BIHAMONIC MAPS AND SUBMANIFOLDS

A. BALMUŞ

This talk is intended to present the main results of my PhD Thesis concerning the theory of biharmonic maps.

A natural generalization for harmonic maps and minimal immersions is obtained by considering the variational problem associated to the integral of the squared norm of the tension field. More precisely, in their first paper on harmonic maps (see [10]), Eells and Sampson suggested a generalization for harmonic maps by defining the *biharmonic maps* as the critical points of the *bienergy* functional

$$E_2 : C^\infty(M,N) \to \mathbb{R}, \quad E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 \, v_g,$$

where $\tau(\phi) = \operatorname{trace} \nabla d\phi$ is the tension field of ϕ which vanishes for harmonic maps. The Euler-Lagrange equation corresponding to E_2 (see [12]) is given by the vanishing of the *bitension field*

$$\tau_2(\phi) = -J^\phi(\tau(\phi)) = -\Delta\tau(\phi) - \operatorname{trace}\ R^N\big(d\phi, \tau(\phi)\big)d\phi,$$

where J^ϕ is formally the Jacobi operator of ϕ. Since J^ϕ is linear, any harmonic map is biharmonic and we call *proper biharmonic* the non-harmonic biharmonic maps.

Equation $\tau_2(\phi) = 0$ is called the *biharmonic equation* and, in local coordinates, yields a 4-th order non-linear system of PDE's. The goal of the thesis was that of investigating this equation in three different geometrical contexts.

A series of non-existence results (see [6, 12]) encouraged the search of examples of proper biharmonic maps. We propose here several new methods, inspired by the Baird-Kamissoko method [1], for constructing proper biharmonic maps starting with harmonic maps and using warped product manifolds (see [3]). The main results are:

i. the condition for the biharmonicity of the inclusion of a Riemannian manifold N into the warped product $M \times_{f^2} N$ and of the projection $\overline{\pi} : M \times_{f^2} N \to M$;
ii. the construction of two new classes of non-harmonic biharmonic maps using products of harmonic maps $\phi = \mathbf{1}_M \times \psi : M \times N \to M \times N$ and warping the metric on their domain or codomain;
iii. the study of axially symmetric biharmonic maps, using the warped product setting.

The biharmonic submanifolds of a non-positive sectional curvature space that have been considered so far (see, for example, [6, 8, 9, 11]) turned out to be trivial (that is minimal), and the attempts that have been made have led to the following

Generalized Chen Conjecture: *biharmonic submanifolds of a non-positive sectional curvature manifold are minimal.*

On the contrary, the class of proper biharmonic submanifolds of the sphere is rather rich (see [6, 7, 13]), but a full understanding of their geometry has not yet been achieved. Our contribution in this direction (see [2, 4]) consists in:

i. the study of the type, in the sense of B-Y. Chen, of compact proper biharmonic submanifolds with constant mean curvature in spheres;
ii. the complete classification of proper biharmonic hypersurfaces with at most two distinct principal curvatures in space forms;
iii. the classification of compact proper biharmonic hypersurfaces of $\mathbb{S}^4$;

iv. some rigidity properties for pseudo-umbilical biharmonic submanifolds of codimension 2 and for biharmonic surfaces with parallel mean curvature vector field in $\mathbb{S}^n$;
v. the characterization and classification of proper biharmonic products of spheres in spheres.

Ever since the characterization result obtained by E. Ruh and J. Vilms in [14] and the remarkable link with constant mean curvature hypersurfaces the study of submanifolds with associated harmonic Gauss map in Euclidean spaces has been a classical problem in harmonic maps theory. We propose a generalization of this problem in [5] and obtain:

i. the characterization of submanifolds of the Euclidean space with biharmonic Gauss map;
ii. examples of hypersurfaces of the Euclidean space with biharmonic Gauss map.

References

[1] P. Baird, D. Kamissoko. On constructing biharmonic maps and metrics. *Ann. Global Anal. Geom.* 23 (2003), 65–75.

[2] A. Balmuş, S. Montaldo, C. Oniciuc. Biharmonic maps between warped product manifolds. *J. Geom. Phys.* 57 (2007), 449–466.

[3] A. Balmuş, S. Montaldo, C. Oniciuc. Classification results for biharmonic submanifolds in spheres. *Israel J. Math.*, to appear.

[4] A. Balmuş, S. Montaldo, C. Oniciuc. Biharmonic hypersurfaces in 4-dimensional space forms. preprint.

[5] A. Balmuş, S. Montaldo, C. Oniciuc. Submanifolds with biharmonic Gauss map. preprint

[6] R. Caddeo, S. Montaldo, C. Oniciuc. Biharmonic submanifolds of $\mathbb{S}^3$. *Internat. J. Math.* 12 (2001), 867–876.

[7] R. Caddeo, S. Montaldo, C. Oniciuc. Biharmonic submanifolds in spheres. *Israel J. Math.* 130 (2002), 109–123.

[8] B-Y. Chen. Some open problems and conjectures on submanifolds of finite type. *Soochow J. Math.* 17 (1991), 169–188.

[9] I. Dimitric. Submanifolds of $\mathbb{E}^m$ with harmonic mean curvature vector. *Bull. Inst. Math. Acad. Sinica* 20 (1992), 53–65.

[10] J. Eells, J.H. Sampson. Harmonic mappings of Riemannian manifolds. *Amer. J. Math.* 86 (1964), 109–160.

[11] Th. Hasanis, Th. Vlachos. Hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field. *Math. Nachr.* 172 (1995), 145–169.

[12] G.Y. Jiang. 2-harmonic isometric immersions between Riemannian manifolds. *Chinese Ann. Math.* Ser. A 7 (1986), 130-144.

[13] S. Montaldo, C. Oniciuc. A short survey on biharmonic maps between Riemannian manifolds. *Rev. Un. Mat. Argentina* 47 (2006), 2, 1–22.

[14] E. Ruh, J. Vilms. The tension field of the Gauss map. *Trans. Amer. Math. Soc.* 149 (1970), 569–573.

[15] The Bibliography of Biharmonic Maps, `http://beltrami.sc.unica.it/ biharmonic/`.

Faculty of Mathematics, "Al.I. Cuza" University of Iasi,
Bd. Carol I Nr. 11, 700506 Iasi, ROMANIA
E-mail address: balmus1@yahoo.com