HOMOTOPY INVARIANCE OF GEOMETRICALLY TAUTNESS OF RIEMANNIAN FLOWS WITH APPLICATION TO SASAKIAN GEOMETRY

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In this talk, we will discuss homotopy invariance of geometrically tautness of 1-dimensional Riemannian foliations (Riemannian flows) and its application to deformation theory of Sasakian metrics.

A foliated manifold $(M, \mathcal{F})$ is called geometrically taut if there exists a Riemannian metric g on M such that every leaf of $\mathcal{F}$ is a minimal submanifold of (M, g). In the case that $(M, \mathcal{F})$ is Riemannian, Alvarez Lopez showed that this property is equivalent to triviality of a basic cohomology class determined by $(M, \mathcal{F})$. We will mention that the cohomology class defined by Alvarez Lopez is algebraic if $\mathcal{F}$ is 1-dimensional and transversely parallelizable. Then, the following theorem follows from this fact immediately:

Theorem 1. *(Homotopy invariance of geometrically tautness of Riemannian flows) Let T be a connected open set of a Euclidean space $\mathbb{R}^L$ and $\{\mathcal{F}_t\}_{t \in T}$ be a smooth family of Riemannian flows on a closed manifold. Then one of the following two cases occurs:*

(1) *For every t in T, $\mathcal{F}_t$ is geometrically taut.*
(2) *For every t in T, $\mathcal{F}_t$ is not geometrically taut.*

We will discuss an application of this theorem to Sasakian geometry. The Reeb flows of Sasakian manifolds are Riemannian and have a transversely holomorphic structure naturally. We consider the following question: If we deform the Reeb flow of a Sasakian manifold as a transversely holomorphic flow, when does there exist a Sasakian metric of which the Reeb flow is isomorphic to the deformed transversely holomorphic flow? This question is concerned with a stability property of Sasakian metrics with respect to deformation of transversely holomorphic flows. In the case of Kähler metrics which have many similar geometric properties to Sasakian metrics, a theorem of Kodaira and Spencer claims Kähler metrics have such stability with respect to deformation of complex structures.

We will explain that a key is the relation of basic Euler classes of Riemannian flows to transverse complex structures and the following theorem holds:

Theorem 2. *Let T be an open neighborhood of 0 in $\mathbb{R}^L$ and $\{(\tau_t, I_t)\}_{t \in T}$ be a smooth family of transversely holomorphic flows on a closed manifold M. Assume that (τ_0, I_0) has a compatible Sasakian metric $\tilde{g}$. We write g for the transverse metric of τ_0 induced by $\tilde{g}$. Then the following are equivalent:*

(1) *There exists an open neighborhood U of 0 in T and a smooth family of Riemannian metrics $\{\tilde{g}_t\}_{t \in U}$ on M such that $\tilde{g}_t$ is a compatible Sasakian metric to (τ_t, I_t) for every t in U and $\tilde{g}_0 = \tilde{g}$.*
(2) *There exists an open neighborhood U of 0 in T such that $\{\tau_t\}_{t \in U}$ has a smooth family of transverse metrics $\{g_t\}_{t \in U}$ which satisfies $g_0 = g$ and the $(0,2)$-part of the basic Euler class of (τ_t, I_t) vanishes for every t in U.*

The total space of the circle bundle associated to a positive line bundle over a compact complex manifold M has a Sasakian metric of which the Reeb flow is the foliation $\mathcal{F}_0$ formed by circle fibers. By Theorem 2, small deformation of $\mathcal{F}_0$ as a transversely holomorphic Riemannian flow has a compatible Sasakian metric if M is Fano.