# Transverse Dirac operators on foliated manifolds and its applications

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We will discuss some problems related with transverse Dirac operators on foliated manifolds. Our main result is an analogue of the Kodaira vanishing theorem for a transverse Spin$^c$ Dirac operator on a compact manifold endowed with a transversely almost complex Riemannian foliation. This result generalizes the famous Kodaira vanishing theorem for the cohomology of the sheaf of sections of a holomorphic vector bundle twisted by a large power of a positive line bundle, which was extended to symplectic manifolds by Borthwick-Uribe and Ma-Marinescu.

Let $M$ be a compact manifold equipped with a smooth Riemannian foliation $\mathcal{F}$ of even codimension $q$. Let $g_M$ be a bundle-like metric on $M$ and $g_Q$ its restriction to the normal bundle $Q = TM/T\mathcal{F}$. Consider an almost complex structure $J$ on $Q$, compatible with $g_Q$. The almost complex structure $J$ defines canonically an orientation of $Q$ and induces a splitting $Q \otimes \mathbb{C} = Q^{(1,0)} \oplus Q^{(0,1)}$, where $Q^{(1,0)}$ and $Q^{(0,1)}$ are the eigenbundles of $J$ corresponding to the eigenvalues $i$ and $-i$ respectively. We also have the corresponding decomposition of the complexified conormal bundle $Q^* \otimes \mathbb{C} = Q^{(1,0)*} \oplus Q^{(0,1)*}$ and the decomposition of the exterior algebra bundles $\Lambda(Q^* \otimes \mathbb{C}) = \oplus_{p,q} \Lambda^{p,q}(Q^* \otimes \mathbb{C})$, where $\Lambda^{p,q}(Q^* \otimes \mathbb{C}) = \Lambda^p Q^{(1,0)*} \otimes \Lambda^q Q^{(0,1)*}$. The transverse Levi-Civita connection $\nabla$ can be written as

$$\nabla = \nabla^{(1,0)} + \nabla^{(0,1)} + A,$$

where $\nabla^{(1,0)}$ and $\nabla^{(0,1)}$ are the canonical Hermitian connections on $Q^{(1,0)}$ and $Q^{(0,1)}$ respectively and $A \in C^\infty(T^*M \otimes \mathrm{End}(Q))$, which satisfies $JA = -AJ$.

Consider a self-adjoint transverse Clifford module

$$\Lambda^{0,*} = \Lambda^{\mathrm{even}} Q^{(0,1)*} \oplus \Lambda^{\mathrm{odd}} Q^{(0,1)*}.$$

The action of any $f \in Q$ with decomposition $f = f_{1,0} + f_{0,1} \in Q^{(1,0)} \oplus Q^{(0,1)}$ on $\Lambda^{0,*}$ is defined as

$$c(f) = \sqrt{2}(\varepsilon_{f^*_{1,0}} - i_{f_{0,1}}),$$

where $\varepsilon_{f^*_{1,0}}$ denotes the exterior product by the covector $f^*_{1,0} \in Q^*_x$ dual to $f_{1,0}$, $i_{f_{0,1}}$ the interior product by $f_{0,1}$. This module has a natural leafwise flat Clifford connection $\nabla^{\Lambda^{0,*}}$. Consider also a Hermitian vector bundle $\mathcal{W}$ equipped with a leafwise flat Hermitian connection $\nabla^{\mathcal{W}}$. Then we get the twisted transverse Clifford module $\Lambda^{0,*} \otimes \mathcal{W}$ equipped with a product leafwise flat Hermitian connection $\nabla^{\Lambda^{0,*} \otimes \mathcal{W}}$.

The associated transverse Dirac operator $D_{\Lambda^{0,*} \otimes \mathcal{W}}$, which will be called the twisted transverse Spin$^c$ Dirac operator, is the first order elliptic differential operator acting on smooth sections of $\Lambda^{0,*} \otimes \mathcal{W}$ as

$$D_{\Lambda^{0,*} \otimes \mathcal{W}} = \sum_{\alpha=1}^{q} (c(f_\alpha) \otimes 1) \left( \nabla^{\Lambda^{0,*} \otimes \mathcal{W}}_{f_\alpha} - \frac{1}{2} g_M(\tau, f_\alpha) \right),$$

where $f_1, \dots, f_q$ is a local orthonormal frame for $T^H M = T\mathcal{F}^\perp \cong Q$, $\tau \in T^H M$ is the mean curvature vector of $\mathcal{F}$. The operator $D_{\Lambda^{0,*}\otimes\mathcal{W}}$ is a self-adjoint operator in the Hilbert space $L^2(M, \Lambda^{0,*} \otimes \mathcal{W})$.

Consider a Hermitian line bundle $\mathcal{L}$ equipped with a leafwise flat Hermitian connection $\nabla^{\mathcal{L}}$. The curvature of $\nabla^{\mathcal{L}}$ is an imaginary valued 2-form $R^{\mathcal{L}} = (\nabla^{\mathcal{L}})^2$ on $M$. Since $\nabla^{\mathcal{L}}$ is leafwise flat, $R^{\mathcal{L}}$ vanishes on $T\mathcal{F}$, and, therefore, defines a 2-form $R^{\mathcal{L}}$ on $Q$. Assume that the 2-form $R^{\mathcal{L}}$ is non-degenerate and $J$-invariant on $Q$. Then it is a symplectic form on $Q$.

Put

$$m = \inf_{u \in Q_x^{(1,0)}, x \in M} \frac{R_x^{\mathcal{L}}(u, \bar{u})}{|u|^2} > 0.$$

Let $D_k$ denote the transverse Spin$^c$ Dirac operator $D_{\Lambda^{0,*}\otimes\mathcal{W}\otimes\mathcal{L}^k}$, and $D_k^-$ the restriction of $D_k$ to the space $C^\infty(M, \Lambda^{odd} Q^{(0,1)*} \otimes \mathcal{W} \otimes \mathcal{L}^k)$.

**Theorem 1.** *Under current assumptions, there exists $C > 0$ such that for $k \in \mathbb{N}$ we have*

$$\operatorname{spec} D_k^2 \subset \{0\} \cup (2km - C, +\infty).$$

*For sufficiently large $k$, we get* $\operatorname{Ker} D_k^- = 0$.

Ma and Marinescu gave a proof of the Kodaira vanishing theorem for symplectic manifolds, which uses only the Lichnerowicz formula for the Spin$^c$ Dirac operator. We follow their approach. So we also prove a Lichnerowicz type formula for a transverse Dirac operator on a compact foliated manifold $(M, \mathcal{F})$, which has its own interest.

We will also discuss some applications of our results in transverse index theory and geometric quantization of foliations.