On transversal Weitzenböck formulas for Riemannian foliations

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Abstract

The transverse geometry of a Riemannian foliation has been intensively studied in the last period of time. In this paper we study the interplay between the foliated structure of a Riemannian foliation and the classical Weitzenböck formula. We obtain a *transversal* Weitzenböck type formula which is an extension of the previous formula for basic forms due to Ph. Tondeur, M. Min-Oo, and E. Ruh [Mi-Ru-To], and is also more general than a recent Weitzenböck formula for transverse fiber bundle due to Y. Kordyukov [Ko].

Let us consider the C^∞ Riemannian foliation $\mathcal{F}$ on a closed manifold M, endowed with a bundle-like metric g. In what follows let us consider $\{f_a\}$, $1 \leq a \leq q$, as being C^∞ local infinitesimal transformation of $(M, \mathcal{F})$ orthogonal to the leaves, while $\{e_i\}$, $1 \leq i \leq p$, will be C^∞ local vector fields tangent to the leaves. Let us consider also the dual coframe $\left\{\alpha^a, \beta^i\right\}$ for $\{f_a, e_i\}$. We denote by $U^{\mathcal{T}}$ the transverse component and by $U^{\mathcal{L}}$ the leafwise component of a local tangent vector field U. We start out by considering the Gray-O'Neill tensors fields A and T:

$$T_U V := \nabla^{\mathcal{L}}_{U^{\mathcal{L}}} V^{\mathcal{T}} + \nabla^{\mathcal{T}}_{U^{\mathcal{L}}} V^{\mathcal{L}},\ A_U V := \nabla^{\mathcal{L}}_{U^{\mathcal{T}}} V^{\mathcal{T}} + \nabla^{\mathcal{T}}_{U^{\mathcal{T}}} V^{\mathcal{L}},$$

where ∇ is the Levi-Civita connection, and U and V local tangent vector fields. In accordance with [Mi-Ru-To], using the metric connection $\widetilde{\nabla}_X Y := \pi_{\mathcal{F}} \nabla_X \pi_{\mathcal{F}} Y + \pi_Q \nabla_X \pi_Q Y$, the Levi Civita connection splits as follows:

$$\nabla_U = \widetilde{\nabla}_{U^{\mathcal{T}}} + \widetilde{\nabla}_{U^{\mathcal{L}}} + A_{U^{\mathcal{T}}} + T_{U^{\mathcal{L}}}. \tag{1}$$

In the classical way we get a bigrading for the de Rham complex (Ω, d), induced by the foliated structure and the bundle-like metric:

$$\Omega^{u,v} = C^\infty \left(\bigwedge^u T\mathcal{F}^{\perp *} \oplus \bigwedge^v T\mathcal{F}^* \right),\ u, v \in \mathbf{Z}. \tag{2}$$

Then, the de Rham derivative and coderivative split into bihomogeneous components as follows:

$$d = d_{0,1} + d_{1,0} + d_{2,-1},\ \ \delta = \delta_{0,-1} + \delta_{1,0} + \delta_{-2,1}, \tag{3}$$

where the indices describe the corresponding bigrading.

Considering a foliated chart $\mathcal{U}$ on M, we get

$$\Omega^{u,v}(\mathcal{U}) = \Omega^u(\mathcal{U}/\mathcal{F}_{\mathcal{U}}) \wedge \Omega^{0,v}(\mathcal{U}) \equiv \Omega^u(\mathcal{U}/\mathcal{F}_{\mathcal{U}}) \otimes \Omega^{0,v}(\mathcal{U}).$$

Here $\Omega^u(\mathcal{U}/\mathcal{F}_\mathcal{U})$ denotes the set of *basic* forms of transversal degree u, defined on $\mathcal{U}$. Then, if we take $\alpha \in \Omega^u(\mathcal{U}/\mathcal{F}_\mathcal{U})$ and $\beta \in \Omega^{0,v}(\mathcal{U})$, after calculations end up with the following relations for the general case of a Riemannian foliation:

$$\begin{aligned} d_{1,0}(\alpha \wedge \beta) &= \sum_a \alpha^a \wedge \widetilde{\nabla}_{f_a}\alpha \wedge \beta + \alpha \wedge (-1)^u \sum_a \alpha^a \wedge \widetilde{\nabla}_{f_a}\beta, \\ \delta_{-1,0}(\alpha \wedge \beta) &= -\sum_a i_{f_a}\widetilde{\nabla}_{f_a}\alpha \wedge \beta + i_{k^\natural}\alpha \wedge \beta - \sum_a i_{f_a}\alpha \wedge \widetilde{\nabla}_{f_a}\beta, \end{aligned} \tag{4}$$

where $g(k^\natural, X) = k(X) := g(X, \sum_i \beta^i \wedge \nabla_{e_i}\beta)$, otherwise said k is the *mean curvature form.* We remark that if $\beta = 0$, then we obtain the formulas presented in [Al].

Now, using the above stated relations and arguing as in [Sl], we get a *transversal* Weitzenböck formula for the *transversal* Laplace operator $\Delta_\perp := d_{0,1}\delta_{-1,0} + \delta_{-1,0}d_{0,1}$:

Theorem 1 *If ω is a differential form of degree r defined on M, then the following relation holds:*

$$\begin{aligned} \langle \Delta_\perp \omega, \omega \rangle &= 2\left\langle \nabla^0_{\mathcal{L},0,0}\omega, \nabla^2_{\mathcal{L},0,0}\omega \right\rangle + \left\|\nabla_{\mathcal{T},0,0}\omega\right\|^2 + \left\|\nabla_{\mathcal{L},1,-1}\omega\right\|^2 \\ &\quad + \left\|\nabla_{\mathcal{L},-1,1}\omega\right\|^2 + \left\langle K^2_{0,0}\omega, \omega \right\rangle. \end{aligned} \tag{5}$$

where the lower indices of the Levi-Civita connection components indicate the transversal-leafwise *splitting and the way the above operators change the bigrading of ω.*

The above formula is more general than the Weitzenböck formula presented in [Mi-Ru-To] which allow the authors to obtain vanishing results concerning the basic de Rham complex of a Riemannian foliation works for basic forms and also more general than *transverse* Weitzenböck type formula in [Ko, Theorem 8] which works on transverse fiber bundle. In certain situations, a useful tool for studying basic de Rham complex is the associated spectral sequence (see e.g [Al-Ko]). The spectral sequence terms do not contain only basic differential forms, so our *transversal* Weitzenböck type formula written for differential forms of arbitrary degree might help us to investigate the cohomology of a Riemannian foliation. Some particular cases are pointed out in the final part of the paper.

References

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