On the uniform perfectness of the group of diffeomorphisms

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A group is perfect if any element is written as a product of commutators. A group is uniformly perfect if any element is written as a product of a bounded number commutators. The identity component $\mathrm{Diff}^r(M^n)_0$ of the group of C^r diffeomorphisms of an n-dimensional manifold M^n $(1 \leq r < n+1,\ n+1 < r \leq \infty)$ was shown to be perfect by Herman, Mather and Thurston.

We show that any element of the identity component $\mathrm{Diff}_c^r(\mathbf{R}^n)_0$ of the group of C^r diffeomorphisms of the n-dimensional Euclidean space $\mathbf{R}^n$ with compact support $(1 \leq r < n+1,\ n+1 < r \leq \infty)$ can be written as a product of two commutators. This statement holds for the interior M^n of a compact n-dimensional manifold which has a handle decomposition only with handles of indices not greater than $(n-1)/2$. These were also shown by Burago, Ivanov and Polterovich.

For the group $\mathrm{Diff}^r(M)$ of C^r diffeomorphisms of a compact manifold M, we show the following for its identity component $\mathrm{Diff}^r(M)_0$. For an even-dimensional compact manifold M^{2m} with handle decomposition without handles of the middle index m, any element of $\mathrm{Diff}^r(M^{2m})_0$ $(1 \leq r < 2m+1,\ 2m+1 < r \leq \infty)$ can be written as a product of four commutators. For an odd-dimensional compact manifold M^{2m+1}, any element of $\mathrm{Diff}^r(M^{2m+1})_0$ $(1 \leq r < 2m+2,\ 2m+2 < r \leq \infty)$ can be written as a product of six commutators. For a 3-dimensional compact manifold M^3, this was also obtained by Burago, Ivanov and Polterovich and we see that their idea works for any odd-dimensional compact manifold.