_j.
  2. **protohomogeneous** if $N_{K_j}(\mathfrak{v})$ acts transitively on the unit sphere in $\mathfrak{v}$.

Nilpotent construction

- Suppose $\Phi = \Lambda \setminus \{\alpha_j\}$. In this case, we write j instead of Φ in subscripts.
- $\Delta_j = \sqcup_{\nu \geq 1} \Delta_j^\nu$, where $\Delta_j^\nu = \left\{ \nu\alpha_j + \sum_{i \neq j} m_i \alpha_i \right\} \rightsquigarrow \mathfrak{n}_j^\nu = \bigoplus_{\lambda \in \Delta_j^\nu} \mathfrak{g}_\lambda$, $\mathfrak{n}_j = \bigoplus_{\nu \geq 1} \mathfrak{n}_j^\nu$.
- Write $K_j = M_j \cap K$. Given a subspace $\mathfrak{v} \subseteq \mathfrak{n}_j^1$, define a subalgebra $\mathfrak{n}_{j,\mathfrak{v}} = \mathfrak{n}_j \ominus \mathfrak{v}$ and $N_{j,\mathfrak{v}} \subseteq N$ the corresponding subgroup. We call $\mathfrak{v}$
  1. **admissible** if $N_{M_j}(\mathfrak{n}_{j,\mathfrak{v}})$ acts transitively on B_j.
  2. **protohomogeneous** if $N_{K_j}(\mathfrak{v})$ acts transitively on the unit sphere in $\mathfrak{v}$.
- If $\mathfrak{v}$ satisfies these conditions and $\dim(\mathfrak{v}) \geq 2$, then $N^0_{M_j}(\mathfrak{n}_{j,\mathfrak{v}})N_{j,\mathfrak{v}}$ acts on M with cohomogeneity 1 and a singular orbit.

Nilpotent construction

- Suppose $\Phi = \Lambda \setminus \{\alpha_j\}$. In this case, we write j instead of Φ in subscripts.
- $\Delta_j = \sqcup_{\nu \geq 1} \Delta_j^\nu$, where $\Delta_j^\nu = \left\{ \nu\alpha_j + \sum_{i \neq j} m_i \alpha_i \right\} \rightsquigarrow \mathfrak{n}_j^\nu = \bigoplus_{\lambda \in \Delta_j^\nu} \mathfrak{g}_\lambda$, $\mathfrak{n}_j = \bigoplus_{\nu \geq 1} \mathfrak{n}_j^\nu$.
- Write $K_j = M_j \cap K$. Given a subspace $\mathfrak{v} \subseteq \mathfrak{n}_j^1$, define a subalgebra $\mathfrak{n}_{j,\mathfrak{v}} = \mathfrak{n}_j \ominus \mathfrak{v}$ and $N_{j,\mathfrak{v}} \subseteq N$ the corresponding subgroup. We call $\mathfrak{v}$
  1. **admissible** if $N_{M_j}(\mathfrak{n}_{j,\mathfrak{v}})$ acts transitively on B_j.
  2. **protohomogeneous** if $N_{K_j}(\mathfrak{v})$ acts transitively on the unit sphere in $\mathfrak{v}$.
- If $\mathfrak{v}$ satisfies these conditions and $\dim(\mathfrak{v}) \geq 2$, then $N^0_{M_j}(\mathfrak{n}_{j,\mathfrak{v}}) N_{j,\mathfrak{v}}$ acts on M with cohomogeneity 1 and a singular orbit.

Nilpotent construction problem

Given M, find all such subspaces $\mathfrak{v}$ for all j. (May assume M is irreducible.)

Nilpotent construction

- Suppose $\Phi = \Lambda \setminus \{\alpha_j\}$. In this case, we write j instead of Φ in subscripts.
- $\Delta_j = \sqcup_{\nu \geq 1} \Delta_j^\nu$, where $\Delta_j^\nu = \left\{ \nu\alpha_j + \sum_{i \neq j} m_i \alpha_i \right\} \rightsquigarrow \mathfrak{n}_j^\nu = \bigoplus_{\lambda \in \Delta_j^\nu} \mathfrak{g}_\lambda$, $\mathfrak{n}_j = \bigoplus_{\nu \geq 1} \mathfrak{n}_j^\nu$.
- Write $K_j = M_j \cap K$. Given a subspace $\mathfrak{v} \subseteq \mathfrak{n}_j^1$, define a subalgebra $\mathfrak{n}_{j,\mathfrak{v}} = \mathfrak{n}_j \ominus \mathfrak{v}$ and $N_{j,\mathfrak{v}} \subseteq N$ the corresponding subgroup. We call $\mathfrak{v}$
  1. **admissible** if $N_{M_j}(\mathfrak{n}_{j,\mathfrak{v}})$ acts transitively on B_j.
  2. **protohomogeneous** if $N_{K_j}(\mathfrak{v})$ acts transitively on the unit sphere in $\mathfrak{v}$.
- If $\mathfrak{v}$ satisfies these conditions and $\dim(\mathfrak{v}) \geq 2$, then $N^0_{M_j}(\mathfrak{n}_{j,\mathfrak{v}})N_{j,\mathfrak{v}}$ acts on M with cohomogeneity 1 and a singular orbit.

Nilpotent construction problem

Given M, find all such subspaces $\mathfrak{v}$ for all j. (May assume M is irreducible.)

- Has been solved in rank 1, some spaces of rank 2, and $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$.

Nilpotent construction

- Suppose $\Phi = \Lambda \setminus \{\alpha_j\}$. In this case, we write j instead of Φ in subscripts.
- $\Delta_j = \sqcup_{\nu \geq 1} \Delta_j^\nu$, where $\Delta_j^\nu = \left\{ \nu\alpha_j + \sum_{i \neq j} m_i \alpha_i \right\} \rightsquigarrow \mathfrak{n}_j^\nu = \bigoplus_{\lambda \in \Delta_j^\nu} \mathfrak{g}_\lambda$, $\mathfrak{n}_j = \bigoplus_{\nu \geq 1} \mathfrak{n}_j^\nu$.
- Write $K_j = M_j \cap K$. Given a subspace $\mathfrak{v} \subseteq \mathfrak{n}_j^1$, define a subalgebra $\mathfrak{n}_{j,\mathfrak{v}} = \mathfrak{n}_j \ominus \mathfrak{v}$ and $N_{j,\mathfrak{v}} \subseteq N$ the corresponding subgroup. We call $\mathfrak{v}$
  1. **admissible** if $N_{M_j}(\mathfrak{n}_{j,\mathfrak{v}})$ acts transitively on B_j.
  2. **protohomogeneous** if $N_{K_j}(\mathfrak{v})$ acts transitively on the unit sphere in $\mathfrak{v}$.
- If $\mathfrak{v}$ satisfies these conditions and $\dim(\mathfrak{v}) \geq 2$, then $N^0_{M_j}(\mathfrak{n}_{j,\mathfrak{v}})N_{j,\mathfrak{v}}$ acts on M with cohomogeneity 1 and a singular orbit.

Nilpotent construction problem

Given M, find all such subspaces $\mathfrak{v}$ for all j. (May assume M is irreducible.)

- Has been solved in rank 1, some spaces of rank 2, and $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$.
- Has yielded new actions only on $\mathbb{C}H^n, \mathbb{H}H^n, \mathbb{O}H^2$, $G_2^2/\mathrm{SO}(4)$, and $G_2(\mathbb{C})/G_2$.

Main theorem

Main theorem

Theorem (S., Sanmartín-Lopez; '24)

The nilpotent construction produces no new actions other than the known examples or their canonical extensions.

Main theorem

Theorem (S., Sanmartín-Lopez; '24)

The nilpotent construction produces no new actions other than the known examples or their canonical extensions.

Theorem (Berndt, Tamaru, Brück, Díaz-Ramos, Domínguez-Vázquez, Otero, Rodríguez-Vázquez, S., Sanmartín-Lopez; '01-'24)

Main theorem

Theorem (S., Sanmartín-Lopez; '24)

The nilpotent construction produces no new actions other than the known examples or their canonical extensions.

Theorem (Berndt, Tamaru, Brück, Díaz-Ramos, Domínguez-Vázquez, Otero, Rodríguez-Vázquez, S., Sanmartín-Lopez; '01-'24)

Let $M = G/K$ be a symmetric space of noncompact type and H a connected Lie group acting on M properly and isometrically.

Main theorem

Theorem (S., Sanmartín-Lopez; '24)

The nilpotent construction produces no new actions other than the known examples or their canonical extensions.

Theorem (Berndt, Tamaru, Brück, Díaz-Ramos, Domínguez-Vázquez, Otero, Rodríguez-Vázquez, S., Sanmartín-Lopez; '01-'24)

Let $M = G/K$ be a symmetric space of noncompact type and H a connected Lie group acting on M properly and isometrically. Then H acts with cohomogeneity one if and only if it its action is orbit-equivalent to one of the following:

Main theorem

Theorem (S., Sanmartín-Lopez; '24)

The nilpotent construction produces no new actions other than the known examples or their canonical extensions.

Theorem (Berndt, Tamaru, Brück, Díaz-Ramos, Domínguez-Vázquez, Otero, Rodríguez-Vázquez, S., Sanmartín-Lopez; '01-'24)

Let $M = G/K$ be a symmetric space of noncompact type and H a connected Lie group acting on M properly and isometrically. Then H acts with cohomogeneity one if and only if it its action is orbit-equivalent to one of the following:

1. The action of H_ℓ for some one-dimensional linear subspace $\ell \subseteq \mathfrak{a}$.

Main theorem

Theorem (S., Sanmartín-Lopez; '24)

The nilpotent construction produces no new actions other than the known examples or their canonical extensions.

Theorem (Berndt, Tamaru, Brück, Díaz-Ramos, Domínguez-Vázquez, Otero, Rodríguez-Vázquez, S., Sanmartín-Lopez; '01-'24)

Let $M = G/K$ be a symmetric space of noncompact type and H a connected Lie group acting on M properly and isometrically. Then H acts with cohomogeneity one if and only if it its action is orbit-equivalent to one of the following:

1. The action of H_ℓ for some one-dimensional linear subspace $\ell \subseteq \mathfrak{a}$.
2. The action of H_{α_i} for some simple root $\alpha_i \in \Lambda$.

Main theorem

Theorem (S., Sanmartín-Lopez; '24)

The nilpotent construction produces no new actions other than the known examples or their canonical extensions.

Theorem (Berndt, Tamaru, Brück, Díaz-Ramos, Domínguez-Vázquez, Otero, Rodríguez-Vázquez, S., Sanmartín-Lopez; '01-'24)

Let $M = G/K$ be a symmetric space of noncompact type and H a connected Lie group acting on M properly and isometrically. Then H acts with cohomogeneity one if and only if it its action is orbit-equivalent to one of the following:

1. The action of H_ℓ for some one-dimensional linear subspace $\ell \subseteq \mathfrak{a}$.
2. The action of H_{α_i} for some simple root $\alpha_i \in \Lambda$.
3. The canonical extension of a C1-action with a totally geodesic singular orbit on an irreducible boundary component B_Φ of M.

Main theorem

Theorem (S., Sanmartín-Lopez; '24)

The nilpotent construction produces no new actions other than the known examples or their canonical extensions.

Theorem (Berndt, Tamaru, Brück, Díaz-Ramos, Domínguez-Vázquez, Otero, Rodríguez-Vázquez, S., Sanmartín-Lopez; '01-'24)

Let $M = G/K$ be a symmetric space of noncompact type and H a connected Lie group acting on M properly and isometrically. Then H acts with cohomogeneity one if and only if it its action is orbit-equivalent to one of the following:

1. The action of H_ℓ for some one-dimensional linear subspace $\ell \subseteq \mathfrak{a}$.
2. The action of H_{α_i} for some simple root $\alpha_i \in \Lambda$.
3. The canonical extension of a C1-action with a totally geodesic singular orbit on an irreducible boundary component B_Φ of M.
4. The canonical extension of a diagonal C1-action on a reducible rank-2 boundary component B_Φ of M whose de Rham factors are homothetic.

Main theorem

Theorem (Berndt, Tamaru, Brück, Díaz-Ramos, Domínguez-Vázquez, Otero, Rodríguez-Vázquez, S., Sanmartín-Lopez; '01-'24)

5. A decomposable action $H_i \times \prod_{j \neq i} G_j \curvearrowright M_i \times \prod_{j \neq i} M_j = M$, where the de Rham factor M_i and the action $H_i \curvearrowright M_i$ are one the following:

Main theorem

Theorem (Berndt, Tamaru, Brück, Díaz-Ramos, Domínguez-Vázquez, Otero, Rodríguez-Vázquez, S., Sanmartín-Lopez; '01-'24)

5. A decomposable action $H_i \times \prod_{j \neq i} G_j \curvearrowright M_i \times \prod_{j \neq i} M_j = M$, where the de Rham factor M_i and the action $H_i \curvearrowright M_i$ are one the following:
   1. M_i is homothetic to
      - $\mathrm{Gr}^*(k, \mathbb{C}^{2k+n})$ $(k, n \geq 1)$,
      - $\mathrm{SO}(2n+1, \mathbb{H})/\mathrm{U}(2n+1)$ $(n \geq 1)$, or
      - $E_6^{-14}/\mathrm{Spin}(10)\mathrm{U}(1)$,

      and $H_i \curvearrowright M_i$ is the canonical extensions of a NC action from the boundary component $\mathbb{C}H^{n+1}$ (resp., $\mathbb{C}H^3$ or $\mathbb{C}H^5$) of M_i.

Main theorem

Theorem (Berndt, Tamaru, Brück, Díaz-Ramos, Domínguez-Vázquez, Otero, Rodríguez-Vázquez, S., Sanmartín-Lopez; '01-'24)

5. A decomposable action $H_i \times \prod_{j \neq i} G_j \curvearrowright M_i \times \prod_{j \neq i} M_j = M$, where the de Rham factor M_i and the action $H_i \curvearrowright M_i$ are one the following:
    1. M_i is homothetic to
        - $\mathrm{Gr}^*(k, \mathbb{C}^{2k+n})\,(k, n \geq 1)$,
        - $\mathrm{SO}(2n+1, \mathbb{H})/\mathrm{U}(2n+1)\,(n \geq 1)$, or
        - $E_6^{-14}/\mathrm{Spin}(10)\mathrm{U}(1)$,

       and $H_i \curvearrowright M_i$ is the canonical extensions of a NC action from the boundary component $\mathbb{C}H^{n+1}$ (resp., $\mathbb{C}H^3$ or $\mathbb{C}H^5$) of M_i.
    2. $M_i \simeq \mathrm{Gr}^*(k, \mathbb{H}^{2k+n})\,(k, n \geq 1)$ and $H_i \curvearrowright M_i$ is the canonical extensions of a NC action from the boundary component $\mathbb{H}H^{n+1}$ of M_i.

Main theorem

Theorem (Berndt, Tamaru, Brück, Díaz-Ramos, Domínguez-Vázquez, Otero, Rodríguez-Vázquez, S., Sanmartín-Lopez; '01-'24)

5. A decomposable action $H_i \times \prod_{j\neq i} G_j \curvearrowright M_i \times \prod_{j\neq i} M_j = M$, where the de Rham factor M_i and the action $H_i \curvearrowright M_i$ are one the following:
   1. M_i is homothetic to
      - $\mathrm{Gr}^*(k, \mathbb{C}^{2k+n})\,(k, n \geq 1)$,
      - $\mathrm{SO}(2n+1, \mathbb{H})/\mathrm{U}(2n+1)\,(n \geq 1)$, or
      - $E_6^{-14}/\mathrm{Spin}(10)\mathrm{U}(1)$,

      and $H_i \curvearrowright M_i$ is the canonical extensions of a NC action from the boundary component $\mathbb{C}H^{n+1}$ (resp., $\mathbb{C}H^3$ or $\mathbb{C}H^5$) of M_i.
   2. $M_i \simeq \mathrm{Gr}^*(k, \mathbb{H}^{2k+n})\,(k, n \geq 1)$ and $H_i \curvearrowright M_i$ is the canonical extensions of a NC action from the boundary component $\mathbb{H}H^{n+1}$ of M_i.
   3. $M_i \simeq \mathbb{O}H^2$ and $H_i \curvearrowright M_i$ is a NC action.

Main theorem

Theorem (Berndt, Tamaru, Brück, Díaz-Ramos, Domínguez-Vázquez, Otero, Rodríguez-Vázquez, S., Sanmartín-Lopez; '01-'24)

5. A decomposable action $H_i \times \prod_{j\neq i} G_j \curvearrowright M_i \times \prod_{j\neq i} M_j = M$, where the de Rham factor M_i and the action $H_i \curvearrowright M_i$ are one the following:
   1. M_i is homothetic to
      - $\mathrm{Gr}^*(k, \mathbb{C}^{2k+n})\,(k, n \geq 1)$,
      - $\mathrm{SO}(2n+1, \mathbb{H})/\mathrm{U}(2n+1)\,(n \geq 1)$, or
      - $E_6^{-14}/\mathrm{Spin}(10)\mathrm{U}(1)$,

      and $H_i \curvearrowright M_i$ is the canonical extensions of a NC action from the boundary component $\mathbb{C}H^{n+1}$ (resp., $\mathbb{C}H^3$ or $\mathbb{C}H^5$) of M_i.
   2. $M_i \simeq \mathrm{Gr}^*(k, \mathbb{H}^{2k+n})\,(k, n \geq 1)$ and $H_i \curvearrowright M_i$ is the canonical extensions of a NC action from the boundary component $\mathbb{H}H^{n+1}$ of M_i.
   3. $M_i \simeq \mathbb{O}H^2$ and $H_i \curvearrowright M_i$ is a NC action.
   4. $M_i \simeq G_2^2/\mathrm{SO}(4)$ and $H_i \curvearrowright M_i$ is the NC action.

Main theorem

Theorem (Berndt, Tamaru, Brück, Díaz-Ramos, Domínguez-Vázquez, Otero, Rodríguez-Vázquez, S., Sanmartín-Lopez; '01-'24)

5. A decomposable action $H_i \times \prod_{j \neq i} G_j \curvearrowright M_i \times \prod_{j \neq i} M_j = M$, where the de Rham factor M_i and the action $H_i \curvearrowright M_i$ are one the following:
   1. M_i is homothetic to
      - $\mathrm{Gr}^*(k, \mathbb{C}^{2k+n})\,(k, n \geq 1)$,
      - $\mathrm{SO}(2n+1, \mathbb{H})/\mathrm{U}(2n+1)\,(n \geq 1)$, or
      - $E_6^{-14}/\mathrm{Spin}(10)\mathrm{U}(1)$,

      and $H_i \curvearrowright M_i$ is the canonical extensions of a NC action from the boundary component $\mathbb{C}H^{n+1}$ (resp., $\mathbb{C}H^3$ or $\mathbb{C}H^5$) of M_i.
   2. $M_i \simeq \mathrm{Gr}^*(k, \mathbb{H}^{2k+n})\,(k, n \geq 1)$ and $H_i \curvearrowright M_i$ is the canonical extensions of a NC action from the boundary component $\mathbb{H}H^{n+1}$ of M_i.
   3. $M_i \simeq \mathbb{O}H^2$ and $H_i \curvearrowright M_i$ is a NC action.
   4. $M_i \simeq G_2^2/\mathrm{SO}(4)$ and $H_i \curvearrowright M_i$ is the NC action.
   5. $M_i \simeq G_2(\mathbb{C})/G_2$ and $H_i \curvearrowright M_i$ is the NC action.

Backbone of the proof

Backbone of the proof

- In the NC, the admissibility condition on $\mathfrak{v}$ has to do with *transitive actions on noncompact symmetric spaces*.

Backbone of the proof

- In the NC, the admissibility condition on $\mathfrak{v}$ has to do with *transitive actions on noncompact symmetric spaces*.
- Onishchik ('66) classified transitive isometric actions on irreducible compact symmetric spaces.

Backbone of the proof

- In the NC, the admissibility condition on $\mathfrak{v}$ has to do with *transitive actions on noncompact symmetric spaces*.
- Onishchik ('66) classified transitive isometric actions on irreducible compact symmetric spaces.
- In the noncompact case, there are infinitely many such actions so no classification result is known.

Backbone of the proof

- In the NC, the admissibility condition on $\mathfrak{v}$ has to do with *transitive actions on noncompact symmetric spaces*.
- Onishchik ('66) classified transitive isometric actions on irreducible compact symmetric spaces.
- In the noncompact case, there are infinitely many such actions so no classification result is known.

Theorem (S., Sanmartín-Lopez; '24)

Backbone of the proof

- In the NC, the admissibility condition on $\mathfrak{v}$ has to do with *transitive actions on noncompact symmetric spaces*.
- Onishchik ('66) classified transitive isometric actions on irreducible compact symmetric spaces.
- In the noncompact case, there are infinitely many such actions so no classification result is known.

Theorem (S., Sanmartín-Lopez; '24)

Let $M = G/K$ be a symmetric space of noncompact type and $H \subseteq G$ a Lie subgroup that acts transitively on M.

Backbone of the proof

- In the NC, the admissibility condition on $\mathfrak{v}$ has to do with *transitive actions on noncompact symmetric spaces*.
- Onishchik ('66) classified transitive isometric actions on irreducible compact symmetric spaces.
- In the noncompact case, there are infinitely many such actions so no classification result is known.

Theorem (S., Sanmartín-Lopez; '24)

Let $M = G/K$ be a symmetric space of noncompact type and $H \subseteq G$ a Lie subgroup that acts transitively on M. Then there exists

1. an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$

Backbone of the proof

- In the NC, the admissibility condition on $\mathfrak{v}$ has to do with *transitive actions on noncompact symmetric spaces*.
- Onishchik ('66) classified transitive isometric actions on irreducible compact symmetric spaces.
- In the noncompact case, there are infinitely many such actions so no classification result is known.

Theorem (S., Sanmartín-Lopez; '24)

Let $M = G/K$ be a symmetric space of noncompact type and $H \subseteq G$ a Lie subgroup that acts transitively on M. Then there exists

1. an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and
2. a maximal abelian subspace $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$

Backbone of the proof

- In the NC, the admissibility condition on $\mathfrak{v}$ has to do with *transitive actions on noncompact symmetric spaces*.
- Onishchik ('66) classified transitive isometric actions on irreducible compact symmetric spaces.
- In the noncompact case, there are infinitely many such actions so no classification result is known.

Theorem (S., Sanmartín-Lopez; '24)

Let $M = G/K$ be a symmetric space of noncompact type and $H \subseteq G$ a Lie subgroup that acts transitively on M. Then there exists

1. an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and
2. a maximal abelian subspace $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$

such that $\mathfrak{h}$ contains a solvable subalgebra of the form $V \oplus \mathfrak{n}$, where $V \subseteq \mathfrak{t}_0 \oplus \mathfrak{a}$ projects surjectively onto $\mathfrak{a}$.

Outline of the proof

Outline of the proof

"Corollary"

In the NC, we may assume $\mathfrak{v} = \bigoplus_{\lambda \in \Delta_j^1} \mathfrak{v}_\lambda$, where $\mathfrak{v}_\lambda = \mathfrak{v} \cap \mathfrak{g}_\lambda$.

Outline of the proof

"Corollary"

In the NC, we may assume $\mathfrak{v} = \bigoplus_{\lambda \in \Delta_j^1} \mathfrak{v}_\lambda$, where $\mathfrak{v}_\lambda = \mathfrak{v} \cap \mathfrak{g}_\lambda$.

"Corollary"

There cannot be two roots in Δ_j^1 of the same height.

Outline of the proof

“Corollary”

In the NC, we may assume $\mathfrak{v} = \bigoplus_{\lambda \in \Delta_j^1} \mathfrak{v}_\lambda$, where $\mathfrak{v}_\lambda = \mathfrak{v} \cap \mathfrak{g}_\lambda$.

“Corollary”

There cannot be two roots in Δ_j^1 of the same height.

Corollary

α_j cannot have more than one neighbor in the Dynkin diagram of Σ.

Outline of the proof

Outline of the proof

We assume that $\mathfrak{v}$ has this form and M is irreducible of $\mathrm{rk}(M) > 1$ from now on.

Outline of the proof

We assume that $\mathfrak{v}$ has this form and M is irreducible of $\mathrm{rk}(M) > 1$ from now on.

Proposition

For the NC action not to be a canonical extension or have a totally geodesic singular orbit, we must have $\mathfrak{v} = \mathfrak{n}_j^1$.

Outline of the proof

We assume that $\mathfrak{v}$ has this form and M is irreducible of $\mathrm{rk}(M) > 1$ from now on.

Proposition

For the NC action not to be a canonical extension or have a totally geodesic singular orbit, we must have $\mathfrak{v} = \mathfrak{n}_j^1$.

Proposition

Let $\delta = \sum_{i=1}^{r} m_i \alpha_i$ be the highest root. For the singular orbit not to be totally geodesic, we must have $m_j \geq 3$.

Outline of the proof

We assume that $\mathfrak{v}$ has this form and M is irreducible of $\mathrm{rk}(M) > 1$ from now on.

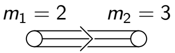
Proposition

For the NC action not to be a canonical extension or have a totally geodesic singular orbit, we must have $\mathfrak{v} = \mathfrak{n}_j^1$.

Proposition

Let $\delta = \sum_{i=1}^r m_i \alpha_i$ be the highest root. For the singular orbit not to be totally geodesic, we must have $m_j \geq 3$.

The only root system satisfying these criteria is G_2 with $j = 2$, which leads to the two C1-actions on $G_2^2/\mathrm{SO}(4)$ and $G_2(\mathbb{C})/G_2$, respectively.



The end

Thanks for your attention!