# A CLASSIFICATION FOR ALMOST CONTACT STRUCTURES

BY

JOSE A. OUBIÑA

Departamento de Geometría y Topología

Facultad de Matemáticas

Universidad de Santiago de Compostela

SPAIN

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# INTRODUCTION

A $(2n+1)$-dimensional differentiable manifold $M$ of class $C^\infty$ is said to have an almost contact structure (J.W. Gray [6]) if the structural group of its tangent bundle reduces to $U(n)\times 1$; equivalently (S. Sasaki and Y. Hatakeyama [8,9]), an almost contact structure is given by a triple $(\varphi,\xi,\eta)$ (also called $(\varphi,\xi,\eta)$-structure) satisfying certain conditions. Many different types of almost contact structures are defined in the literature (normal, Sasakian, cosymplectic, nearly Sasakian, nearly cosymplectic, quasi-Sasakian, etc., [1,2]).

If $M$ has an almost contact structure, on $M\times\mathbb{R}$ there exists a canocally associated almost complex structure. The purpose of this paper is to give a classification, which contains Tashiro's [11,12], of the different types of almost contact structures on a manifold $M$ through the types of the associated almost complex structures on $M\times\mathbb{R}$, by using Gray-Hervella's classification of almost Hermitian manifolds [5].

In §1 we give, first of all, the basic definitions and some results from the theory of almost contact structures. Next, we consider the product metric $h$ on $M\times\mathbb{R}$ of a Riemannian metric $g$ on a manifold $M$ with a $(\varphi,\xi,\eta)$-structure and the Euclidean metric on $\mathbb{R}$, and we consider a certain Riemannian metric $h^\circ$ confomally related to $h$. We give some formulas for the later use. In §2, we define and characterize the almost contact metric structures through the almost Hermitian manifolds $(M\times\mathbb{R},h)$ and $(M\times\mathbb{R},h^{o})$; those obtained from $h$ generalize cosymplectic structures and those obtained from $h^{o}$ generalize Sasakian structures; furthermore, there appears a new class of almost contact structures which contains, and in a certain sense separates, both cosymplectic and Sasakian and that we call trans-Sasakian structures. In §3 we relate the different types of almost contact metric structures, we compare quasi-Sasakian and trans-Sasakian structures and illustrate the inclusion relations with a diagram. Finally, the case $n = 1$ is considered.

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## 1. REVIEW OF DEFINITIONS AND NEEDED RESULTS

A $(2n+1)$-dimensional real differentiable manifold $M$ of class $C^{\infty}$ is said to have a $(\varphi,\xi,\eta)$-*structure* or an *almost contact structure* if it admits a field $\varphi$ of endomorphisms of the tangent spaces, a vector field $\xi$, and a 1-form $\eta$ satisfyng

(1.1) $$\eta(\xi) = 1,$$

(1.2) $$\varphi^2 = - I + \eta\otimes\xi,$$

where I denotes the identity transformation [8] . Then $\varphi\xi = 0$ and $\eta\circ\varphi = 0$; moreover, the endomorphism $\varphi$ has rank $2n$ [2].

Denote by $\chi(M)$ the Lie algebra of $C^\infty$-vector fields on $M$. If a manifold $M$ with a $(\varphi,\xi,\eta)$-structure admits a Riemannian metric $g$ such that

$$(1.3)\qquad g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$

where $X,Y \in \chi(M)$, then $M$ is said to have a $(\varphi,\xi,\eta,g)$-*structure* or an *almost contact metric structure* and $g$ is called a *compatible metric* [8]. An immediate consequence is $\eta(X) = g(X,\xi)$. A manifold with a $(\varphi,\xi,\eta)$-structure admits a compatible metric $g$ [8]. The 2-form $\Phi$ on $M$ defined by

$$(1.4)\qquad \Phi(X,Y) = g(X,\varphi Y)$$

is called the *fundamental 2-form* of the almost contact metric structure.

If $\nabla$ is the Riemannian connection of $g$, it is easy to prove

$$(1.5)\qquad (\nabla_X\eta)Y = g(Y,\nabla_X\xi),$$

and hence $\nabla_X\xi = 0$ if and only if $\nabla_X\eta = 0$;

$$(1.6)\qquad (\nabla_X\Phi)(Y,Z) = g(Y,(\nabla_X\varphi)Z) = -g((\nabla_X\varphi)Y,Z),$$

$$(1.7)\qquad (\nabla_X\Phi)(Y,\varphi Z)-(\nabla_X\Phi)(\varphi Y,Z) = \eta(Y)(\nabla_X\eta)Z+\eta(Z)(\nabla_X\eta)Y,$$

$$(1.8)\qquad (\nabla_X\Phi)(Y,Z)+(\nabla_X\Phi)(\varphi Y,\varphi Z) = \eta(Z)(\nabla_X\eta)\varphi Y-\eta(Y)(\nabla_X\eta)\varphi Z,$$

$$(1.9)\qquad (\nabla_X\eta)Y = (\nabla_X\Phi)(\xi,\varphi Y),$$

$$(1.10)\qquad (\nabla_X\eta)\varphi Y = (\nabla_X\Phi)(Y,\xi).$$

The exterior derivatives of $\eta$ and $\Phi$ are given by

$$(1.11) \quad 2d\eta(X,Y) = (\nabla_X \eta)Y - (\nabla_Y \eta)X,$$

$$(1.12) \quad 3d\Phi(X,Y,Z) = \mathfrak{S}(\nabla_X \Phi)(Y,Z),$$

where $\mathfrak{S}$ denotes the cyclic sum over $X,Y,Z \in \mathfrak{X}(M)$. If $\{X_i, \varphi X_i, \xi;\ i=1,\dots,n\}$ is a local orthonormal basis, defined on an open subset of $M$, the coderivatives of $\eta$ and $\Phi$ are computed to be

$$(1.13) \quad \delta\eta = -\sum_{i=1}^{n} \{(\nabla_{X_i}\eta)X_i + (\nabla_{\varphi X_i}\eta)\varphi X_i\},$$

$$(1.14) \quad \delta\Phi(X) = -\sum_{i=1}^{n} \{(\nabla_{X_i}\Phi)(X_i,X) + (\nabla_{\varphi X_i}\Phi)(\varphi X_i,X)\} - (\nabla_\xi \Phi)(\xi,X).$$

Let $M$ be a manifold with an almost contact structure $(\varphi,\xi,\eta)$ and consider the manifold $M \times \mathbb{R}$. We denote a vector field on $M\times\mathbb{R}$ by $(X, a\frac{d}{dt})$, where $X \in \mathfrak{X}(M)$, $t$ is the coordinate of $\mathbb{R}$ and $a$ is a $C^\infty$ function on $M\times\mathbb{R}$. S. Sasaki and Y. Hatakeyama [9] define an almost complex structure $J$ on $M\times\mathbb{R}$ by

$$(1.15) \quad J(X, a\frac{d}{dt}) = (\varphi X - a\xi, \eta(X)\frac{d}{dt})$$

and they prove that $J$ is integrable if and only if

$$(1.16) \quad [\varphi,\varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi,\varphi]$ is the Nijenhuis torsion of $\varphi$. An almost contact structure is said to be *normal* if $J$ is integrable.

Different kinds of almost contact metric structures have been defined. A $(\varphi,\xi,\eta,g)$-structure is said to be *cosymplectic* [1,2] if it is normal with $\Phi$ and $\eta$ closed, *nearly cosymplectic* [2] if $(\nabla_X\varphi)Y + (\nabla_Y\varphi)X = 0$, *nearly Sasakian* [2] if $(\nabla_X\varphi)Y + (\nabla_Y\varphi)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X$, *quasi-Sasakian* [1] if it is normal with $d\Phi = 0$. An almost contact metric structure with $\Phi = d\eta$ is called a *contact metric structure* [2,8]; a contact metric structure such that $\xi$ is a Killing vector field with respect to $g$ is said to be a *K-contact structure* [2,7]. If a contact metric structure is normal, it is called a *Sasakian structure* [2].

We shall denote by $|N|$, $|C|$, $|nC|$, $|nS|$, $|qS|$, $|S|$, $|Kc|$ and $|c|$ the classes of normal, cosymplectic, nearly cosymplectic, nearly Sasakian, quasi-Sasakian, Sasakian, K-contact and contact metric structures, respectively.

Now, if $g$ is a Riemannian metric on the manifold $M$ with a $(\varphi,\xi,\eta)$-structure, we define a Riemannian metric on $M\times\mathbb{R}$ by

$$(1.17) \qquad h((X,a\frac{d}{dt}),(Y,b\frac{d}{dt})) = g(X,Y) + ab$$

and another by

$$(1.18) \qquad h^{o} = e^{2\sigma}h,$$

where $\sigma : M\times\mathbb{R} \longrightarrow \mathbb{R}$ is defined by $\sigma(x,t) = t$ for all $(x,t) \in M\times\mathbb{R}$. Then, the identity of $M\times\mathbb{R}$ is a conformal diffeomorphism between the Riemannian manifolds $(M\times\mathbb{R},h)$ and $(M\times\mathbb{R},h^{o})$.

LEMMA 1.1. *The following conditions are equivalent:*

*(i)* $g$ *is a compatible metric with the* $(\varphi,\xi,\eta)$*-structure.*

*(ii)* $h$ *is a Hermitian metric on* $(M\times\mathbb{R},J)$.

*(iii)* $h^{o}$ *is a Hermitian metric on* $(M\times\mathbb{R},J)$.

*Proof*: We have

$$h(J(X,a\tfrac{d}{dt}),J(Y,b\tfrac{d}{dt})) = g(\varphi X,\varphi Y) + \eta(X)\eta(Y) + ab.$$

The equivalences follow from (1.3),(1.17) and (1.18).

Hereafter, we suppose that the equivalent conditions of Lemma 1.1 are satisfied.

The Kaehler form $F$ of $(M\times\mathbb{R},J,h)$ is given by

$$F((X,a\tfrac{d}{dt}),(Y,b\tfrac{d}{dt})) = h((X,a\tfrac{d}{dt}),J(Y,b\tfrac{d}{dt})).$$

Hence,

$$(1.19)\qquad F((X,a\tfrac{d}{dt}),(Y,b\tfrac{d}{dt})) = \Phi(X,Y) - b\eta(X) + a\eta(Y),$$

and for the Kaehler form $F^{\circ}$ of $(M\times\mathbb{R},J,h^{\circ})$, we have

$$(1.20)\qquad F^{\circ}((X,a\tfrac{d}{dt}),(Y,b\tfrac{d}{dt})) = e^{2\sigma}\{\Phi(X,Y)-b\eta(X)+a\eta(Y)\}.$$

If $(x_{\circ},t_{\circ})\in M\times\mathbb{R}$, we consider injections $i:M \longrightarrow M\times\mathbb{R}$ and $j:\mathbb{R} \longrightarrow M\times\mathbb{R}$ defined by $i(x) = (x,t_{\circ})$ qnd $j(t) = (x_{\circ},t)$; if $X\in\mathfrak{X}(M)$ and $a$ is a $C^{\infty}$ function on $M\times\mathbb{R}$, $X(a\circ i)$ and $\frac{d}{dt}(a\circ j)$ will be simply denoted $X(a)$ and $\frac{da}{dt}$.

Let $\nabla$, $D$ and $D^{\circ}$ be the Riemannian connections of $(M,g)$, $(M\times\mathbb{R},h)$ and $(M\times\mathbb{R},h^{\circ})$, respectively.

PROPOSITION 1.2. *Let* $X,Y\in\mathfrak{X}(M)$, $a,b$ $C^{\infty}$ *functions on* $M\times\mathbb{R}$. *Then,*

$$(1.21)\quad D_{(X,a\frac{d}{dt})}(Y,b\tfrac{d}{dt}) = (\nabla_X Y,\{X(b) + a\tfrac{db}{dt}\}\tfrac{d}{dt}),$$

$$(1.22)\quad D^{\circ}_{(X,a\frac{d}{dt})}(Y,b\tfrac{d}{dt}) = (\nabla_X Y+bX+aY,\{-g(X,Y)+X(b)+a\tfrac{db}{dt}+ab\}\tfrac{d}{dt}).$$

*Proof*: (1.21) follows from (1.17) and from the following formula:

$$2h(D_{(X,a\frac{d}{dt})}(Y,b\frac{d}{dt}),(Z,c\frac{d}{dt})) =$$

$$(X,a\frac{d}{dt})(h((Y,b\frac{d}{dt}),(Z,c\frac{d}{dt}))) + (Y,b\frac{d}{dt})(h((X,a\frac{d}{dt}),(Z,c\frac{d}{dt})))$$

$$- (Z,c\frac{d}{dt})(h((X,a\frac{d}{dt}),(Y,b\frac{d}{dt}))) + h([(X,a\frac{d}{dt}),(Y,b\frac{d}{dt})],(Z,c\frac{d}{dt}))$$

$$+ h([(Z,c\frac{d}{dt}),(X,a\frac{d}{dt})],(Y,b\frac{d}{dt})) + h((X,a\frac{d}{dt}),[(Z,c\frac{d}{dt}),(Y,b\frac{d}{dt})]).$$

(1.22) is obtained from the analogous formula for $D^\circ$ or from (1.21) and the formula expressing the Riemannian connection $D^\circ$ in terms of D [3].

PROPOSITION 1.3. *Let* $X, Y \in \mathfrak{X}(M)$, $a,b$ $C^\infty$ *functions on* $M\times\mathbb{R}$. *Then,*

$$(1.23)\quad (D_{(X,a\frac{d}{dt})}J)(Y,b\frac{d}{dt}) = ((\nabla_X\varphi)Y - b\nabla_X\xi,(\nabla_X\eta)Y\frac{d}{dt}),$$

$$(1.24)\quad (D^\circ_{(X,a\frac{d}{dt})}J)(Y,b\frac{d}{dt}) =$$

$$((\nabla_X\varphi)Y+\eta(Y)X-g(X,Y)\xi-b\varphi X-b\nabla_X\xi,\{-\Phi(X,Y)+(\nabla_X\eta)Y\}\frac{d}{dt}).$$

*Proof*: (1.23) follows from (1.15) and (1.21); (1.24) follows from (1.4), (1.15) and (1.22).

PROPOSITION 1.4. *Let* $X,Y,Z\in\mathfrak{X}(M)$, $a,b,c$ $C^\infty$ *functions on* $M\times\mathbb{R}$. *Then,* (1.25) $(D_{(X,a\frac{d}{dt})}F)((Y,b\frac{d}{dt}),(Z,c\frac{d}{dt})) =$

$$(\nabla_X\Phi)(Y,Z) - c(\nabla_X\eta)Y + b(\nabla_X\eta)Z,$$

(1.26) $(D^{\circ}_{(X,a\frac{d}{dt})}F^{\circ})((Y,b\frac{d}{dt}),(Z,c\frac{d}{dt})) = e^{2\sigma}\{(\nabla_X\Phi)(Y,Z) +$

$$\eta(Z)g(X,Y)-\eta(Y)g(X,Z)+c\Phi(X,Y)-b\Phi(X,Z)-c(\nabla_X\eta)Y+b(\nabla_X\eta)Z\}$$

*Proof*: (1.25) follows from (1.5), (1.6), (1.17), (1.23) and from formula

$$(D_{(X,a\frac{d}{dt})}F)((Y,b\frac{d}{dt}),(Z,c\frac{d}{dt})) = h((Y,b\frac{d}{dt}),(D_{(X,a\frac{d}{dt})}J)(Z,c\frac{d}{dt})).$$

In a similar way, replacing D, $h$ and F by D°, $h^{\circ}$ and $F^{\circ}$, respectively, in the formula above, and using (1.18) and (1.24) we obtain (1.26).

PROPOSITION 1.5. *Let* $X,Y,Z \in \mathfrak{X}(M)$, $a,b,c$ $C^{\infty}$ *functions on* $M\times\mathbb{R}$. *Then,*

(1.27) $3dF((X,a\frac{d}{dt}),(Y,b\frac{d}{dt}),(Z,c\frac{d}{dt})) =$

$$3d\Phi(X,Y,Z) - 2\{cd\eta(X,Y)+ad\eta(Y,Z)+bd\eta(Z,X)\},$$

(1.28) $3dF^{\circ}((X,a\frac{d}{dt}),(Y,b\frac{d}{dt}),(Z,c\frac{d}{dt})) = e^{2\sigma}(3d\Phi(X,Y,Z) +$

$$2c\{\Phi(X,Y)-d\eta(X,Y)\}+2b\{\Phi(Z,X)-d\eta(Z,X)\}+2a\{\Phi(Y,Z)-d\eta(Y,Z)\}).$$

*Proof*: (1.27) follows from (1.12), (1.25) and from formula

$$3dF((X,a\frac{d}{dt}),(Y,b\frac{d}{dt}),(Z,c\frac{d}{dt})) = \mathfrak{S}\,(D_{(X,a\frac{d}{dt})}F)((Y,b\frac{d}{dt}),(Z,c\frac{d}{dt})).$$

Similarly, (1.28) follows from (1.12), (1.26) and from the formula above replacing $F$ and $D$ by $F^{\circ}$ and $D^{\circ}$, respectively.

We denote by $\delta$, $\bar{\delta}$ and $\bar{\delta}^{\circ}$ the coderivative operators of $(M,g)$, $(M\times\mathbb{R},h)$ and $(M\times\mathbb{R},h^{\circ})$, respectively.

PROPOSITION 1.6. *Let* $X \in \mathfrak{X}(M)$ *and* $a$ *a* $C^{\infty}$ *function on* $M\times\mathbb{R}$. *Then,*

$$(1.29) \qquad \bar{\delta}F(X,a\frac{d}{dt}) = \delta\Phi(X) - a\delta\eta,$$

$$(1.30) \qquad \bar{\delta}^{\circ}F^{\circ}(X,a\frac{d}{dt}) = \delta\Phi(X) - 2n\eta(X) - a\delta\eta.$$

*Proof*: We consider a local orthonormal basis $\{X_1,\dots,X_n,\varphi X_1,\dots,\varphi X_n,\xi\}$ defined on an open subset $U$ of $M$. Then, $\{(X_1,0),\dots,(X_n,0),(\varphi X_1,0),\dots,(\varphi X_n,0),(\xi,0),(0,\frac{d}{dt})\}$ is an orthonormal frame field with respect to $h$ on the open $U\times\mathbb{R}$ of $M\times\mathbb{R}$. The coderivative of $F$ is given by

$$\bar{\delta}F(X,a\frac{d}{dt}) =$$

$$-\sum_{i=1}^{n}\{(D_{(X_i,0)}F)((X_i,0),(X,a\frac{d}{dt}))+(D_{(\varphi X_i,0)}F)((\varphi X_i,0),(X,a\frac{d}{dt}))\}$$

$$-(D_{(\xi,0)}F)((\xi,0),(X,a\frac{d}{dt})) - (D_{(0,\frac{d}{dt})}F)((0,\frac{d}{dt}),(X,a\frac{d}{dt})).$$

By (1.13), (1.14) and (1.25) we obtain (1.29). Similarly, we prove (1.30), using (1.26) and the fact that

$\{e^{-\sigma}(X_1,0),\ldots,e^{-\sigma}(X_n,0),e^{-\sigma}(\varphi X_1,0),\ldots,e^{-\sigma}(\varphi X_n,0),$

$e^{-\sigma}(\xi,0),e^{-\sigma}(0,\frac{d}{dt})\}$ is an orthonormal frame field with respect to $h^o$ on $U\times\mathbb{R}$.

## 2. NEW TYPES OF ALMOST CONTACT STRUCTURES

The classification of A. Gray and L.M. Hervella [5] of almost Hermitian manifolds has been accomplished by means of a representation of the unitary group on a certain space $\mathcal{W}$, which can be interpreted as the space of tensors which satisfy the same identities as the covariant derivative of the Kaehler form on an almost Hermitian manifold. This representation has four irreducible components, $\mathcal{W}=\mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_3\oplus\mathcal{W}_4$, and it is possible to form sixteen different invariant subspaces from these four and each one of them corresponds to a different class of almost Hermitian manifolds. So, $\mathcal{W}_1$ corresponds to the class of nearly Kaehler manifolds, $\mathcal{W}_2$ to that of almost Kaehlerian, $\mathcal{W}_1\oplus\mathcal{W}_2$ to that of quasi-Kaehlerian, $\mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_3$ to that of semi-Kaehlerian, $\mathcal{W}_3\oplus\mathcal{W}_4$ to that of Hermitian, $\mathcal{W}_1\oplus\mathcal{W}_3\oplus\mathcal{W}_4$ to that of $G_1$-manifolds, $\mathcal{W}_2\oplus\mathcal{W}_3\oplus\mathcal{W}_4$ to that of $G_2$-manifolds and $\mathcal{W}_4$ to a class which contains locally conformal Kaehler manifolds. Furthermore, in [5] it is proved that every class containing $\mathcal{W}_4$ is preserved under a conformal diffeomorphism.

Let $M$ be a manifold with an almost contact metric structure $(\varphi,\xi,\eta,g)$. Then, by Lemma 1.1, we may consider the almost Hermitian manifolds $(M\times\mathbb{R},J,h)$ and $(M\times\mathbb{R},J,h^\circ)$.

The manifold $(M\times\mathbb{R},J,h)$ is Hermitian $(W_3\oplus W_4)$ if $J$ is integrable; a useful alternate characterization of Hermitian manifolds is given by the condition (cf. [3])

$$(2.1)\quad (D_{(X,a\frac{d}{dt})}J)(Y,b\tfrac{d}{dt}) - (D_{J(X,a\frac{d}{dt})}J)(J(Y,b\tfrac{d}{dt})) = 0.$$

THEOREM 2.1. *The following conditions are equivalent:*

*(i) The almost contact metric structure* $(\varphi,\xi,\eta,g)$ *is normal.*

*(ii)* $(M\times\mathbb{R},J,h)$ *is Hermitian.*

*(iii)* $(M\times\mathbb{R},J,h^\circ)$ *is Hermitian.*

*(iv)* $[\varphi,\varphi] + 2d\eta\otimes\xi = 0.$

*(v)* $(\nabla_X\varphi)Y - (\nabla_{\varphi X}\varphi)\varphi Y + \eta(Y)\nabla_{\varphi X}\xi = 0$ *for* $X,Y\in\mathfrak{X}(M)$.

*(vi)* $(\nabla_X\Phi)(Y,Z) - (\nabla_{\varphi X}\Phi)(\varphi Y,Z) - \eta(Y)(\nabla_{\varphi X}\eta)Z = 0$ *for* $X,Y,Z\in\mathfrak{X}(M)$.

*Proof:* *(i)* $\Longleftrightarrow$ *(ii)* follows from the definition of normal structure; *(ii)* $\Longleftrightarrow$ *(iii)* because the class of Hermitian manifolds is preserved under a conformal diffeomorphism. *(i)* $\Longleftrightarrow$ *(iv)* by (1.16). *(v)* $\Longleftrightarrow$ *(vi)* by (1.5) and (1.6). We shall prove *(ii)* $\Longleftrightarrow$ *(v)*. By (1.23), (2.1) is separated into the equations

$$(a)\quad (\nabla_X\varphi)Y - (\nabla_{\varphi X}\varphi)\varphi Y + \eta(Y)\nabla_{\varphi X}\xi = 0,$$

$$(b)\quad (\nabla_X\eta)Y - (\nabla_{\varphi X}\eta)\varphi Y = 0,$$

$$(c)\quad -\nabla_X\xi + (\nabla_{\varphi X}\varphi)\xi = 0,$$

(*d*) $(\nabla_\xi \varphi)\varphi Y - \eta(Y)\nabla_\xi \xi = 0,$

(*e*) $(\nabla_\xi \varphi)\xi = 0,$

(*f*) $(\nabla_\xi \eta)\varphi Y = 0.$

All these equations can be deduced from (*a*). In fact, setting $X=\xi$ in (*a*) we obtain $\nabla_\xi \varphi = 0$; thus we obtain (*d*), (*e*) and (*f*). Setting $Y=\xi$ in (*a*) we deduce (*c*). Finally, (*b*) is obtained from (*a*) using (1.6), (1.9) and (1.10).

The manifold $(M\times\mathbb{R}, J, h)$ (resp. $(M\times\mathbb{R}, J, h^\circ)$) is almost Kaehlerian ($W_2$) if $dF = 0$ (resp. $dF^\circ = 0$). By Proposition 1.5, we have

THEOREM 2.2. (*i*) $(M\times\mathbb{R}, J, h)$ *is almost Kaehlerian if and only if* $d\Phi = 0$ *and* $d\eta = 0$.

(*ii*) $(M\times\mathbb{R}, J, h^\circ)$ *is almost Kaehlerian if and only if* $(\varphi, \xi, \eta, g)$ *is a contact metric structure.*

The manifold $(M\times\mathbb{R}, J, h)$ is Kaehlerian if

$$(D_{(X, a\frac{d}{dt})} J)(Y, b\frac{d}{dt}) = 0. \tag{2.2}$$

THEOREM 2.3. *The following conditions are equivalent:*

(*i*) *The almost contact metric structure* $(\varphi, \xi, \eta, g)$ *is cosymplectic.*

(*ii*) $(M\times\mathbb{R}, J, h)$ *is Kaehlerian.*

(*iii*) $\nabla_X \varphi = 0$ *for* $X \in \mathfrak{X}(M)$.

*Proof:* $(i) \Longleftrightarrow (ii)$ is obtained from Theorems 2.1 and 2.2 and from the fact that a manifold is Kaehlerian if and only if it is almost Kaehlerian and Hermitian. $(ii) \Longleftrightarrow (iii)$ follows from (1.23) and (2.2).

Similarly, using (1.24), we obtain

THEOREM 2.4. *The following conditions are equivalent:*
*(i) The almost contact metric structure* $(\varphi,\xi,\eta,g)$ *is Sasakian.*
*(ii)* $(M\times\mathbb{R},J,h^{\circ})$ *is Kaehlerian.*
*(iii)* $(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X$ *for* $X,Y \in \chi(M)$.

$$(\nabla_X\Phi)(Y,Z) = -g(X,Y)\eta(Z) + g(X,Z)\eta(Y)$$

DEFINITION 2.5. *An almost contact metric structure* $(\varphi,\xi,\eta,g)$ *is called almost-cosymplectic if* $(M\times\mathbb{R},J,h)$ *is almost Kaehlerian.*

We shall denote by $|aC|$ the class of almost-cosymplectic structures.

The manifold $(M\times\mathbb{R},J,h)$ is nearly Kaehlerian $(W_1)$ if

$$(2.3)\qquad (D_{(X,a\frac{d}{dt})}J)(Y,b\tfrac{d}{dt}) + (D_{(Y,b\frac{d}{dt})}J)(X,a\tfrac{d}{dt}) = 0.$$

DEFINITION 2.6. *An almost contact metric structure* $(\varphi,\xi,\eta,g)$ *is called nearly-K-cosymplectic* $|nKC|$ *(resp. nearly-K-Sasakian* $|nKS|$*) if* $(M\times\mathbb{R},J,h)$ *(resp.* $(M\times\mathbb{R},J,h^{\circ})$*) is nearly Kaehlerian.*

THEOREM 2.7. *(i)* $(\varphi,\xi,\eta,g)$ *is nearly-K-cosymplectic if and only if*

(2.4) $(\nabla_X\varphi)Y + (\nabla_Y\varphi)X = 0,$ $(\nabla_X \Phi)(X,Y)=0$

(2.5) $\nabla_X\xi = 0,$

*for all* $X,Y \in \mathfrak{X}(M)$.

*(ii)* $(\varphi,\xi,\eta,g)$ *is nearly-K-Sasakian if and only if*

(2.6) $(\nabla_X\varphi)Y + (\nabla_Y\varphi)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X,$

(2.7) $\nabla_X\xi = -\varphi X,$

*for all* $X,Y \in \mathfrak{X}(M)$.

*Furthermore, if (2.4) (resp. (2.6)) is satisfied, the condition (2.5) (resp. (2.7)) is equivalent to*

(2.8) $\nabla_\xi\varphi = 0.$

*Proof:* *(i)* Proposition 1.3 implies that the equation (2.3) is separated into

$$(\nabla_X\varphi)Y + (\nabla_Y\varphi)X = 0, \quad \nabla_X\xi = 0, \quad (\nabla_X\eta)Y + (\nabla_Y\eta)X = 0.$$

Since $\nabla_X\xi = 0$ implies $\nabla_X\eta = 0$, the third equation follows from the second. Setting $Y=\xi$ in (2.4) we see that (2.5) is equivalent to (2.8).

*(ii)* Similarly, $(M\times\mathbb{R},J,h^\circ)$ is nearly Kaehlerian if and only if

$$(\nabla_X\varphi)Y+(\nabla_Y\varphi)X+\eta(X)Y+\eta(Y)X-2g(X,Y)\xi=0, \quad \nabla_X\xi+\varphi X=0, \quad (\nabla_X\eta)Y+(\nabla_Y\eta)X=0.$$

By (1.4) and (1.5), the second equation implies

(2.9) $\quad (\nabla_X\eta)Y = \Phi(X,Y)$

and hence we obtain the third. Setting $Y=\xi$ in (2.6) we see that (2.7) is equivalent to (2.8).

The manifold $(M\times\mathbb{R},J,h)$ is quasi-Kaehlerian $(W_1\oplus W_2)$ if

(2.10) $\quad (D_{(X,a\frac{d}{dt})}J)(Y,b\frac{d}{dt}) + (D_{J(X,a\frac{d}{dt})}J)(J(Y,b\frac{d}{dt})) = 0.$

DEFINITION 2.8. *An almost contact metric structure* $(\varphi,\xi,\eta,g)$ *is called quasi-K-cosymplectic* $|qKC|$ *(resp. quasi-K-Sasakian* $|qKS|$*) if* $(M\times\mathbb{R},J,h)$ *(resp.* $(M\times\mathbb{R},J,h^\circ)$*) is quasi-Kaehlerian.*

THEOREM 2.9. *(i)* $(\varphi,\xi,\eta,g)$ *is quasi-K-cosymplectic if and only if* $(\nabla_X\Phi)(Y,Z)+(\nabla_{\varphi X}\Phi)(\varphi Y,Z)+\eta(Y)(\nabla_{\varphi X}\eta)Z=0$

(2.11) $\quad (\nabla_X\varphi)Y + (\nabla_{\varphi X}\varphi)\varphi Y = \eta(Y)\nabla_{\varphi X}\xi$

*for all* $X,Y \in \mathfrak{X}(M)$.

*(ii)* $(\varphi,\xi,\eta,g)$ *is quasi-K-Sasakian if and only if*

(2.12) $\quad (\nabla_X\varphi)Y + (\nabla_{\varphi X}\varphi)\varphi Y = 2g(X,Y)\xi + \eta(Y)\nabla_{\varphi X}\xi - 2\eta(Y)X$

$(\nabla_X\Phi)(Y,Z)+(\nabla_{\varphi X}\Phi)(\varphi Y,Z)+2g(X,Y)\eta(Z)+\eta(Y)(\nabla_{\varphi X}\eta)Z-2\eta(Y)g(X,Z)=0$

*for all* $X,Y \in \mathfrak{X}(M)$.

*Proof*: We shall prove *(i)*. By Proposition 1.3, (2.10) is separated into the equations

*(a)* $\quad (\nabla_X\varphi)Y + (\nabla_{\varphi X}\varphi)\varphi Y - \eta(Y)\nabla_{\varphi X}\xi = 0,$

*(b)* $\quad (\nabla_X\eta)Y + (\nabla_{\varphi X}\eta)\varphi Y = 0,$

(*c*) $\nabla_X \xi + (\nabla_{\varphi X} \varphi)\xi = 0$,

(*d*) $-(\nabla_\xi \varphi)\varphi Y + \eta(Y)\nabla_\xi \xi = 0$,

(*e*) $(\nabla_\xi \varphi)\xi = 0$,

(*f*) $(\nabla_\xi \eta)\varphi Y = 0$,

where all of these equations follow from the first. In fact, setting $X=\xi$ in (*a*) we obtain (*e*) and hence $\nabla_\xi \xi = 0$. Therefore, we have (*d*) and (*f*). Setting $Y=\xi$ in (*a*) we obtain (*c*). (*b*) follows from (*a*) using (1.6), (1.9) and (1.10). (*ii*) is proved in a similar way.

The manifold $(M\times\mathbb{R}, J, h)$ (resp. $(M\times\mathbb{R}, J, h^\circ)$) is semi-Kaehlerian ($W_1\oplus W_2\oplus W_3$) if $\overline{\delta}F = 0$ (resp. $\overline{\delta}^\circ F^\circ = 0$).

DEFINITION 2.10. *An almost contact metric structure* $(\varphi, \xi, \eta, g)$ *is called semi-cosymplectic* $|sC|$ *(resp. semi-Sasakian* $|sS|$*) if* $(M\times\mathbb{R}, J, h)$ *(resp.* $(M\times\mathbb{R}, J, h^\circ)$*) is semi-Kaehlerian.*

By Proposition 1.6, we have

THEOREM 2.11. (*i*) $(\varphi, \xi, \eta, g)$ *is semi-cosymplectic if and only if*

$$\delta\Phi = 0, \qquad \delta\eta = 0. \tag{2.13}$$

(*ii*) $(\varphi, \xi, \eta, g)$ *is semi-Sasakian if and only if*

$$\eta = \frac{1}{2n}\delta\Phi. \tag{2.14}$$

An almost Hermitian manifold is a $W_3$-manifold if it is semi-Kaehlerian and Hermitian.

DEFINITION 2.12. *An almost contact metric structure* $(\varphi,\xi,\eta,g)$ *is called semi-cosymplectic normal* $|sCN|$ (*resp. semi-Sasakian normal* $|sSN|$) *if* $(M\times\mathbb{R},J,h)$ (*resp.* $(M\times\mathbb{R},J,h^\circ)$) *is a* $W_3$*-manifold.*

By Theorems 2.1 and 2.11, we have

THEOREM 2.13. (*i*) $(\varphi,\xi,\eta,g)$ *is semi-cosymplectic normal if and only if*

$(\nabla_X\Phi)(Y,Z)-(\nabla_{\varphi X}\Phi)(\varphi Y,Z)-\eta(Y)(\nabla_{\varphi X}\eta)Z=0$

$$\delta\Phi=0,\quad \delta\eta=0,\quad (\nabla_X\varphi)Y-(\nabla_{\varphi X}\varphi)\varphi Y+\eta(Y)\nabla_{\varphi X}\xi=0.$$

(*ii*) $(\varphi,\xi,\eta,g)$ *is semi-Sasakian normal if and only if*

$$\eta=\frac{1}{2n}\delta\Phi,\qquad (\nabla_X\varphi)Y-(\nabla_{\varphi X}\varphi)\varphi Y+\eta(Y)\nabla_{\varphi X}\xi=0.$$

Since the class $W_4$ of almost Hermitian manifolds is preserved under a conformal diffeomorphism, $(M\times\mathbb{R},J,h^\circ)$ is a $W_4$-manifold if and only if $(M\times\mathbb{R},J,h)$ is a $W_4$-manifold, and this is equivalent to (cf. [5])

$$(2.15)\quad (D_{(X,a\frac{d}{dt})}F)((Y,b\tfrac{d}{dt}),(Z,c\tfrac{d}{dt}))=$$

$$-\frac{1}{2n}\{h((X,a\tfrac{d}{dt}),(Y,b\tfrac{d}{dt}))\bar{\delta}F(Z,c\tfrac{d}{dt})-h((X,a\tfrac{d}{dt}),(Z,c\tfrac{d}{dt}))\bar{\delta}F(Y,b\tfrac{d}{dt})$$

$$-h((X,a\tfrac{d}{dt}),J(Y,b\tfrac{d}{dt}))\bar{\delta}F(J(Z,c\tfrac{d}{dt}))+h((X,a\tfrac{d}{dt}),J(Z,c\tfrac{d}{dt}))\bar{\delta}F(J(Y,b\tfrac{d}{dt}))\}.$$

Note that $\dim(M\times\mathbb{R})=2n+2$. (locally conformal Kähler mfds ($n\geq 2$))

DEFINITION 2.14. *An almost contact metric structure* $(\varphi,\xi,\eta,g)$ *is called trans-Sasakian* $|tS|$ *if* $(M\times\mathbb{R},J,h)$ *is a* $W_4$*-manifold.*

$(\nabla_X\Phi)(Y,Z) = -\frac{1}{2n}\{g(X,Y)\eta(Z)-g(X,Z)\eta(Y)\delta\Phi(\xi)+g(X,\varphi Y)\eta(Z)\delta\eta-g(X,\varphi Z)\eta(Y)\delta\eta\}$

THEOREM 2.15. $(\varphi,\xi,\eta,g)$ *is trans-Sasakian if and only if*

(2.16) $\quad (\nabla_X\Phi)(Y,Z) =$

$$-\frac{1}{2n}\{g(X,Y)\delta\Phi(Z)-g(X,Z)\delta\Phi(Y)+g(X,\varphi Y)\eta(Z)\delta\eta-g(X,\varphi Z)\eta(Y)\delta\eta\}$$

*for all* $X,Y,Z\in\chi(M)$. | and $\varphi^*(\delta\Phi)=0$

*Proof*: By using (1.15), (1.17), (1.25) and (1.29), the equation (2.15) is separated into

(a) $(\nabla_X\Phi)(Y,Z) = -\frac{1}{2n}\{g(X,Y)\delta\Phi(Z)-g(X,Z)\delta\Phi(Y)-g(X,\varphi Y)\delta\Phi(\varphi Z)$

$+g(X,\varphi Y)\eta(Z)\delta\eta+g(X,\varphi Z)\delta\Phi(\varphi Y)-g(X,\varphi Z)\eta(Y)\delta\eta\}$,

(b) $(\nabla_X\eta)Y = -\frac{1}{2n}\{g(X,Y)\delta\eta-g(X,\varphi Y)\delta\Phi(\xi)+\eta(X)\delta\Phi(\varphi Y)-\eta(X)\eta(Y)\delta\eta\}$,

(c) $-\eta(Y)\delta\Phi(\varphi Z)+\eta(Z)\delta\Phi(\varphi Y) = 0$,

(d) $-\delta\Phi(Y)+\eta(Y)\delta\Phi(\xi) = 0$.

Setting $Z=\xi$ in (c) we obtain $\delta\Phi(\varphi Y) = 0$, and hence, by (1.2) we have $\delta\Phi(Y) = \delta\Phi(-\varphi^2Y)+\delta\Phi(\eta(Y)\xi) = \eta(Y)\delta\Phi(\xi)$; therefore (c) implies (d). Conversely, replacing $Y$ by $\varphi Y$ in (d), we obtain $\delta\Phi(\varphi Y) = 0$ and (c) becomes an identity. Hence, the conditions (c), (d) and $\delta\Phi(\varphi Y) = 0$ are mutually equivalent. Therefore, the system of equations (a)-(d) is equivalent to the system of equations

Now, it is easy to prove that (a') and (c') imply (2.16). Conversely, from (2.16) and making use of (1.14) we obtain (c') and hence (a'). Finally, setting $Z=\xi$ and replacing $Y$ by $\varphi Y$ in (a') and using (1.9) we obtain (b'). Thus, equation[19] (2.16) is equivalent to the system of equations (a') - (c')

$$\text{(a')} \quad (\nabla_X \Phi)(Y,Z) = -\frac{1}{2n}\{g(X,Y)\delta\Phi(Z) - g(X,Z)\delta\Phi(Y) + g(X,\varphi Y)\eta(Z)\delta\eta - g(X,\varphi Z)\eta(Y)\delta\eta\},$$

$$\text{(b')} \quad (\nabla_X \eta)Y = -\frac{1}{2n}\{g(\varphi X,\varphi Y)\delta\eta - g(X,\varphi Y)\delta\Phi(\xi)\},$$

$$\text{(c')} \quad \delta\Phi(\varphi X) = 0.$$

Setting $Z=\xi$ and replacing $Y$ by $\varphi Y$ in $(a')$ and making use of (1.9) we obtain $(b')$. Setting $X=\xi$ in $(b')$ we have $\nabla_\xi \eta = 0$. On the other hand, setting $X=Y=\xi$ and replacing $Z$ by $\varphi Z$ in $(a')$ we obtain

$$(\nabla_\xi \Phi)(\xi,\varphi Z) = -\frac{1}{2n}\delta\Phi(\varphi Z).$$

By (1.9), we have

$$(\nabla_\xi \eta)Z = (\nabla_\xi \Phi)(\xi,\varphi Z)$$

from which we obtain $(c')$. Thus $(a')$ is the only independent condition.

Note that condition $\varphi^*(\delta\xi)=0$ in the Theorem above may be substituted by the condition $\nabla_\xi \eta = 0$. In fact, setting $X=\xi$ in (b') we have $(\nabla_\xi \eta)Y = -\frac{1}{2n}\delta\Phi(\varphi Y)$

The almost Hermitian manifold $(M\times\mathbb{R},J,h)$ is a $G_1$-manifold $(\mathcal{W}_1\oplus\mathcal{W}_3\oplus\mathcal{W}_4)$ if

$$(2.17) \quad (D_{(X,a\frac{d}{dt})}J)(X,a\tfrac{d}{dt}) + J(D_{J(X,a\frac{d}{dt})}J)(X,a\tfrac{d}{dt}) = 0.$$

DEFINITION 2.16. *An almost contact metric structure* $(\varphi,\xi,\eta,g)$ *is called* $G_1$*-Sasakian* $|G_1S|$ *if* $(M\times\mathbb{R},J,h)$ *(or, equivalently,* $(M\times\mathbb{R},J,h^\circ)$*) is a* $G_1$*-manifold.*

THEOREM 2.17. $(\varphi,\xi,\eta,g)$ *is* $G_1$*-Sasakian if and only if*

(2.18) $(\nabla_X\varphi)X - (\nabla_{\varphi X}\varphi)\varphi X + \eta(X)\nabla_{\varphi X}\xi = 0$

*for all* $X \in \mathfrak{X}(M)$. $\Leftrightarrow (\nabla_X\Phi)(Y,X) - (\nabla_{\varphi X}\Phi)(\varphi X,Y) - \eta(X)(\nabla_{\varphi X}\eta)Y = 0$

*Proof:* By (1.23), (2.17) is separated into the equations

(*a*) $(\nabla_X\varphi)X + \varphi(\nabla_{\varphi X}\varphi)X - \{(\nabla_{\varphi X}\eta)X\}\xi = 0,$

(*b*) $(\nabla_X\eta)X + \eta((\nabla_{\varphi X}\varphi)X) = 0,$

(*c*) $\nabla_X\xi + \varphi(\nabla_\xi\varphi)X + \varphi(\nabla_{\varphi X}\xi) - \{(\nabla_\xi\eta)X\}\xi = 0,$

(*d*) $\eta((\nabla_\xi\varphi)X) = 0,$

(*e*) $\varphi(\nabla_\xi\xi) = 0.$

All the previous equations can be deduced from (*a*). In fact, setting $X=\xi$ in (*a*), we obtain $(\nabla_\xi\varphi)\xi = 0$ and hence $\nabla_\xi\xi = 0$; thus we obtain (*e*). Furthermore,

$$\eta((\nabla_\xi\varphi)X) = -(\nabla_\xi\eta)\varphi X = 0,$$

which is the equation (*d*). By (1.6), (*a*) implies

$$(\nabla_X\Phi)(Y,X) - (\nabla_{\varphi X}\Phi)(\varphi Y,X) - \eta(Y)(\nabla_{\varphi X}\eta)X = 0.$$

Setting $Y=\xi$ in the above equation and using (1.10), we obtain

$$(\nabla_X\eta)\varphi X + (\nabla_{\varphi X}\eta)X = 0.$$

Replacing $X$ by $X+\varphi X$ in the last equation we obtain

$$(\nabla_X\eta)X - (\nabla_{\varphi X}\eta)\varphi X = 0,$$

which is equivalent to *(b)*. Now, replacing $X$ by $X+\xi$ in *(a)*, we have

$$(\nabla_X\varphi)\xi + (\nabla_\xi\varphi)X + \varphi(\nabla_{\varphi X}\varphi)\xi = 0.$$

Applying $\varphi$ to this equation we obtain *(c)*. Therefore *(a)* is the only independent condition. Taking into account that

$$(\nabla_{\varphi X}\varphi)\varphi+\varphi(\nabla_{\varphi X}\varphi) = \nabla_{\varphi X}\varphi^2 = \nabla_{\varphi X}(-I+\eta\otimes\xi) = (\nabla_{\varphi X}\eta)\otimes\xi+\eta\otimes\nabla_{\varphi X}\xi,$$

we may conclude that *(a)* is equivalent to (2.18).

The almost Hermitian manifold $(M\times\mathbb{R},J,h)$ is a $G_2$-manifold $(\mathcal{W}_2\oplus\mathcal{W}_3\oplus\mathcal{W}_4)$ if

$$(2.19)\quad dF((X,a\tfrac{d}{dt}),J(Y,b\tfrac{d}{dt}),J(Z,c\tfrac{d}{dt}))+dF(J(X,a\tfrac{d}{dt}),(Y,b\tfrac{d}{dt}),J(Z,c\tfrac{d}{dt}))$$

$$+dF(J(X,a\tfrac{d}{dt}),J(Y,b\tfrac{d}{dt}),(Z,c\tfrac{d}{dt}))-dF((X,a\tfrac{d}{dt}),(Y,b\tfrac{d}{dt}),(Z,c\tfrac{d}{dt})) = 0.$$

DEFINITION 2.18. *An almost contact metric structure* $(\varphi,\xi,\eta,g)$ *is called* $G_2$*-Sasakian* $|G_2S|$ *if* $(M\times\mathbb{R},J,h)$ *(or, equivalently,* $(M\times\mathbb{R},J,h^\circ)$*) is a* $G_2$*-manifold.*

THEOREM 2.19. *The following conditions are equivalent:*

*(i)* $(\varphi,\xi,\eta,g)$ *is* $G_2$*-Sasakian.*

*(ii)* $3d\Phi(X,\varphi Y,\varphi Z)+3d\Phi(\varphi X,Y,\varphi Z)+3d\Phi(\varphi X,\varphi Y,Z)-3d\Phi(X,Y,Z)$

$$-2\eta(Z)(d\eta(X,\varphi Y)+d\eta(\varphi X,Y))-2\eta(Y)(d\eta(Z,\varphi X)+d\eta(\varphi Z,X))$$

$$-2\eta(X)(d\eta(Y,\varphi Z)+d\eta(\varphi Y,Z)) = 0 \qquad \textit{for all } X,Y,Z \in \mathfrak{X}(M).$$

*(iii)* $\mathfrak{S}\{(\nabla_X\Phi)(Y,Z)-(\nabla_{\varphi X}\Phi)(\varphi Y,Z)-\eta(Y)(\nabla_{\varphi X}\eta)Z\} = 0$

*for all* $X,Y,Z \in \mathfrak{X}(M)$.

*Proof*: *(ii)* $\Longleftrightarrow$ *(iii)* follows from (1.7), (1.8) and (1.12). We shall prove *(i)* $\Longleftrightarrow$ *(ii)*. By (1.27), (2.19) is separated into the conditions *(ii)* and

$$-3d\Phi(X,\varphi Y,\xi) + 2\eta(Y)d\eta(\xi,X) - 3d\Phi(\varphi X,Y,\xi)$$

$$+ 2\eta(X)d\eta(Y,\xi) - 2d\eta(\varphi X,\varphi Y) + 2d\eta(X,Y) = 0.$$

This equation follows from *(ii)* replacing $X$ by $\varphi X$ and setting $Z=\xi$.

An almost Hermitian manifold is a $(W_1 \oplus W_3)$-manifold if it is a $G_1$-manifold and semi-Kaehlerian.

DEFINITION 2.20. *An almost contact metric structure* $(\varphi,\xi,\eta,g)$ *is called* $G_1$*-semi-cosymplectic* $|G_1 sC|$ *(resp.* $G_1$*-semi-Sasakian* $|G_1 sS|$*) if* $(M\times\mathbb{R},J,h)$ *(resp.* $(M\times\mathbb{R},J,h^\circ)$*) is a* $(W_1 \oplus W_3)$*-manifold.*

By Theorems 2.11 and 2.17, we have

THEOREM 2.21. *(i)* $(\varphi,\xi,\eta,g)$ *is* $G_1$*-semi-cosymplectic if and only if*

$$(\nabla_X \Phi)(X,Y) - (\nabla_{\varphi X}\Phi)(\varphi X,Y) - \eta(X)(\nabla_{\varphi X}\eta)Y$$

$$\Updownarrow$$

$$\delta\Phi = 0, \quad \delta\eta = 0, \quad (\nabla_X\varphi)X - (\nabla_{\varphi X}\varphi)\varphi X + \eta(X)\nabla_{\varphi X}\xi = 0.$$

*(ii)* $(\varphi,\xi,\eta,g)$ *is* $G_1$*-semi-Sasakian if and only if*

$$\eta = \frac{1}{2n}\delta\Phi, \quad (\nabla_X\varphi)X - (\nabla_{\varphi X}\varphi)\varphi X + \eta(X)\nabla_{\varphi X}\xi = 0.$$

An almost Hermitian manifold is a $(W_2 \oplus W_3)$-manifold if it is a $G_2$-manifold and semi-Kaehlerian.

DEFINITION 2.22. *An almost contact metric structure* $(\varphi, \xi, \eta, g)$ *is called* $G_2$*-semi-cosymplectic* $|G_2 sC|$ *(resp.* $G_2$*-semi-Sasakian* $|G_2 sS|$*) if* $(M\times\mathbb{R}, J, h)$ *(resp.* $(M\times\mathbb{R}, J, h^\circ)$*) is a* $(W_2 \oplus W_3)$*-manifold.*

By Theorems 2.11 and 2.19, we have

THEOREM 2.23. *(i)* $(\varphi, \xi, \eta, g)$ *is* $G_2$*-semi-cosymplectic if and only if*

$$\delta\Phi=0, \quad \delta\eta=0, \quad \mathfrak{S}\{(\nabla_X\Phi)(Y,Z)-(\nabla_{\varphi X}\Phi)(\varphi Y,Z)-\eta(Y)(\nabla_{\varphi X}\eta)Z\} = 0.$$

*(ii)* $(\varphi, \xi, \eta, g)$ *is* $G_2$*-semi-Sasakian if and only if*

$$\eta = \frac{1}{2n}\delta\Phi, \qquad \mathfrak{S}\{(\nabla_X\Phi)(Y,Z)-(\nabla_{\varphi X}\Phi)(\varphi Y,Z)-\eta(Y)(\nabla_{\varphi X}\eta)Z\} = 0.$$

The almost Hermitian manifold $(M\times\mathbb{R}, J, h)$ is a $(W_1 \oplus W_4)$-manifold if

$$(2.20) \quad (D_{(X,a\frac{d}{dt})}F)((X,a\tfrac{d}{dt}),(Y,b\tfrac{d}{dt}))=-\frac{1}{2n}\{h((X,a\tfrac{d}{dt}),(X,a\tfrac{d}{dt}))\overline{\delta}F(Y,b\tfrac{d}{dt})$$
$$-h((X,a\tfrac{d}{dt}),(Y,b\tfrac{d}{dt}))\overline{\delta}F(X,a\tfrac{d}{dt})-h(J(X,a\tfrac{d}{dt}),(Y,b\tfrac{d}{dt}))\overline{\delta}F(J(X,a\tfrac{d}{dt}))\}.$$

DEFINITION 2.24. *An almost contact metric structure* $(\varphi, \xi, \eta, g)$ *is called nearly-trans-Sasakian* $|ntS|$ *if* $(M\times\mathbb{R}, J, h)$ *(or, equivalently,* $(M\times\mathbb{R}, J, h^\circ)$*) is a* $(W_1 \oplus W_4)$*-manifold.*

THEOREM 2.25. $(\varphi,\xi,\eta,g)$ *is nearly-trans-Sasakian if and only if it satisfies the following conditions:*

(2.21) $(\nabla_X\Phi)(X,Y)=-\frac{1}{2n}\{g(X,X)\delta\Phi(Y)-g(X,Y)\delta\Phi(X)+g(\varphi X,Y)\eta(X)\delta\eta\}$,

(2.22) $(\nabla_X\eta)Y = -\frac{1}{2n}\{g(\varphi X,\varphi Y)\delta\eta + g(\varphi X,Y)\delta\Phi(\xi)\}$,

*for all* $X,Y \in \mathfrak{X}(M)$.

*Proof*: By (1.25) and (1.29), the equation (2.20) is separated into

(a) $(\nabla_X\Phi)(X,Y) =$

$-\frac{1}{2n}\{g(X,X)\delta\Phi(Y)-g(X,Y)\delta\Phi(X)-g(\varphi X,Y)\delta\Phi(\varphi X)+g(\varphi X,Y)\eta(X)\delta\eta\}$,

(b) $(\nabla_X\eta)Y = -\frac{1}{2n}\{g(\varphi X,\varphi Y)\delta\eta+g(\varphi X,Y)\delta\Phi(\xi)+\eta(Y)\delta\Phi(\varphi X)\}$,

(c) $\delta\Phi(X) = \eta(X)\delta\Phi(\xi)$.

We shall prove that (a) and (b) imply (c). Setting $X=\xi$ in (a), we obtain

$$(\nabla_\xi\Phi)(\xi,Y) = -\frac{1}{2n}\{\delta\Phi(Y) - \eta(Y)\delta\Phi(\xi)\}.$$

By (1.10),

$$(\nabla_\xi\Phi)(\xi,Y) = -(\nabla_\xi\eta)\varphi Y.$$

Setting $X=\xi$ in (b) we have $\nabla_\xi\eta = 0$; thus, from the last two equations we obtain (c). Moreover, the condition (c) implies $\delta\Phi(\varphi X) = 0$. Thus, the system of equations (a)-(c) is equivalent to the two equations (2.21)-(2.22).

The almost Hermitian manifold $(M\times\mathbb{R},J,h)$ is a $(W_2\oplus W_4)$-manifold if

(2.23) $$dF = F\wedge\theta,$$

"locally conformal almost Kähler condition if $\dim(M\times\mathbb{R})\geq 6$ (If $\dim(M\times\mathbb{R})=4$ one has to add $d\theta=0$)

where $\theta$ is the Lee form of $(M\times\mathbb{R},J,h)$, given by

$$\theta(X,a\frac{d}{dt}) = \frac{1}{n}\delta F(J(X,a\frac{d}{dt})),$$

which, by (1.29), may be written

(2.24) $$\theta(X,a\frac{d}{dt}) = \frac{1}{n}(\delta\Phi(\varphi X)-\eta(X)\delta\eta-a\delta\Phi(\xi)).$$

We define a 1-form $\omega$ on $M$ by

(2.25) $$\omega(X) = \theta(X,0) = \frac{1}{n}(\delta\Phi(\varphi X)-\eta(X)\delta\eta).$$

DEFINITION 2.26. *An almost contact metric structure* $(\varphi,\xi,\eta,g)$ *is called almost-trans-Sasakian* $|atS|$ *if* $(M\times\mathbb{R},J,h)$ *(or, equivalently,* $(M\times\mathbb{R},J,h^\circ)$*) is a* $(W_2\oplus W_4)$*- manifold.*

THEOREM 2.27. $(\varphi,\xi,\eta,g)$ *is almost-trans-Sasakian if and only if the following conditions are satisfied:*

(2.26) $$d\Phi = \Phi\wedge\omega,$$

(2.27) $$d\eta = \frac{1}{2n}\{\delta\Phi(\xi)\Phi - 2\eta\wedge\varphi(\delta\Phi)\}.$$

*Proof*: By (1.19), (1.27), (1.29) and (2.24), the equation (2.23) is separated into the equations (2.26) and (2.27).

The almost Hermitian manifold $(M\times\mathbb{R},J,h)$ is a $(W_1\oplus W_2\oplus W_4)$-manifold if

$$(2.28)\quad (D_{(X,a\frac{d}{dt})}F)((Y,b\frac{d}{dt}),(Z,c\frac{d}{dt}))+(D_{J(X,a\frac{d}{dt})}F)(J(Y,b\frac{d}{dt}),(Z,c\frac{d}{dt}))$$

$$= -\frac{1}{n}\{h((X,a\frac{d}{dt}),(Y,b\frac{d}{dt}))\bar{\delta}F(Z,c\frac{d}{dt})-h((X,a\frac{d}{dt}),(Z,c\frac{d}{dt}))\bar{\delta}F(Y,b\frac{d}{dt})$$

$$-h((X,a\frac{d}{dt}),J(Y,b\frac{d}{dt}))\bar{\delta}F(J(Z,c\frac{d}{dt}))+h((X,a\frac{d}{dt}),J(Z,c\frac{d}{dt}))\bar{\delta}F(J(Y,b\frac{d}{dt}))\}.$$

DEFINITION 2.28. *An almost contact metric structure* $(\varphi,\xi,\eta,g)$ *is called quasi-trans-Sasakian* $|qtS|$ *if* $(M\times\mathbb{R},J,h)$ *(or, equivalently,* $(M\times\mathbb{R},J,h^\circ)$*) is a* $(W_1\oplus W_2\oplus W_4)$*-manifold.*

THEOREM 2.29. $(\varphi,\xi,\eta,g)$ *is quasi-trans-Sasakian if and only if*

$$(2.29)\quad (\nabla_X\Phi)(Y,Z) + (\nabla_{\varphi X}\Phi)(\varphi Y,Z) + \eta(Y)(\nabla_{\varphi X}\eta)Z =$$

$$-\frac{1}{n}\{g(X,Y)\delta\Phi(Z) - g(X,Z)\delta\Phi(Y)$$

$$- g(X,\varphi Y)(\delta\Phi(\varphi Z)-\eta(Z)\delta\eta) + g(X,\varphi Z)(\delta\Phi(\varphi Y)-\eta(Y)\delta\eta)\}$$

*for all* $X,Y,Z\in\mathfrak{X}(M)$.

*Proof*: By (1.10), (1.25) and (1.29), the equation (2.28) is separated into

$$(a)\quad (\nabla_X\Phi)(Y,Z) + (\nabla_{\varphi X}\Phi)(\varphi Y,Z) + \eta(Y)(\nabla_{\varphi X}\eta)Z =$$

$$-\frac{1}{n}\{g(X,Y)\delta\Phi(Z) - g(X,Z)\delta\Phi(Y) - g(X,\varphi Y)\delta\Phi(\varphi Z)$$

$$+ g(X,\varphi Y)\eta(Z)\delta\eta + g(X,\varphi Z)\delta\Phi(\varphi Y) - g(X,\varphi Z)\eta(Y)\delta\eta\},$$

(*b*) $(\nabla_X\eta)Y + (\nabla_{\varphi X}\eta)\varphi Y =$

$$-\frac{1}{n}\{g(X,Y)\delta\eta - g(X,\varphi Y)\delta\Phi(\xi) + \eta(X)\delta\Phi(\varphi Y) - \eta(X)\eta(Y)\delta\eta\},$$

(*c*) $(\nabla_\xi\Phi)(\varphi Y,Z) + \eta(Y)(\nabla_\xi\eta)Z = -\frac{1}{n}\{\eta(Y)\delta\Phi(\varphi Z)-\eta(Z)\delta\Phi(\varphi Y)\},$

(*d*) $(\nabla_\xi\eta)\varphi Y = -\frac{1}{n}\{-\delta\Phi(Y) + \eta(Y)\delta\Phi(\xi)\},$

from which, the first is the only independent equation. In fact, setting $Z=\xi$ in (*a*), replacing $Y$ by $\varphi Y$ and using (1.9) and (1.10) we obtain (*b*). Setting $X=\xi$ in (*a*) and replacing $Y$ by $\varphi Y$, we have

$$(\nabla_\xi\Phi)(\varphi Y,Z) = \frac{1}{n}\{\eta(Z)\delta\Phi(\varphi Y)\}.$$

Setting $X=\xi$ in (*b*), interchanging $Y$ and $Z$ and multiplying by $\eta(Y)$, we obtain

$$\eta(Y)(\nabla_\xi\eta)Z = -\frac{1}{n}\{\eta(Y)\delta\Phi(\varphi Z)\}.$$

From the last two equations we obtain (*c*).

## 3. INCLUSION RELATIONS

Before comparing the classes of the almost contact metric structures, we shall define a new class which contains some of the other classes.

DEFINITION 3.1. *An almost contact metric structure* $(\varphi,\xi,\eta,g)$ *is called an almost-K-contact structure if* $\nabla_\xi\varphi=0$.

We shall denote by $|aKc|$ the class of the almost-K-contact structures. By Theorems 2.1 and 2.9, we have

PROPOSITION 3.2. $|N| \cup |qKC| \cup |qKS| \subset |aKc|$.

Keeping in mind that an almost Hermitian manifold is Kaehlerian if and only if it is almost Kaehlerian and nearly Kaehlerian, that an almost Kaehler or nearly Kaehler manifold is quasi-Kaehlerian and that a quasi-Kaehler manifold is semi-Kaehlerian, we have

THEOREM 3.3. (*i*) $|C| = |aC| \cap |nKC| \subset |aC| \cup |nKC| \subset |qKC| \subset |sC|$.
(*ii*) $|S| = |c| \cap |nKS| \subset |c| \cup |nKS| \subset |qKS| \subset |sS|$.

THEOREM 3.4. *Every nearly cosymplectic structure is semi-cosymplectic.*

*Proof*: If $(\varphi,\xi,\eta,g)$ is a nearly cosymplectic structure then, by (1.6), $(\nabla_X\Phi)(X,Y) = 0$, and hence, by (1.14), $\delta\Phi = 0$. Furthermore, $(\nabla_\xi\varphi)\xi = 0$ and so $\nabla_\xi\xi = 0$. Differentiating the compatibility condition of the metric (1.3) with respect to $\xi$ we have $g((\nabla_\xi\varphi)X,\varphi X) = 0$, from which the nearly cosymplectic condition gives $g((\nabla_X\varphi)\xi,\varphi X) = 0$ and, by using (1.6) and (1.9), $(\nabla_X\eta)X = 0$; therefore, by (1.13), $\delta\eta=0$. The conclusion follows from Theorem 2.11.

THEOREM 3.5. *Every nearly Sasakian structure is semi-Sasakian.*

*Proof*: Let $X$ be orthogonal to $\xi$ with $g(X,X) = 1$. Then, by (1.6) and the nearly Sasakian condition, we have

$$(\nabla_X\Phi)(X,Y)=-g((\nabla_X\varphi)X,Y)=-g(g(X,X)\xi-\eta(X)X,Y)=-g(\xi,Y)=-\eta(Y).$$

On the other hand,

$$(\nabla_\xi\Phi)(\xi,Y) = -g((\nabla_\xi\varphi)\xi,Y) = 0.$$

Hence, $\delta\Phi(Y) = 2n\eta(Y)$. From Theorem 2.11, we may conclude that the structure is semi-Sasakian.

Since the identity is a non homothetic conformal diffeomorphism between the almost Hermitian manifolds $(M\times\mathbb{R},J,h)$ and $(M\times\mathbb{R},J,h^\circ)$ (or by Theorem 2.11), we have that if one of them is semi-Kaehlerian, the other is never semi-Kaehlerian [3]. As a consequence, we have

THEOREM 3.6. *$|sC|\cap|sS|$ is the empty class. Then if $(\varphi,\xi,\eta,g)$ is in one of the classes $|C|$, $|aC|$, $|nKC|$, $|qKC|$, $|nC|$, $|sC|$, it is never in any of the classes $|S|$, $|c|$, $|nKS|$, $|qKS|$, $|nS|$, $|sS|$.*

If an almost Hermitian manifold is Kaehlerian then it is a $W_4$-manifold and if a manifold is a $W_4$-manifold then it is Hermitian. The nearly Kaehler manifolds are $G_1$-manifolds and the almost Kaehler manifolds are $G_2$-manifolds. Furthermore, an almost Hermitian manifold is Hermitian if and only if it is a $G_1$-manifold and a $G_2$-manifold. Thus we have

THEOREM 3.7. $|C| \cup |S| \subset |tS| \subset |N|$,
$|nKC| \cup |nKS| \cup |N| \subset |G_1S|$, $|aC| \cup |c| \cup |N| \subset |G_2S|$,
$|G_1S| \cap |G_2S| = |N|$.

THEOREM 3.8. $|nKC| = |nC| \cap |aKc| = |nC| \cap |qKC| = |nC| \cap |G_1S| = |nC| \cap |qtS|$.

*Proof:* Since every nearly-K-cosymplectic structure is nearly cosymplectic, quasi-K-cosymplectic, $G_1$-Sasakian and quasi-trans-Sasakian and making use of Proposition 3.2, we obtain that $|nKC|$ is contained in the four intersections of classes above. $|nC| \cap |qKc| \subset |nC| \cap |aKc|$ follows from Proposition 3.2 and $|nC| \cap |aKc| \subset |nKC|$ from Theorem 2.7. If $(\varphi,\xi,\eta,g)$ is nearly cosymplectic then, by Theorem 3.4, it is semi-cosymplectic; if it is also quasi-trans-Sasakian, then it is quasi-K-cosymplectic, since an almost Hermitian manifold is quasi-Kaehlerian if and only if it is semi-Kaehlerian and it is a $(W_1 \oplus W_2 \oplus W_4)$-manifold. Therefore, $|nC| \cap |qtS| \subset |nC| \cap |qKC|$. We shall now prove $|nC| \cap |G_1S| \subset |nKC|$. If $(\varphi,\xi,\eta,g)$ is nearly cosymplectic, $(\nabla_X\varphi)X = 0$ and hence, by Theorem 2.17, the $G_1$-Sasakian condition (2.18) becomes $\eta(X)\nabla_{\varphi X}\xi = 0$, from which, replacing $X$ by $X+Y$, we obtain $\eta(X)\nabla_{\varphi Y}\xi+\eta(Y)\nabla_{\varphi X}\xi = 0$; setting $Y=\xi$, we have $\nabla_{\varphi X}\xi = 0$ and hence $\nabla_{\varphi^2 X}\xi = 0$; since $\nabla_\xi\xi = 0$, by (1.2), we obtain $\nabla_X\xi = 0$; applying Theorem 2.7, we may conclude that $(\varphi,\xi,\eta,g)$ is nearly-K-cosymplectic.

THEOREM 3.9. $|C| = |aC| \cap |nKC| = |aC| \cap |nC| = |nC| \cap |N| = |aC| \cap |N| = |qKC| \cap |N| = |sC| \cap |tS|$.

*Proof:* It is evident that the class $|C|$ is contained in all the intersections of classes above. The first equality is in Theorem 3.3. If $(\varphi, \xi, \eta, g)$ is an almost-cosymplectic structure, then it is quasi-K-cosymplectic and hence, by Proposition 3.2, it is an almost-K-contact structure; thus, if it is also nearly cosymplectic then, by Theorem 3.8, it is nearly-K-cosymplectic. Then $|aC| \cap |nC| \subset |aC| \subset |nKC|$. $|qKC| \cap |N| \subset |C|$ since a quasi-Kaehler and Hermitian manifold is Kaehlerian. $|nC| \cap |N| \subset |qKC| \cap |N|$ since, by Theorem 2.7 and Proposition 3.2, a normal nearly cosymplectic structure is nearly-K-cosymplectic and hence quasi-K-cosymplectic. $|aC| \cap |N| \subset |qKC| \cap |N|$ because every almost-cosymplectic structure is quasi-K-cosymplectic. $|sC| \cap |tS| = |C|$ since an almost Hermitian manifold is semi-Kaehlerian and it is a $W_4$-manifold if and only if it is Kaehlerian.

PROPOSITION 3.10. *Every nearly-K-Sasakian structure is a contact metric structure.*

*Proof:* This is immediate from (1.11) and (2.9).

THEOREM 3.11. $|S| = |nKS| = |aKc| \cap |nS| = |c| \cap |nS| = |Kc| \cap |nS| = |nS| \cap |N| = |nS| \cap |qKS| = |nS| \cap |G_1 S| = |nS| \cap |qtS| = |qKS| \cap |N| = |sS| \cap |tS|$.

*Proof:* $|S| = |nKS|$ follows from (*ii*) of Theorem 2.2, Proposition 3.10 and from the fact that an almost Hermitian manifold is Kaehlerian if and only if it is nearly Kaehlerian and almost Kaehlerian. The proof of the other equalities is similar to those of Theorems 3.8 and 3.9.

PROPOSITION 3.12. *If* $(\varphi,\xi,\eta,g)$ *is a quasi-K-Sasakian structure,*

(3.1) $\quad d\eta(X,Y) + d\eta(\varphi X,\varphi Y) = 2\Phi(X,Y)$

*for* $X,Y \in \mathfrak{X}(M)$.

*Proof*: By using (1.6), (1.9) and (1.10), from (2.12) it follows that

$$(\nabla_X\eta)Y + (\nabla_{\varphi X}\eta)\varphi Y = 2\Phi(X,Y)$$

from which, by (1.11), we obtain (3.1).

COROLLARY 3.13. *A quasi-K-Sasakian structure* $(\varphi,\xi,\eta,g)$ *is a contact metric structure if and only if* $d\eta(X,Y)=d\eta(\varphi X,\varphi Y)$ *for all* $X,Y \in \mathfrak{X}(M)$. (equivalently, $N^{(2)}=0$ ; Blair p.51)

Now, we shall prove some properties of quasi-Sasakian structures and their relation with trans-Sasakian structures. For a quasi-Sasakian structure $(\varphi,\xi,\eta,g)$ we have (cf. [1])

(3.2) $\quad (\nabla_X\eta)Y = -(\nabla_Y\eta)X.$

From (1.13) and (3.2), we have

PROPOSITION 3.14. *If* $(\varphi,\xi,\eta,g)$ *is a quasi-Sasakian structure then* $\delta\eta = 0$.

LEMMA 3.15. *Let* $\{X_i,\varphi X_i,\xi;\ i=1,\dots,n\}$ *be a local orthonormal basis defined on an open subset of the manifold*

*M with an almost contact metric structure* $(\varphi,\xi,\eta,g)$. *Then,*

$$(3.3)\quad \delta\Phi(\varphi X) = -3\sum_{i=1}^{n}\{d\Phi(X_i,\varphi X_i,X)\} + \eta(X)\delta\eta - (\nabla_\xi\eta)X$$

*for all* $X \in \mathfrak{X}(M)$.

*Proof*: By (1.14), we have

$$(3.4)\quad \delta\Phi(\varphi X) = -\sum_{i=1}^{n}\{(\nabla_{X_i}\Phi)(X_i,\varphi X)+(\nabla_{\varphi X_i}\Phi)(\varphi X_i,\varphi X)\}-(\nabla_\xi\Phi)(\xi,\varphi X).$$

By using (1.7) and since $\eta(X_i) = g(X_i,\xi) = 0$, it follows that

$$(3.5)\quad (\nabla_{X_i}\Phi)(X_i,\varphi X) = (\nabla_{X_i}\Phi)(\varphi X_i,X) + \eta(X)(\nabla_{X_i}\eta)X_i.$$

By (1.8),

$$(3.6)\quad (\nabla_{\varphi X_i}\Phi)(\varphi X_i,\varphi X) = -(\nabla_{\varphi X_i}\Phi)(X_i,X) + \eta(X)(\nabla_{\varphi X_i}\eta)\varphi X_i.$$

On the other hand, from (1.12) we have

$$(3.7)\qquad (\nabla_{X_i}\Phi)(\varphi X_i,X) - (\nabla_{\varphi X_i}\Phi)(X_i,X) =$$

$$3d\Phi(X_i,\varphi X_i,X) - (\nabla_X\Phi)(X_i,\varphi X_i) = 3d\Phi(X_i,\varphi X_i,X),$$

since, by (1.6),

$$(\nabla_X\Phi)(X_i,\varphi X_i) = -g(\nabla_X\varphi X_i,\varphi X_i) + g(\varphi\nabla_X X_i,\varphi X_i)$$

$$= -g(\nabla_X\varphi X_i,\varphi X_i)+g(\nabla_X X_i,X_i)-\eta(\nabla_X X_i)\eta(X_i) = 0.$$

Furthermore, by (1.9) we have

$$(3.8)\qquad (\nabla_\xi\Phi)(\xi,\varphi X) = (\nabla_\xi\eta)X.$$

Therefore, from (3.4), (3.5), (3.6), (3.7), (3.8) and (1.13) we obtain (3.3).

PROPOSITION 3.16. *If* $(\varphi,\xi,\eta,g)$ *is a quasi-Sasakian structure then* $\delta\Phi(\varphi X)=0$ *and so* $\delta\Phi(X) = \eta(X)\delta\Phi(\xi)$ *for all* $X \in \chi(M)$.

*Proof:* By definition, a quasi-Sasakian structure is normal; then if we set $X=Y=\xi$ in (v) of Theorem 2.1, we obtain $(\nabla_\xi\varphi)\xi = 0$, from which $\nabla_\xi\xi=0$ and hence $\nabla_\xi\eta=0$. By Proposition 3.14, $\delta\eta=0$. Furthermore, $d\Phi=0$. Therefore, from Lemma 3.15 we obtain $\delta\Phi(\varphi X) = 0$.

LEMMA 3.17. *If* $(\varphi,\xi,\eta,g)$ *is a trans-Sasakian structure then*

$$(3.9) \qquad d\Phi = -\frac{1}{n}\delta\eta(\Phi\wedge\eta).$$

*Proof:* This follows from (1.12) and Theorem 2.15.

THEOREM 3.18. *If* $(\varphi,\xi,\eta,g)$ *is a trans-Sasakian structure satisfying* $\delta\eta = 0$ *then it is quasi-Sasakian.*

*Proof:* Since a trans-Sasakian structure is normal, this is immediate from Lemma 3.17.

In particular, note that both cosymplectic and Sasakian structures are trans-Sasakian satisfying $\delta\eta = 0$, since both structures are semi-cosymplectic and semi-Sasakian, respectively.

PROPOSITION 3.19. *A quasi-Sasakian structure* $(\varphi,\xi,\eta,g)$ *is trans-Sasakian if and only if*

$$(3.10)\qquad (\nabla_X\Phi)(Y,Z) = -\frac{1}{2n}\{g(X,Y)\eta(Z) - g(X,Z)\eta(Y)\}\delta\Phi(\xi).$$

*Proof:* For a quasi-Sasakian structure, $\delta\eta = 0$ by Proposition 3.14. If it is also trans-Sasakian, ~~by Theorem 2.15,~~ ~~we have~~ from Theorem 2.15 it follows (3.10).

~~$$(\nabla_X\Phi)(Y,Z) = -\frac{1}{2n}\{g(X,Y)\delta\Phi(Z) - g(X,Z)\delta\Phi(Y)\}.$$~~

~~Now, from Proposition 3.16 it follows (3.10)~~. Conversely, if a quasi-Sasakian structure satisfies (3.10), Theorem 2.15 and Propositions 3.14 ~~and 3.16~~ imply that it is trans-Sasakian.

For a cosymplectic structure, $\delta\Phi = 0$ and the expression (3.10) becomes $(\nabla_X\Phi)(Y,Z) = 0$. A Sasakian structure is semi-Sasakian and hence, by Theorem 2.11, $\delta\Phi(\xi)=2n\eta(\xi)=2n$; then, in this case, the expression (3.10) becomes

$$(\nabla_X\Phi)(Y,Z) = g(X,Z)\eta(Y) - g(X,Y)\eta(Z),$$

which was proved, in another way, in [10].

PROPOSITION 3.20. *If* $(\varphi,\xi,\eta,g)$ *is a quasi-Sasakian structure then*

$$(3.11)\qquad (\nabla_X\Phi)(Y,Z) = d\eta(X,\varphi Y)\eta(Z) - d\eta(X,\varphi Z)\eta(Y)$$

*for all* $X,Y,Z \in \mathfrak{X}(M)$.

The proof is a very lengthy computation but similar to that of [10] for a Sasakian structure.

THEOREM 3.21. *A quasi-Sasakian structure* $(\varphi, \xi, \eta, g)$ *is trans-Sasakian if and only if*

$$(3.12) \qquad d\eta(X,Y) = \frac{1}{2n}\Phi(X,Y)\delta\Phi(\xi)$$

*for all* $X, Y \in \mathfrak{X}(M)$.

*Proof*: If $(\varphi,\xi,\eta,g)$ is quasi-Sasakian and trans-Sasakian, setting $Z=\xi$ in (3.10) and (3.11) and equalizing both expressions, we obtain

$$(3.13) \qquad d\eta(X,\varphi Y) = -\frac{1}{2n}\{g(X,Y) - \eta(X)\eta(Y)\}\delta\Phi(\xi).$$

Furthermore, since a quasi-Sasakian structure is normal, $\nabla_\xi \eta = 0$, and hence, by (1.11), $d\eta(X,\xi) = 0$. Then, by (1.2), (1.4) and (3.13),

$$d\eta(X,Y) = - d\eta(X,\varphi^2 Y) + \eta(Y)d\eta(X,\xi) = -d\eta(X,\varphi^2 Y)$$

$$= \frac{1}{2n}g(X,\varphi Y)\delta\Phi(\xi) = \frac{1}{2n}\Phi(X,Y)\delta\Phi(\xi).$$

Conversely, if (3.12) is satisfied for a quasi-Sasakian structure, then, from Propositions 3.19 and 3.20 it follows that it is trans-Sasakian.

Diagram I summarizes the inclusion relations among the different classes of almost contact metric structures, where the arrows ($\longrightarrow$) mean inclusions ( $\subset$ ).

DIAGRAM I

Now, we suppose that $M$ is a 3-dimensional ($n=1$) manifold with a $(\varphi,\xi,\eta,g)$-structure. Then $(M\times\mathbb{R},J,h)$ and $(M\times\mathbb{R},J,h^\circ)$ are almost Hermitian manifolds of dimension 4. In [4] it is proved that a 4-dimensional nearly-Kaehler manifold is Kaehlerian, and, similarly, it can be proved that a 4-dimensional semi-Kaehler manifold is quasi-Kaehlerian and that a 4-dimensional quasi Kaehler manifold is almost Kaehlerian. Furthermore, a 4-dimensional Hermitian manifold is a $W_4$-manifold [5]. Thus, the 4-dimensional $G_1$-manifolds are Hermitian and all the 4-dimensional almost Hermitian manifolds are $G_2$-manifolds. As a consequence, the relations among the classes of almost contact metric structures on a 3-dimensional manifold are those represented in Diagram II, for which we also use the following

THEOREM 3.22. *(i) A nearly cosymplectic structure on a 3-dimensional manifold is cosymplectic.*

*(ii) A nearly Sasakian structure on a 3-dimensional manifold is Sasakian.*

*Proof*: It follows from Theorem 2.7, Proposition 3.2 and the first equality in *(i)* and *(ii)* of Theorem 3.3.

Finally, from [1] it follows that a quasi-Sasakian structure on a 3-dimensional manifold is either cosymplectic or Sasakian.

DIAGRAM II

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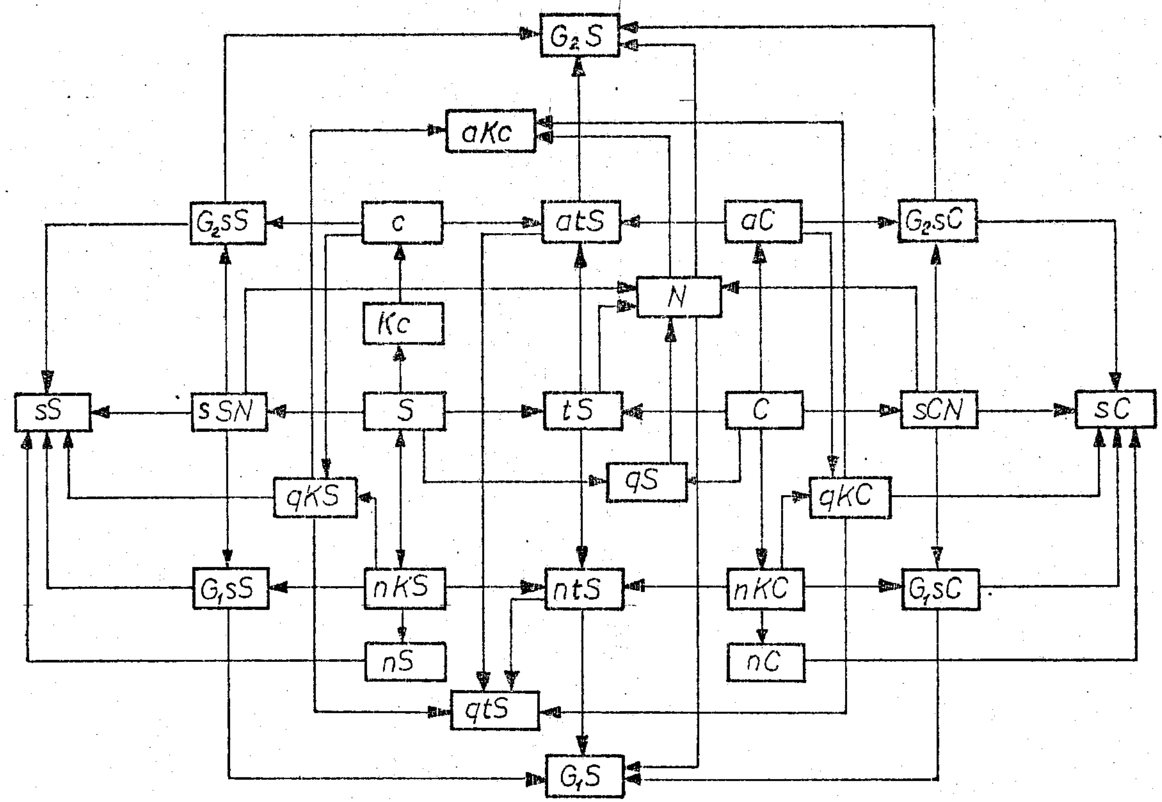

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DIAGRAM I

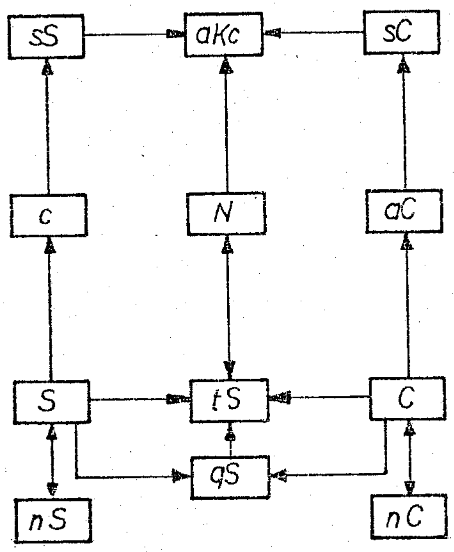


DIAGRAM II