Analogies between coverings of certain foliations and algebraic field extensions

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We want to extend the dictionary between algebraic number theory and certain foliations discovered by Christopher Deninger [1]. For this sake consider a tuple $(M, \mathcal{F}, \phi, K_1, \ldots, K_n)$, where

- M is an oriented, connected, 3-dimensional, compact, smooth manifold,
- $\mathcal{F}$ is a leafwise oriented foliation on M by Riemann surfaces,
- $K_1, \ldots, K_n$ are compact leaves and
- ϕ is a foliation-invariant flow (i.e. an $\mathbb{R}$-action by foliated maps) such that $\phi^t(K_i) \subset K_i$ for all t and all i. Furthermore we assume that $\phi^t|_{M \setminus (K_1 \cup \cdots \cup K_n)}$ is transverse to the leaves.

Let $(M', \mathcal{F}', \phi', K'_1, \ldots, K'_{n'})$ be a second tuple of the above type and $p : M' \to M$ be a foliated map with $\phi^t \circ p = p \circ \phi'^t$ for all t. Assume that for each choice of local coordinates we can write locally

$$p : \mathbb{D} \times \mathbb{R} \supset U \to V \subset \mathbb{D} \times \mathbb{R}, \quad (z, x) \mapsto (p_1(z, x), p_2(x))$$

with $p_1(., x)$ a non-constant holomorphic map and p_2 a local diffeomorphism. If the ramification points of p lie in a finite union of closed orbits, we call p a covering. In this case for a closed orbit $\Gamma \subset M'$ the ramification index $e_{\Gamma|p(\Gamma)}$ along Γ is constant.

Let now $\mathfrak{o}$ be the ring of integers in an algebraic number field. We get the following dictionary:

algebraic object	foliated analogue
Spec $\mathfrak{o}$	$(M, \mathcal{F}, \phi, K_1, \ldots, K_n)$ as above
finite prime $\mathfrak{p}$	closed orbit γ
$\mathcal{N}(\mathfrak{p})$	$e^{l(\gamma)}$
infinte prime coming from a real/complex embedding	compact leaf in/not in the boundary
$a \in \mathfrak{o}$	$f : M \to \hat{\mathbb{C}}$ meromorphic with poles and zeros only on the closed orbits

$i : \mathfrak{o} \to \mathfrak{o}'$	$p : M' \to M$ covering
$i(a) \in \mathfrak{o}'$	$f \circ p : M' \to \hat{\mathbb{C}}$
$e_{\mathfrak{P}\vert\mathfrak{p}}$	ramification index $e_{\Gamma\vert\gamma}$
$f_{\mathfrak{P}\vert\mathfrak{p}}$	$f_{\Gamma\vert\gamma} = l(\Gamma)/l(\gamma)$
$\nu_{\mathfrak{p}}(a)$	$\mathrm{ord}_{\gamma}(f)$
$N_{L\vert K}(a)$	$N_{M'\vert M}(f)(z) = \prod_{p(z')=z} f(z')^{e_{z'}}$

We prove "foliated translations" of various well-known algebraic formulas like the fundamental identity $\sum e_{\mathfrak{p}} f_{\mathfrak{p}} = [L : K]$, the Riemann Hurwitz formula or a Grothendieck-Riemann-Roch of relative dimension zero. Furthermore we extend our dictionary to some basic notions of Arakelov theory. Note that in the unramified case some of these results are special cases of [2].

References

[1] Deninger, Christopher: Number theory and dynamical systems on foliated spaces. Jber. d. Dt. Math.-Verein. 103 (2001), 79-100.

[2] Sunada, Toshikazu Geodesic flows and geodesic random walks. Geometry of geodesics and related topics (Tokyo, 1982), 47–85, Adv. Stud. Pure Math., 3, North-Holland, Amsterdam, 1984.