Affability of tiling dynamical systems

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An *Euclidean tiling* is a partition of $\mathbb{R}^m$ into *tiles*, which are polyhedra touching face to face. These tiles are obtained by translation from a finite set of *prototiles*. A tiling is said to be *aperiodic* if it has no translation symmetries. It is said to be *repetitive* if for any patch M, there exists $R > 0$ such that any ball of radius R contains a translated copy of M. Let $\mathfrak{T}(\mathcal{P})$ be the set of tilings $\mathcal{T}$ obtained from a finite set of prototiles $\mathcal{P}$. Then it is possible to endow $\mathfrak{T}(\mathcal{P})$ with the *Gromov-Hausdorff topology* [1, 2] generated by the basic neighbourhoods $U^r_{\varepsilon,\varepsilon'} = \{\mathcal{T}' \in \mathfrak{T}(\mathcal{P}) \;/\; \exists\; v, v' \in \mathbb{R}^m : \| v \| < \varepsilon, \| v' \| < \varepsilon', R(\mathcal{T} + v, \mathcal{T}' + v') > r\}$ where $R(\mathcal{T}, \mathcal{T}')$ is the supremum of radii $R > 0$ such that $\mathcal{T}$ and $\mathcal{T}'$ coincide on the ball $B(0, R)$. Thus $\mathfrak{T}(\mathcal{P})$ becomes a compact metrizable space which is foliated by the orbits $L_{\mathcal{T}}$ of the natural action of $\mathbb{R}^m$. For each $\mathcal{T} \in \mathfrak{T}(\mathcal{P})$, let $D_{\mathcal{T}}$ be the Delone set associated to a choice of base points in the prototiles. Now $\Sigma = \{\mathcal{T} \in \mathfrak{T}(\mathcal{P})/0 \in D_{\mathcal{T}}\}$ is a totally disconnected closed subspace which meets all the leaves. For any aperiodic and repetitive tiling $\mathcal{T} \in \mathfrak{T}(\mathcal{P})$, the *continuous hull* of $\mathcal{T}$ is the closure of its orbit. In the Euclidean case, any tiling in $\mathfrak{X} = \overline{L}_{\mathcal{T}}$ is also aperiodic and repetitive and therefore $X = \Sigma \cap \mathfrak{X}$ is homeomorphic to the Cantor set. In particular, $\mathfrak{X}$ is a minimal saturated set having trivial holonomy.

The aim of this poster is to show that the transverse dynamics of $\mathfrak{X}$ is represented (up to orbital equivalence) by the tail equivalence relation on the infinite path space of a Bratteli diagram. Such an equivalence relation is orbit equivalent to a minimal action of $\mathbb{Z}$ on the Cantor set [3, 4]. In other words, we prove:

Theorem.- *Any equivalence relation $\mathcal{R}$ on a Cantor set X arising from the continous hull $\mathfrak{X}$ of an aperiodic and repetitive Euclidean tiling is affable in the sense of [4].*

This theorem generalizes a result of H. Matui [6] proved for a subclass of substitution tilings. The proof is made up of three stages:

i) First, we construct an affable equivalence relation $\mathcal{R}_\infty \subset \mathcal{R}$ and we introduce its boundary $\partial\mathcal{R}_\infty$. It is a nowhere dense closed subset in which the $\mathcal{R}$-classes split into at least two $\mathcal{R}_\infty$-classes. We use the *inflation* or *zooming process* developed in [1] to define a sequence of decompositions $\mathcal{B}^{(n)}$ of $\mathfrak{X}$ in a finite number of compact flow boxes. Now $\mathcal{R}_\infty$ is the inductive limit of an increasing sequence of compact discrete equivalence relations $\mathcal{R}_n$.

ii) Secondly we prove that $\partial\mathcal{R}_\infty$ is $\mathcal{R}$-*thin* [4], i.e. $\mu(\partial\mathcal{R}_\infty) = 0$ for every $\mathcal{R}$-invariant probability measure μ. In [7], C. Series has proved that any foliation with polynomial growth is hyperfinite . Here, we will use the same outline (which reminds the proof of the Rohlin lemma).

iii) Finally, in the Euclidean case, we pove that $\mathcal{R}_\infty$ is minimal and every $\mathcal{R}$-class split into a finite number of $\mathcal{R}_\infty$-classes. This will allow us to conclude by applying theorem 4.18 of [4].

In fact, this proof applies to the broader class of *tilable laminations* [1] and we deduce the following result (which extends the main theorem of [5]):

Corollary.- *Any free minimal action of $\mathbb{Z}^m$ on the Cantor set is affable.*

References

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