Homogeneous geodesics in homogeneous manifolds with an affine connection

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Homogeneous geodesics were studied first on homogeneous Riemannian manifolds. See for example [3], [4], [13], [16], [17], [19], [21]. In this situation, a geodesic $\gamma(t)$ through the origin $p \in M$ is *homogeneous* if it is an orbit of a one-parameter group of isometries. Hence, $\gamma(t) = \exp(tX)(p)$, were $p \in M$ is the origin and $X \in \mathfrak{g}$ is a vector from the Lie algebra of the isometry group G. The vector X is called a *geodesic vector.* There is an algebraic formula which characterizes geodesic vectors. This formula uses the fact that any Riemannian homogeneous manifold is reductive and also the scalar product on the tangent space of p.

Every Riemannian homogeneous manifold admits at least one homogeneous geodesics (see [16]) and there are examples which admit just one (see [19]). If all geodesics on M are homogeneous, M is called a *g.o. manifold.* Riemannian g.o. manifolds were studied for example in [1], [6], [9], [12], [17]. The study is based on geodesic graphs. Roughly spoken, a *geodesic graph* is an equivariant map which assigns to every geodesic a geodesic vector which generates it. For a g.o. space, a geodesic graph must exist on all $T_p(M)$.

On pseudo-Riemannian homogeneous manifolds, the study of homogeneous geodesics started with the papers [10] and [23] written by phisicists and continued in the papers [5], [7], [8]. The algebraic formula characterizing geodesic vectors must be generalized and it is applicable only for reductive manifolds. (There exist also non-reductive homogeneous pseudo-Riemannian manifolds.) The generalization concerns the fact, that the affine parameter of a null homogeneous geodesic $\gamma(s)$ may be different from the parameter of the corresponding 1-parameter group $\exp(tX)$ of isometries. (In the Riemannian situation, the parameter must be the same.)

For pseudo-Riemannian homogeneous manifolds with noncompact isotropy group, we observe one more interesting fenomenon - an *almost g.o. space*. It is the manifold, where geodesic graph can be defined on an open dense subset of $T_p(M)$, but not on all $T_p(M)$ (see [5]).

Recently, in [20], O. Kowalski and Z. Vlášek studied homogeneous geodesics on affine homogeneous manifolds in dimension 2 using the classification of homogeneous affine connections obtained in [2]. They propose a new simple method of investigation which cannot be used in full generality, but still works well in dimension two and maybe also in special situations in higher dimensions. As a main result, they constructed a family of two-dimensional homogeneous affine manifolds admitting no homogeneous geodesics and they also solved the problem of a canonical re-parametrization of homogeneous affine geodesics in dimension two.

The study of homogeneous geodesics on examples of affine homogeneous manifolds in dimension 3 is in progress now. In the talk, we are going to review the older results on homogeneous geodesics on Riemannian and pseudo-Riemannian manifolds, show the new method of investigation of homogeneous geodesics in affine homogeneous manifolds and present the new results in dimension 3.

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