Variational formulae for the total mean curvatures of a codimension-one distribution

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Given quadratic matrices $B_1, \dots B_m$ of order n over $\mathbb{R}$ and a unit matrix I_n one can consider the determinant $\det(I_n + t_1 B_1 + \dots + t_m B_m)$ and express it as a polynomial of real variables $\mathbf{t} = (t_1, \dots t_m)$. Namely, $\sigma_{\boldsymbol\lambda}(B_1, \dots B_m)$ are defined by the formula [2]

$$\det(I_n + t_1 B_1 + \dots + t_m B_m) = \sum\nolimits_{|\boldsymbol\lambda| \le n} \sigma_{\boldsymbol\lambda}(B_1, \dots B_m)\, \mathbf{t}^{\boldsymbol\lambda}, \tag{1}$$

where $\boldsymbol\lambda = (\lambda_1, \dots \lambda_m)$, a sequence of nonnegative integers with $|\boldsymbol\lambda| = \lambda_1 + \dots + \lambda_m \le n$. In particular, $\sigma_j(A)$ is the coefficient at t^j of the determinant $\det(I_n + tA) = \sum_j \sigma_j(A)\, t^j$.

A **problem** we are interested in is to develop variational formulae for the functionals

$$I_m : D \mapsto \int_M \sigma_m(D)\, d\,\mathrm{vol}, \tag{2}$$

where (M^n, g) is a fixed compact oriented n-dimensional Riemannian manifold with the curvature tensor R, D ranges over the space of all *codimension-one distributions* (plane fields) on M, $m = 1, 2, \dots n{-}1$ (especially, m even), N the unit normal of D, and $\sigma_m(D)$ is the m-symmetric function of the shape operator $A_N : D \to D$. For example, the minimal value of $I_2(D)$ can be used for estimation from below of the energy $\mathcal{E}(N)$ of N, because [1],

$$\mathcal{E}(N) \ge \tfrac{1}{2n-2} \int_M \mathrm{Ric}(N, N)\, d\,\mathrm{vol} + \tfrac{n+1}{2}\, \mathrm{vol}(M). \tag{3}$$

The Ricci curvature in direction N is $\mathrm{Ric}(N, N) = \sum_i \langle R(e_i, N)N, e_i\rangle$, where $\{e_i\}$ is a local orthonormal basis of D. Denote by $R_N = R(\cdot, N)N$ (the Jacobi operator).

In order to develop variational formulae for (2), we use the integral formulae for a codimension-one distribution in our work [2]

$$\int_M g_m\, d\,\mathrm{vol} = 0, \qquad g_m = \sum\nolimits_{\|\boldsymbol\lambda\| = m} \sigma_{\boldsymbol\lambda}\left(B_1, \dots B_m\right), \tag{4}$$

where $\|\boldsymbol\lambda\| = \lambda_1 + 2\lambda_2 + \dots + m\lambda_m$ and for locally symmetric spaces

$$B_{2k} = \tfrac{(-1)^k}{(2k)!}\, R_N^k, \quad B_{2k+1} = \tfrac{(-1)^k}{(2k+1)!}\, R_N^k\, A_N \qquad (k \ge 0). \tag{5}$$

For example, $B_2 = -(1/2) R_N$, $B_3 = -(1/6) R_N A_N$, $B_4 = -(1/24) R_N^2$, and

$$\begin{gathered} g_2 = \sigma_2(A_N) + \sigma_1(B_2), \qquad g_3 = \sigma_3(A_N) + \sigma_{(1,1)}(A_N, B_2) + \sigma_1(B_3), \\ g_4 = \sigma_4(A_N) + \sigma_{(2,1)}(A_N, B_2) + \sigma_{(1,1)}(A_N, B_3) + \sigma_2(B_2) + \sigma_1(B_4), \quad \text{and so on.} \end{gathered} \tag{6}$$

Denote by div $K=\sum_i(\nabla_{e_i}K)e_i$ the *divergence of a field of linear operators* $K:TM\to TM$, where $\{e_i\}$ is a local base of TM, orthonormal at a point under consideration. Denote by $\widetilde{\rm Ric}(N)=\sum_i R(N,e_i)\,e_i$ – the vector field dual *via* g to the linear form $X\mapsto {\rm Ric}(N,X)=\sum_i\langle R(e_i,N)X,e_i\rangle$ for any $X\in D$. Denote by

$R_1(S)$ the vector valued bilinear form $(X,Y)\mapsto R(SY,N)X$,
$R_2(S)$ the vector valued bilinear form $(X,Y)\mapsto R(X,SY)N$,
$R_3(S)$ the vector valued bilinear form $(X,Y)\mapsto R(X,N)SY$,

where S is a linear operator on D. By the first Bianchi identity, we have $R_3=R_1+R_2$.

For $\boldsymbol{\lambda}=(\lambda_1,\dots\lambda_k)$ denote by $\boldsymbol{\lambda}_i=\boldsymbol{\lambda}-\boldsymbol{e}_i=(\lambda_1,\dots\lambda_i-1,\dots\lambda_k)$. The range of indices $i_1,\dots i_j$ in what follows is $1\dots m$. Set for further convenience $R_N^{-1}=0$.

The main result of the present work is the following.

Theorem 1. *Let D be a codimension one distribution (with the unit normal N) on a compact locally symmetric space M^{n+1}. The distribution D is a critical point of the functional (2) if and only if the following Euler-Lagrange equation holds:*

$$\begin{aligned}\sum_{\substack{\|\boldsymbol{\lambda}\|=m,\\ \lambda_1<m}}\sum_{j\ge1}(-1)^{j-1}\sum_{i_2,\dots i_j}\sum_{k\ge0}\Big[\tfrac{(-1)^k}{(2k+1)!}(\widetilde{B}R_N^k)^t\,\nabla\sigma_{\boldsymbol{\lambda}_{2k+1,i_2\dots i_j}}(\vec B)\\ +\tfrac{(-1)^k}{(2k+1)!}\,\sigma_{\boldsymbol{\lambda}_{2k+1,i_2\dots i_j}}(\vec B)\big[\,{\rm div}((\widetilde{B}R_N^k)^t)+k\,{\rm Tr}\,\big((R_1+R_3)A_N\widetilde{B}R_N^{k-1}\big)\big]\\ +\tfrac{(-1)^k}{(2k)!}\,k\,\sigma_{\boldsymbol{\lambda}_{2k,i_2\dots i_j}}(\vec B)\,{\rm Tr}\,\big((R_1+R_3)\widetilde{B}R_N^{k-1}\big)\Big]^\perp=0,\end{aligned}\qquad(7)$$

where $\vec B=(B_1,\dots B_m)$, see (5), $\widetilde{B}=B_{i_2}\dots B_{i_j}$, and $[\ \cdot\ ]^\perp$ denotes N-orthogonal component.

Corollary 1. *A distribution D orthogonal to a unit vector field N is a critical point for the functional I_2 if and only if $\widetilde{\rm Ric}(N)^\perp=0$, N being the unit normal for D and $(\cdot)^\perp$ the orthogonal projection onto D. It is a point of local minimum if the form $\mathcal{I}_{2,N}$ is positive definite on the space of sections of D.*

Remark 1. The method can be extended for $m\ge2$ to calculate the variations of the total mixed scalar curvature of a distribution. The total mixed scalar curvature can be applied to the problem of minimization the energy and bending of distributions.

Corollary 2. *A distribution D orthogonal to a unit vector field N is a critical point of the functional I_3 if and only if*

$$\Big[\,{\rm Tr}(A_N)\widetilde{\rm Ric}(N)+\tfrac12\nabla\,{\rm Ric}(N,N)+{\rm Tr}\,\big(R_3(A_N)\big)\Big]^\perp=0.\qquad(8)$$

References

[1] F. Brito and P. Walczak, On the energy of unit vector fields with isolated singularities. *Annales Polonici Mathematici*, LXXIII.316 (2000) 269–274.

[2] V. Rovenski and P. Walczak, Integral formulae on foliated symmetric spaces, Preprint, University of Lodz, Fac. Math. Comp. Sci., 2007/13, 27 pp. (2007).