A characterization of geodesically invariant distributions

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In this talk we provide a characterization of distributions which are geodesic invariant with respect to an affine connection using the geometry of the linear frame bundle. We also provide a definition of the geodesic spray of the connection in this framework. Since the page limit for this abstract is two pages, proofs of statements are not provided but are available if needed.

1. Connections on the linear frame bundle

Let M be an n-dimensional smooth manifold and $\pi_M : L(M) \to M$ the bundle of linear frames with total space $L(M)$, base space M and structure group $GL(n;\mathbb{R})$. Recall that a linear frame u at x is an isomorphism $u : \mathbb{R}^n \to T_xM$. For *fixed* $\xi \in \mathbb{R}^n$, this defines the *association map* $\Phi_\xi : L(M) \to TM$ by $\Phi_\xi(u) = u\xi$. The canonical form θ of $L(M)$ is a one-form on $L(M)$ defined by $\theta(X_u) = u^{-1}(\pi_M(X_u))$, $X_u \in T_uL(M)$.

A principal connection ω on the bundle of linear frames over M is called a *linear connection*. There is a one-to-one correspondence between linear connections ω and affine connections ∇ on M (see [1]).

Given a linear connection ω, for each $\xi \in \mathbb{R}^n$, the *standard horizontal vector field corresponding to* ξ, denoted by $B(\xi) : L(M) \to TL(M)$, is defined as follows. For each $u \in L(M)$, the vector $B(\xi)_u$ is the unique horizontal vector at u with the property that $T_u\pi_M(B(\xi)_u) = u\xi$.

The following result provides an explicit relationship between ∇ and the connection one-form ω of the corresponding linear connection.

1.1 Proposition: *Let M be a manifold with a connection ∇ with the corresponding linear connection one-form ω. Given vector fields X and Y on M, let $\widetilde{X}$ and $\widetilde{Y}$ be the natural lifts onto $L(M)$ of X and Y respectively. Then,*

$$\nabla_X Y(x) = [X,Y](x) + u\left(\omega \otimes \theta\left(\widetilde{X}(u), \widetilde{Y}(u)\right)\right), \quad \pi_M(u) = x \tag{1.1}$$

2. The geodesic spray of an affine connection

Given a linear connection ω on M, for *fixed* $\xi \in \mathbb{R}^n$, let $\Phi_\xi : L(M) \to TM$ be the association map. Denote the tangent bundle projection by $\tau_M : TM \to M$. We define a (second-order) vector field $Z : TM \to TTM$ called the *geodesic spray* as follows.

$$Z(v) = T_u\Phi_\xi(B(\xi)_u), \qquad v \in TM, \tag{2.1}$$

where, $u \in L_{\tau_M(v)}(M)$ and $\xi \in \mathbb{R}^n$ are such that $u\xi = v$, and $B(\xi)$ is the standard horizontal vector field corresponding to ξ for the linear connection ω associated with ∇. We have the following result.

2.1 Proposition: *The map Z defined in (2.1) is a second-order vector field on TM. The coordinate expression for Z, in terms of the canonical tangent bundle coordinates (x^i, v^i) is given by*

$$Z = v^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk} v^j v^k \frac{\partial}{\partial v^i} \tag{2.2}$$

The following result provides a decomposition of the geodesic spray in terms of the tangent and the vertical lifts of vector fields (on to TM) respectively.

2.2 Proposition: *Let $v \in T_xM$ for some $x \in M$, and X_v be an arbitrary vector field that has the value v at x. Then,*

$$Z(v) = (X_v)^T(v) - \mathrm{vlft}_v(\nabla_{X_v} X_v(x)). \tag{2.3}$$

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3. Geodesic invariance

Let M be a smooth manifold with an affine connection ∇ and D a smooth p-dimensional distribution on M. The distribution D is called *geodesically invariant* (with respect to ∇) if, for any geodesic $c : [a, b] \to M$, $\dot{c}(a) \in D_{c(a)}$ implies that $\dot{c}(t) \in D_{c(t)}, \forall t \in [a, b]$.

Given vectors fields X and Y on M, the symmetric product $\langle X : Y \rangle$ is defined as follows.

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X.$$

The vector fields taking values in D will be represented by $\Gamma(D)$.

The following corollary to Proposition 1.1 gives a description of the symmetric product using linear frame bundle geometry. In the following, we use $\mathrm{Sym}(\omega \otimes \theta)$ to denote the symmetrization of $\omega \otimes \theta$.

3.1 COROLLARY: *Let M be a manifold with a connection ∇, let ω be the associated linear connection and θ the canonical form on $L(M)$ respectively. If X and Y are vector fields on M and $\widetilde{X}$ and $\widetilde{Y}$ the respective natural lifts on $L(M)$, then*

$$\langle X : Y \rangle (x) = 2u \left(\mathrm{Sym}(\omega \otimes \theta)(\widetilde{X}, \widetilde{Y}) \right), \quad \pi_M(u) = x. \tag{3.1}$$

3.2 REMARK: The object $\mathrm{Sym}(\omega \otimes \theta)$ defines a quadratic form $\Sigma_u : T_u L(M) \times T_u L(M) \to T_u L(M)$, $u \in L(M)$ as follows.

$$\Sigma_u(X, Y) = \widetilde{Z_{X,Y}}(u), \quad X, Y \in T_u L(M)$$

where $\widetilde{Z_{X,Y}}$ is the natural lift onto $L(M)$ of the vector field $Z_{X,Y}$ on M given by

$$Z_{X,Y}(x) = u(\mathrm{Sym}(\omega \otimes \theta)(X, Y)), \quad \pi(u) = x.$$

This is seen to be well-defined.

We are now ready to present a frame bundle version of the characterization of geodesically invariant distributions by Lewis [2].

3.3 THEOREM: *Let D be a p-dimensional distribution on a manifold M with a linear connection ω. Let $\widetilde{D}$ be the natural lift of D onto $L(M)$ and $L(M, D)$ be the bundle adapted to D with base space M and structure group H. The following statements are equivalent.*

(i) *D is geodesically invariant;*

(ii) $\mathrm{Sym}(\omega \otimes \theta)$ *is an $\mathbb{R}^p$-valued quadratic form on $\widetilde{D}|_{L(M,D)}$;*

(iii) $\omega(\widetilde{X}(u)) \in \mathfrak{h}$ for all $\widetilde{X} \in \Gamma(\widetilde{D})$, $u \in L(M, D)$.

We also have the following characterization in terms of the geodesic spray.

3.4 PROPOSITION: *The distribution D is geodesically invariant if and only if, for each $\xi \in \mathbb{R}^p$, the restriction $B(\xi \oplus \mathbf{0}_{n-p})|_{L(M,D)}$ is a vector field on $L(M, D)$.*

References

[1] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry*, volume 1 of *Tracts in pure and applied mathematics*. Interscience Publishers, New York, London, 1964.

[2] Andrew D. Lewis. Affine connections and distributions with applications to nonholonomic mechanics. *Rep. Math. Phys.*, 42(1-2):135–164, 1998.