THE GEOMETRY OF RECURSION OPERATORS

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It is now well known that many geometric structures, particularly on four-manifolds, can be defined in terms of pairs of two-forms; see for example Donaldson [3]. In this work we study the fields of endomorphisms intertwining such pairs of forms. This leads to a natural generalization of Moser's theorem [8] on isotopies of symplectic forms and to a generalization of known geometric structures on four-manifolds to arbitrary dimensions.

Suppose we are given two non-degenerate 2-forms ω and η on the same manifold M. Then there exists a unique field of invertible endomorphisms A of the tangent bundle TM defined by the equation

$$i_X\omega \;=\; i_{AX}\eta\;. \tag{1}$$

The important special case when the two 2-forms involved are closed, and therefore symplectic, is very interesting both from the point of view of physics, where it arises in the context of bi-Hamiltonian systems, and from a purely mathematical viewpoint. In physics the field of endomorphisms A is called a recursion operator, and we shall adopt this terminology here. We shall study the global geometry and topology of a manifold endowed with two (or more) symplectic forms, which we discuss using the associated recursion operator A. For local considerations in the case when the Nijenhuis tensor of A vanishes see Turiel [9].

We shall show that the recursion operator neatly encapsulates the necessary and sufficient condition for the existence of a simultaneous isotopy of two families of symplectic forms. We consider the simplest examples, where the recursion operator A satisfies $A^2 = \pm 1$. We shall find that these most basic cases correspond precisely to the symplectic pairs studied in [1], and to holomorphic symplectic forms respectively. Our discussion of holomorphic symplectic structures in terms of recursion operators generalizes the work of Geiges [4] on conformal symplectic couples from dimension four to arbitrary dimensions. Then we introduce the four geometries defined by triples of symplectic forms whose pairwise recursion operators all satisfy $A^2 = \pm 1$. Throughout our point of view is that of symplectic geometry, taking as our geometric data only the symplectic forms and the recursion operators they define. Nevertheless, we shall see that in two of the four cases the data encoded by the triple of symplectic forms define a pseudo-Riemannian metric leading to the kind of geometry that is used in supersymmetric string theory; see for example [2, 7]. One of these cases is that of hypersymplectic structures in the sense of Hitchin [6], the other one is a symplectic analogue of hyper-Kähler structures. We will show that this symplectic formulation of hyper-Kähler geometry is not equivalent to the usual one, because the symplectic data does not force the associated pseudo-Riemannian metric to be definite. Hyper-Kähler geometry corresponds precisely to the special case in which the natural metric is definite. We shall also discuss briefly the geometries defined by triples of symplectic forms with recursion operators of square ± 1 which do not have natural pseudo-Riemannian metrics attached to them. These structures have more to do with foliations than with differential geometry.

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