Cartan sub-C*-algebras, pseudogroups and orbifolds

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10 July 2008

1. **Non commutative geometry and C*-algebra constructions.**
2. **Pseudogroups and groupoids of germs.**
3. **Cartan subalgebras in C*-algebras.**
4. **Orbifolds.**

Non commutative geometry.

The basic idea of NCG is to replace a space X by an algebra A. Commutative algebras will reflect classical spaces while non commutative algebras will reflect quantum spaces. For example, we replace the locally compact Hausdorff space X by the algebra $A = C_0(X)$ of complex-valued continuous functions on X vanishing at infinity. This algebra is an example of a C*-algebra.

Definition

A C*-algebra is a complex involutive algebra A endowed with a complete norm satisfying

- $\|ab\| \leq \|a\|\|b\|$;
- $\|a^*\| = \|a\|$;
- $\|a^*a\| = \|a\|^2$.

In our case, the norm is the sup norm $\|a\| = \sup_X |a(x)|$. Moreover, $C_0(X)$ is commutative: $ab = ba$. In fact, this is the most general example of a commutative C*-algebra.

Theorem (Gelfand)

Let A be a commutative C-algebra. Then there exists a locally compact Hausdorff space X and an isomorphism $A \cong C_0(X)$.*

This theorem gives in fact an equivalence of categories. The space X is the spectrum of A, i.e. the space of characters of A, and the isomorphism is the Gelfand transform. There is no such a theorem for NC C*-algebras.

What I will present is a mild generalization of Gelfand's theorem. It is based on a construction which associates a C*-algebra to a locally compact groupoid with a Haar system.

Groupoids

Definition

A groupoid is a small category $(G, G^{(0)})$ such that every arrow is invertible.

$$r, s : G \to G^{(0)}$$

$$\begin{array}{ccc} G^{(2)} & \to & G \\ (\gamma, \gamma') & \mapsto & \gamma\gamma' \end{array} \qquad \begin{array}{ccc} G & \to & G \\ \gamma & \mapsto & \gamma^{-1} \end{array}$$

Example: action of a group Γ on a space X: $\begin{array}{ccc} X \times \Gamma & \to & X \\ (x, g) & \mapsto & xg \end{array}$

$$G = \{(x, g, y) \in X \times \Gamma \times X : y = xg\}$$

$$r(x, g, y) = x \qquad s(x, g, y) = y$$

$$(x, g, y)(y, h, z) = (x, gh, z) \qquad (x, g, y)^{-1} = (y, g^{-1}, x)$$

Haar systems

Definition

Let G be locally compact Hausdorff topological groupoid. A Haar system $\lambda = (\lambda^x)$ is a family of measures λ^x with support $G^x = r^{-1}(x)$ satisfying

- (continuity) $\forall f \in C_c(G), x \mapsto \int f d\lambda^x$ is continuous;
- (left invariance) $\forall \gamma \in G$, $\gamma\lambda^{s(\gamma)} = \lambda^{r(\gamma)}$.

In the previous example of a group action (Γ, X), a left Haar measure λ on Γ defines a Haar system (λ^x) on G such that $\int f d\lambda^x = \int f(x, g, xg) d\lambda(g)$.

Definition

We say that the topological groupoid G is étale if the range map $r : G \to G^{(0)}$ is a local homeomorphism.

An étale locally compact Hausdorff groupoid has a natural Haar system, given by $\int f d\lambda^x = \sum_{r(\gamma)=x} f(\gamma)$.

The C*-algebra $C_r^*(G)$

Let (G,λ) be a locally compact Hausdorff groupoid endowed with a Haar system. The following operations turn the space $C_c(G)$ of compactly supported complex-valued continuous functions on G into an involutive algebra:

$$f*g(\gamma)=\int f(\gamma\gamma')g(\gamma'^{-1})d\lambda^{s(\gamma)}(\gamma');$$

$$f^*(\gamma)=\overline{f(\gamma^{-1})}.$$

For each $x\in G^{(0)}$, one defines the representation π_x of $C_c(G)$ on the Hilbert space $L^2(G_x,\lambda_x)$, where $G_x=s^{-1}(x)$ and $\lambda_x=(\lambda^x)^{-1}$, by $\pi_x(f)\xi=f*\xi$. One defines the reduced norm $\|f\|_r=\sup\|\pi_x(f)\|$. The reduced C*-algebra $C_r^*(G)$ is the completion of $C_c(G)$ for the reduced norm.

Some examples

groupoid	**C*-algebra**
G	$C_r^*(G)$
X	$C_0(X)$
$\{1,\dots,n\}\times\{1,\dots,n\}$	$M_n(\mathbb{C})$
$X\times X$	$\mathcal{K}(L^2(X))$
tail equivalence relation on $\{1,\dots,n\}^{\mathbb{N}}$	$UHF(n^\infty)=\otimes_{\mathbb{N}}M_n(\mathbb{C})$
on a Bratteli diagram	$AF-\text{algebra}$
one-sided shift on $\{1,\dots,n\}^{\mathbb{N}}$	Cuntz algebra O_n
rotation of angle θ	rotation algebra A_θ

The C*-algebra $C_r^*(G,E)$

We shall need a slight generalization of the above construction.

Definition (twist)

Let G be a groupoid. A twist over G is a groupoid extension

$$\mathbb{T} \times X \rightarrowtail E \twoheadrightarrow G$$

where $X = G^{(0)} = E^{(0)}$ and $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

For example, a 2-Čech cocycle $\sigma = (\sigma_{ijk})$ relative to an open cover (U_i) of a topological space X defines a twist E_σ over the groupoid $G = \{(i,x,j) : x \in U_i \cap U_j\}$ of the open cover.

We replace the complex-valued functions by the sections of the associated complex line bundle. Essentially the same formulas as above provide the C*-algebra $C_r^*(G,E)$.

Reconstruction

When passing from the groupoid G to the C*-algebra $A = C_r^*(G)$, much information is lost. However, some extra piece of structure allows to recover G from A. In the case of a group, the coproduct does the job. When G is étale, the commutative C*-algebra $B = C_0(X)$, where $X = G^{(0)}$ is a subalgebra of $A = C_r^*(G, E)$. Thus our construction provides a pair (A, B) where A is a C*-algebra and B is a commutative sub-C*-algebra rather than just a C*-algebra. We shall study the case when the pair (A, B) completely determines the twisted groupoid (G, E).

Proposition (R 80)

Let G be an étale second countable locally compact Hausdorff groupoid. Then $B = C_0(X)$ is maximal abelian self-adjoint in $A = C_r^*(G, E)$ iff G is topologically principal.

Let X be a topological space. A partial homeomorphism of X is a homeomorphism $S : \mathrm{D}(S) \to \mathrm{R}(S)$, where $\mathrm{D}(S)$ and $\mathrm{R}(S)$ are open subsets of X. One defines the composition ST and the inverse S^{-1}. A **pseudogroup** on X is a family $\mathcal{G}$ of partial homeomorphisms of X closed under composition and inverse.

Given a partial homeomorphism S and $y \in \mathrm{D}(S)$, we denote by $[Sy, S, y]$ the germ of S at y.

One can associate to a pseudogroup $\mathcal{G}$ on X its **groupoid of germs** G. Its elements are the germs of $\mathcal{G}$. Its groupoid structure is:

$$r([x, S, y]) = x \quad s([x, S, y]) = y$$

$$[x, S, y][y, T, z] = [x, ST, z]$$

$$[x, S, y]^{-1} = [y, S^{-1}, x]$$

Its topology is the topology of germs. It turns G into a topological groupoid, which is locally compact if X is so, but not necessarily Hausdorff.

Conversely, let G be an étale groupoid with $G^{(0)} = X$. Its open bisections define a pseudogroup $\mathcal{G}$ on X, hence a groupoid of germs H. We have the groupoid extension:

$$\mathrm{Int}(G') \rightarrowtail G \twoheadrightarrow H$$

where $G' = \{\gamma \in G : r(\gamma) = s(\gamma)\}$ and $\mathrm{Int}(G')$ is its interior.

Definition

An étale groupoid G is said to be

- effective if it is isomorphic to its groupoid of germs;
- topologically principal if the set of units without isotropy is dense in $G^{(0)}$.

Proposition

Let G be an étale groupoid.

- *topologically principal $\Rightarrow$ effective;*
- *if G is second countable Hausdorff and if $G^{(0)}$ is a Baire space, then effective $\Rightarrow$ topologically principal.*

Examples

- Tranverse holonomy groupoids of foliated manifolds.
- The groupoid of a topologically free semi-group action

$$T : X \times N \to X :$$

$$G(X,T) = \{(x, m-n, y) : T(m)x = T(n)y\}.$$

- Minimal Cantor systems.
- Markov chains satisfying Cuntz-Krieger condition (I).

Definition

Definition (Kumjian 86)

Let B be a sub-C*-algebra of a C*-algebra A. One says that B is regular if its normalizer $N(B) = \{a \in A : aBa^* \subset B \quad a^*Ba \subset B\}$ generates A as a C*-algebra.

Definition (cf. Vershik, Feldman-Moore 77)

Let B be an abelian sub-C*-algebra of a C*-algebra A containing an approximate unit of A. One says that B is a Cartan subalgebra if

- B is maximal abelian self-adjoint (i.e. $B' = B$);
- B is regular;
- there exists a faithful positive linear map $P : A \to B$ such that $P(b) = b$ for all $b \in B$.

Main theorem

Theorem (R 08)

- *Let (G, E) be a twist with G étale, second countable locally compact Hausdorff and topologically principal. Then $C_0(G^{(0)})$ is a Cartan subalgebra of $C^*_r(G, E)$.*
- *Let B be a Cartan sub-algebra of a separable C*-algebra A. Then, there exists a twist (G, E) with G étale, second countable locally compact Hausdorff and topologically principal and an isomorphism of $C^*_r(G, E)$ onto A carrying $C_0(G^{(0)})$ onto B.*

This theorem is a C*-algebraic version of a well-known theorem of Feldman-Moore (77) about von Neumann algebras. The main difference is that the measured equivalence relation of the von Neumann case has to be replaced by a topologically principal groupoid.

It is an improvement of a theorem of Kumjian (86) who deals with the principal case (i.e. étale equivalence relation) and introduces the stronger notion of a diagonal.

Definition

One says that a sub-C*-algebra B of a C*-algebra A has the unique extension property if pure states of B extend uniquely to pure states of A. A Cartan subalgebra which has the unique extension property is called a diagonal.

Then one has

Proposition (Kumjian 86, R 08)

Let B be a Cartan sub-algebra of a separable C-algebra A. Let (G, E) be the associated twist. Then,*

$$G \text{ is principal} \quad \Leftrightarrow \quad B \text{ has the unique extension property.}$$

Existence and uniqueness of Cartan subalgebras

There are deep theorems about the existence and the uniqueness of Cartan subalgebras in the von Neumann algebras case. For example

- the hyperfinite factors have a Cartan subalgebra which is unique up to conjugacy (Krieger+Connes-Feldman-Weiss 81);
- the free group factors $L(\mathbf{F}_n)$ do not have Cartan subalgebras for $n \geq 2$ (Voiculescu 96);
- there are II_1 factors which have uncountably many non-conjugate Cartan subalgebras (Popa 90).

Much less is known about Cartan subalgebras in C*-algebras. Here are a few partial results.

- AF C*-algebras have a privileged AF diagonal which is unique up to conjugacy (Krieger 80) but have other kinds of diagonals (Blackadar 90) and Cartan subalgebras;
- the C*-algebra of a Cantor minimal system has uncountably many diagonals distinguished by the entropy (Giordano-Putnam-Skau 95).

Definition

One says that a C*-algebra A is CCR if if for all irreducible representations $\pi : A \to \mathcal{L}(H)$, we have $\pi(A) \subset \mathcal{K}(H)$.

Theorem (Fack 84, Clark 08)

Let G be an étale locally compact Hausdorff groupoid . TFAE

- $C_r^*(G)$ *is CCR;*
- $G^{(0)}/G$ *is* T_1 *and the isotropy subgroups are almost abelian.*

Definition

One says that a topological groupoid G is proper if $(r,s) : G \to G^{(0)} \times G^{(0)}$ is a proper map.

If G is a proper étale locally compact Hausdorff groupoid, $G^{(0)}/G$ is Hausdorff and the isotropy subgroups are finite. Therefore $C_r^*(G)$ is CCR.

Moerdijk and Pronk have introduced (in 1997) the notion of an orbifold groupoid:

Definition

An orbispace groupoid is a proper, effective, étale, second countable, locally compact and Hausdorff groupoid.

They arise in the following related situations:

- orbifolds;
- foliated manifolds for which all the leaves are compact with finite holonomy.

The explicit constructions involve some choices but provide equivalent groupoids. It seems appropriate to define an orbispace as an equivalence class of proper (effective) groupoids.

The C*-algebra of an orbifold groupoid is a CCR algebra admitting a Cartan subalgebra but the converse does not hold.
Here is a very simple example: the C*-algebra of the map $x \mapsto -x$ on $\mathbb{R}$:

$$A = \{f : [0,\infty] \to M_2(\mathbb{C}) \quad \text{continuous},$$

$$f_{1,1}(0) = f_{2,2}(0), \quad f_{1,2}(0) = f_{2,1}(0), \quad f_{i,j}(\infty) = 0\}$$

$$B = \{f \in A : f_{1,2}(t) = f_{2,1}(t) = 0 \quad \forall t \in [0,\infty]\}$$

The subalgebra B does not have the unique extension property since the states

$$f \mapsto f_{1,1}(0) + f_{1,2}(0) \quad \text{and}$$

$$f \mapsto f_{1,1}(0) - f_{1,2}(0)$$

both extend the pure state

$$f \mapsto f_{1,1}(0) = f_{2,2}(0)$$

of B.

In the case G has no isotropy, the situation is clearer.

Definition

One says that a C*-algebra A has continuous trace if

- the space $\hat{A}$ of equivalence classes of irreducible representations is Hausdorff;
- the set of elements $a \in A^+$ such that $\pi \mapsto \mathrm{Trace}(\pi(a))$ is continuous on $\hat{A}$ is dense in A^+.

Proposition

Let B be a Cartan subalgebra of a continuous-trace C^*-algebra A. Then B has the unique extension property.

Theorem (Green 77, Muhly-R-Williams 94)

Let (G, λ) be a locally compact principal groupoid with a Haar system . TFAE

- $C_r^*(G, \lambda)$ *has continuous trace;*
- G *is a proper groupoid.*

Remark A continuous-trace C*-algebra A carries an invariant, its Dixmier-Douady class, which is an element of $H^3(\hat{A}, \mathbb{Z})$. Let (G, E) be a twist where G is principal and proper. Then the Dixmier-Douady class of $C_r^*(G, E)$ is the image of $[E]$ in $H^3(G^{(0)}/G, \mathbb{Z})$. It is known that the Dixmier-Douady class classifies continuous-trace C*-algebras up to Morita equivalence. One deduces that each continuous-trace C*-algebra is Morita equivalent to a C*-algebra which has a diagonal. However, as shown by an example of [Natsume 85], there exist continuous-trace C*-algebras without diagonals.