# The vacuum weighted Einstein field equations: Properties and rigidity of solutions

Diego Mojón Álvarez

Galician Centre for Mathematical Research and Technology (CITMAga)
Universidade de Santiago de Compostela

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Symmetry and Shape

Joint work with Miguel Brozos Vázquez

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# Outline

# Context: Smooth metric measure spaces

## Smooth metric measure space

Triple $\boldsymbol{(M, g, h\, dvol_g)}$ where

- $M$: smooth manifold
- $g$: semi-Riemannian metric. We focus on Lorentzian metrics.
- $dvol_g$: Riemannian volume element
- $h \in C^{\infty}(M)$: positive density function ($\nabla h \neq 0$)

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## Some notation

- $\rho$: Usual Ricci tensor ($\rho_{ij} = R^k{}_{ikj}$)
- $\mathrm{Ric}$: Ricci operator ($\mathrm{Ric}^i{}_j = g^{ik}\rho_{kj}$)
- $\tau$: Scalar curvature ($\tau = \rho^i{}_i$)
- $\mathrm{Hes}_h$: Hessian tensor of $h$ $\mathrm{Hes}_h(X, Y) = g(\nabla_X \nabla h, Y)$
- $\Delta h$: Laplacian of $h$ ($\Delta h = (\mathrm{Hes}_h)^i{}_i$)

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**Variational approach:**
The Einstein tensor is obtained through a variation of the *Einstein-Hilbert action*:

$$\mathcal{S} = \int_{\mathcal{V}} \tau \, dvol_g$$

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- Critical points of this functional under variations

  $$g[t] = g + t\delta g, \qquad h[t] = h + t\delta h$$

  Variations of the metric and its first derivatives vanish at the integration boundary

- We want to preserve the distinguished measure $\boldsymbol{dV = h\, dvol_g}$, whose variation is $dV[t] = h[t] dvol_{g[t]}$

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$$\delta\mathcal{S}_h = \int_M \langle h\rho + \Delta h g - \mathrm{Hes}_h, \delta g\rangle dvol_g = \int_M \langle D\tau_g^*(h), \delta g\rangle dvol_g$$

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Properties:

- Symmetric
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Taking its trace, we have

$$\Delta h = -\frac{h\tau}{n-1} \quad \Rightarrow \quad \mathrm{Hes}_h = h\left(\rho - \frac{\tau}{n-1}g\right)$$

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## Second aim

To understand the geometry of solutions to the vacuum weighted Einstein field equation.

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The Jordan form of the **Ricci operator** also plays a role:

Type Ia

$$\mathrm{Ric} = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}$$

Type Ib

$$\mathrm{Ric} = \begin{pmatrix} a & b & \\ -b & a & \\ & & \ddots \end{pmatrix}$$

$\{e_1, \ldots, e_n\}$ orthonormal basis

Type II

$$\mathrm{Ric} = \begin{pmatrix} \alpha & 0 & \\ \varepsilon & \alpha & \\ & & \ddots \end{pmatrix}$$

Type III

$$\mathrm{Ric} = \begin{pmatrix} \alpha & 0 & 1 & \\ 0 & \alpha & 0 & \\ 0 & 1 & \alpha & \\ & & & \ddots \end{pmatrix}$$

$\{u, v, e_1, \ldots, e_{n-2}\}$ pseudo-ortho. basis
$(g(u,u) = g(v,v) = 0,\ g(u,v) = 1)$

We assume the Jordan form is constant

# Isotropic solutions

## Lemma

*Let $(M, g, h\, dvol_g)$ be an isotropic solution of the vacuum weighted Einstein field equation. Then* $\mathrm{Ric}$ *is nilpotent and* $\Delta h = 0$.

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1. $(M, g)$ *is Ricci-flat and* $\mathrm{Hes}_h = 0$
2. *The Ricci operator is 2-step nilpotent and $(M, g)$ is a Brinkmann wave*

**Brinkmann wave:** $(M, g)$ with a recurrent lightlike geodesic vector field $V$ ($\nabla_X V = \alpha(X)V$, for a 1-form $\alpha$)

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3. *The Ricci operator is* 3*-step nilpotent and* $(M, g)$ *is a Kundt spacetime*

**Kundt spacetime:** $(M, g)$ with a geodesic lightlike vector field which is

| Expansion-free | Shear-free | Twist-free |
|---|---|---|
| $\theta = \frac{1}{n-2}\nabla_i V^i$ | $\sigma^2 = (\nabla^i V^j)\nabla_{(i} V_{j)} - (n-2)\theta^2$ | $\omega^2 = (\nabla^i V^j)\nabla_{[i} V_{j]}$ |

# Locally conformally flat solutions

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*Let $(M, g, h)$ be a locally conformally flat solution.*

1. *If $g(\nabla h, \nabla h) \neq 0$ at a point $p$, then, on a neighborhood of $p$, $(M, g, h)$ is locally isometric to a* warped product *$(I \times N, dt^2 \oplus \varphi^2 g^N)$, where*
   - *$N$ has constant sectional curvature*
   - *$h(t)$ and $\varphi(t)$ satisfy the following system of ODEs:*

$$\begin{aligned} 0 &= h'\varphi' - h\varphi'', \\ 0 &= h'' + (n-1)h\frac{\varphi''}{\varphi} + \varepsilon\frac{\tau}{n-1}h. \end{aligned}$$

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2. *If $g(\nabla h, \nabla h) = 0$ on an open subset $\mathfrak{U} \subset M$, then $(\mathfrak{U}, g|_{\mathfrak{U}})$ is a plane wave with the metric*

$$g(u, v, x_1, \dots, x_{n-2}) = 2dvdu + F(v, x_1, \dots, x_{n-2})dv^2 + \sum_{i=1}^{n-2} dx_i^2,$$

*where $F(v, x_1, \dots, x_{n-2}) = -\frac{h''(v)}{(n-2)h(v)} \sum_{i=1}^{n-2} x_i^2 + \sum_{i=1}^{n-2} b_i(v)x_i + c(v)$.*

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$$\left.\begin{array}{r} g \text{ is positive definite on } \nabla h^\perp \\ \\ \operatorname{Ric} \text{ is self-adjoint} \\ \\ \operatorname{Ric}(\nabla h) = \lambda_1 \nabla h \end{array}\right\} \Rightarrow \operatorname{Ric} \text{ is diagonalizable (Type Ia)}$$

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- If $\nabla h$ is spacelike, $\operatorname{Ric}$ does not diagonalize in general
- We focus on 4-dimensional solutions

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   1. *If $g(\nabla h, \nabla h) = 0$, then $\mathrm{Ric}$ is nilpotent and $\nabla h$ determines the lightlike parallel line field. Moreover, if $\mathrm{Ric}$ vanishes or is 2-step nilpotent, the underlying manifold is a pp-wave.*

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   2. *If $g(\nabla h, \nabla h) \neq 0$, then $\nabla h$ is spacelike and the distinguished lightlike vector field is orthogonal to $\nabla h$.*

# Sketch of the proof (non-isotropic case)

Type Ia

$$\mathrm{Ric} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$$

Type Ib

$$\mathrm{Ric} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & -b & a & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$$

$\{e_1, \ldots, e_n\}$ orthonormal basis with $e_1 = \nabla h / |\nabla h|$

# Sketch of the proof (non-isotropic case)

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- **Type Ib:** Use $\mathrm{div}\, R = 0$ and the Einstein equations to obtain information on the eigenvalues, the curvature and the Christoffel symbols
  - Polynomial system on 5 variables
  - Show $b = 0$
  - There are solutions with $\mathrm{div}\, R \neq 0$

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# Thank you!


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