D'Atri spaces of Iwasawa type

A complete Riemannian manifold M is said to be a D'Atri space if geodesic symmetries are volume-preserving, up to sign. Equivalently,

$$\det A_v(t) = \det A_{-v}(t) \text{ for any } v \in SM \text{ and } t > 0\ (t \sim 0),$$

where $A_v(t)$ is the Lagrange tensor associated to the geodesic $\gamma_v(t)$ determined by the condition $A_v(0) = 0$, $A_v^{'}(0) = \mathrm{Id}$, that is defined by the equation $A_v''(t) + R_{\gamma_v(t)} \circ A_v(t) = 0$ with $A_v'(t)^{-1} \circ A_v(t)$ a symmetric operator.

We study homogeneous spaces M of Iwasawa type and in particular, those of algebraic rank one. That is, M is represented as a solvable Lie group S, with left invariant metric associated to a metric Lie algebra $\mathfrak{s}$. This Lie algebra is expressed orthogonaly as $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ where $\mathfrak{a} \perp \mathfrak{n} = [\mathfrak{s},\mathfrak{s}]$ is an abelian subalgebra of $\mathfrak{s}$ satisfying that

(i) $\mathrm{ad}_H|_{\mathfrak{n}}$ are symmetric for all $H \in \mathfrak{a}$

(ii) for some $H \in \mathfrak{a}$, $\mathrm{ad}_H|_{\mathfrak{n}}$ has all positive eigenvalues.

In the case of algebraic rank one, we have $\mathfrak{s} = \mathfrak{n} \oplus \mathbf{R}H$, $|H| = 1$, $H \perp \mathfrak{n}$.
We give a characterization of D'Atri spaces of Iwasawa type in terms of the Lie subalgebra $\mathfrak{a}$. In the particular case of rank one we show,

A D'Atri space of Iwasawa type of algebraic rank one is a Damek-Ricci space.

The property of M be a D'Atri space implies that the function on $v \in SM$, $\det A_v(t)$ is invariant under the geodesic flow. This fact and the distinguished element H in the Lie álgebra $\mathfrak{a}$, allows us to show that for any $v \in SM$

$$\det A_v(t) = \det A_H(t) \quad \text{for every } t > 0$$

for some $H \in \mathfrak{a}$. Thus, a D'Atri space of Iwasawa type and algebraic rank one is a harmonic space $(*)$. As a consequence, by applying a result of Heber J.; On harmonic and asymptotically harmonic homogeneous spaces, GAFA 16, 2006 (869-890), it follows that M is a Damek-Ricci space.
The result above $(*)$ was also proved by Heber J. in the case of homogeneous spaces of nonpositive curvature using techniques which are proper of these spaces (Hadamard solvmanifolds).

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