Comparison theorems for volumes of geodesic celestial spheres in Lorentzian geometry [1]

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Abstract

We discuss some volume comparison results for geodesic celestial spheres, and the corresponding objects in Lorentzian space forms. Also, some rigidity results are shown which allow one to detect locally isotropic Lorentzian manifolds by some instrinsic properties of their celestial geodesic spheres.

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1 Introduction

In order to study the geometry of a semi–Riemannian manifold it is usual to consider geometric objects associated to the metric structure of such manifold. The geodesic spheres of a Riemannian manifold are a good example of this in the definite setting. We could define a geodesic sphere in a Riemannian manifold as the set of points which are at a fixed Riemannian distance from another point called the center. One important problem in Riemannian geometry consists of answering the following question: 'to what extend do the properties of a geodesic sphere influence or even determine the geometry of the ambient manifold?'. This problem is difficult to solve in such a generality although some results where obtained. See for example [2] and [3].

When we turn our attention to Lorentzian manifolds several difficulties arise. The Riemannian distance function is continuous and it induces the topology of the Riemannian manifold as a topological manifold. Also, this function has several nice properties. For example, its level sets are compact submanifolds of the ambient manifold. In the Lorentzian setting there is no such distance. Certainly, in a spacetime a distance–like function can be defined, but its properties are completely different from those of its Riemannian counterpart. For example, this 'Lorentzian distance' may fail to be continuous or bounded and geometric objects defined from it usually have a weird behavior. Furthermore, the level sets of this distance function are no longer compact and they do not seem to be adequate for the investigation of volume properties.

As a consequence, one important issue in Lorentzian geometry is to find an interesting family of submanifolds whose properties reflect those of the ambient manifold and which are defined from the Lorentzian structure itself. Previous attempts have been done with the following families: truncated light cones [11], slabs bounded by spacelike hypersurfaces [1], compact geodesic wedges in the chronological future of some point [6] and some neighborhoods covered by timelike geodesics emanating from a given point [7].

In [4] a different family was introduced, the so called geodesic celestial spheres. Intuitively speaking, a geodesic celestial sphere is the set of the points reached after a fixed time, called the radius, travelling along radial geodesics emanating from a fixed point, called the center, which are orthogonal to a given timelike direction. Such a timelike direction represents in General Relativity the infinitesimal Newtonian universe where the observer perceives particles as Newtonian particles relative to its rest position. Thus, a geodesic celestial sphere is exactly the image by the exponential map of the celestial sphere in the infinitesimal restspace. Indeed, we will see how the geometry of Lorentzian manifolds of constant sectional curvature can be characterized by means of the volume of geodesic celestial spheres and what we will call the total scalar curvatures of geodesic celestial spheres.

The paper is organized as follows. In Section 2 we introduce the basic concepts and notation. We define a geodesic celestial sphere in detail and establish the concept of simple Weyl invariant together with the total scalar curvatures of a geodesic celestial sphere associated to a simple Weyl invariant. Also, we give some formulae which we will use in Section 3 to achieve the desired characterization. In fact, in Section 3 we remind some previous results obtained in [4] and then we state Theorem 3.4 which is the core of this paper. We also give some consequences of this theorem which essentially consist of the characterization of constant curvature Lorentzian manifold by means of integrals of low order Weyl invariants along geodesic celestial spheres.

2 Geodesic Celestial Spheres and simple Weyl invariants

Let $\left(M^{n+1}, g\right)$ be a Lorentzian manifold of dimension $n+1$ and signature $(-+\cdots+)$. We will always supose $n>2$. We denote by ∇ the Levi-Civita connection of M. The curvature tensor R is defined by using the convention $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ and $R_{XYVW} = g(R_{XY}V, W)$, where X, Y, V and W are vector fields on M. We will also denote by $\rho(x,y) = trace\{z \mapsto R(x,z)y\}$ and $\tau = trace\,\rho$ the Ricci tensor and the scalar curvature, respectively. With respect to an orthonormal basis $\{e_i\}$ they are written as

$$\rho(X,Y) = \sum_{i=0}^{n} \epsilon_i R_{Xe_iYe_i} \quad \text{and} \quad \tau = \sum_{i=0}^{n} \epsilon_i \rho_{e_ie_i},$$

where $\epsilon_i = g(e_i, e_i)$.

A unit timelike vector $\xi \in T_mM$ is called an *instantaneous observer*, and $\xi^\perp$ is called the *restspace* of ξ. The *celestial sphere of radius* r of ξ is defined by $\mathcal{S}^\xi(r) = \{x \in \xi^\perp; g(x,x) = 1\}$ (c.f. [10]). If $\mathfrak{U}$ is a sufficiently small neighborhood of the origin in T_mM, $\widetilde{M} = \exp_m(\mathfrak{U} \cap \xi^\perp)$ is an embedded Riemannian submanifold of M, where $\exp_m : T_mM \to M$ denotes the exponential map of M at m. We will denote by $\widetilde{\nabla}$ its Levi-Civita connection, $\widetilde{R}$ is the curvature tensor, and in general, we use the symbol $\widetilde{\ }$ to denote the corresponding geometric objects in $\widetilde{M}$. We define the *geodesic celestial sphere of radius* r *associated to* ξ as

$$S^\xi(r) = exp_m\left(\{x \in \xi^\perp; \|x\| = r\}\right) = exp_m(\mathcal{S}^\xi(r)). \tag{1}$$

For r sufficiently small, $S^\xi(r)$ is a compact Riemannian submanifold of $\widetilde{M}$. From now on, all geometric objects defined on a geodesic celestial sphere will be denoted using the symbol $\hat{\ }$.

Following [9], we say that a *simple Weyl invariant of order* 2ν, W, is a differentiable map

$$\begin{array}{rcl} W : M & \longrightarrow & \mathbb{R} \\ x & \mapsto & W(x) = trace(\otimes^\nu R) \end{array}$$

where *trace* is a product of traces with respect to some permutation of the indexes of the ν–fold tensor product of the curvature tensor $\otimes^\nu R$. We note here that the above construction does not require the manifold M to be a Lorentzian manifold.

More generally, a Weyl invariant is a differentiable map which consists of the product of traces of the tensor product of the curvature tensor and its covariant derivatives. The *order* of a Weyl invariant is the number of derivatives of the metric tensor involved in its construction. A scalar curvature invariant is a linear combination of Weyl invariants, and hence, the vector space of scalar curvature invariants is generated by Weyl invariants. As a consequence, a simple Weyl invariant is nothing but a Weyl invariant whose definition does not involve covariant derivatives of the curvature tensor. We say that two simple Weyl invariants W_1 and W_2 are equal if and only if their value is the same for each point $x \in M$ and each manifold M.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis. The easiest simple Weyl invariant is the scalar curvature $\tau = \sum_{i,j} R_{ijij}$, which is a simple Weyl invariant of order 2. Moreover, we have the following basis for the vector space of scalar curvature invariants of order 4:

$$\begin{array}{lllll} \|R\|^2 & = & \sum_{ijkl} \epsilon_i\epsilon_j\epsilon_k\epsilon_l R^2_{ijkl}, & \tau^2, & \\ \|\rho\|^2 & = & \sum_{ij} \epsilon_i\epsilon_j \rho^2_{ij}, & \Delta\tau \;=\; \sum_i \epsilon_i \nabla^2_{ii}\tau, & \end{array} \tag{2}$$

where $\epsilon_i = g(e_i, e_i)$. Among them, $\|R\|^2$, $\|\rho\|^2$ and τ^2 are simple Weyl invariants.

Remark 2.1 Let us define $W = trace(\otimes^\nu R)$ a simple Weyl invariant of order 2ν. In a manifold of constant sectional curvature λ the curvature tensor can be written as $R = \lambda\, R^0$ where

$$R^0_{xyvw} = g(x, v)g(y, w) - g(x, w)g(y, v).$$

It is clear that

$$W = trace(\otimes^\nu R) = \lambda^\nu trace(\otimes^\nu R^0) = \bar{A}_W(n+1)\lambda^\nu,$$

where $n+1$ is the dimension of the manifold and $\bar{A}_W$ is a polynomial that only depends on W. Moreover, if the manifold has dimension 0 or 1 then $R = 0$ and hence we have $\bar{A}_W(0) = \bar{A}_W(1) = 0$. Thus, $\bar{A}_W$ can be written as $\bar{A}_W(n+1) = n(n+1)A_W(n)$, where A_W is another polynomial. Then, for the constant curvature case

$$W = n(n+1)A_W(n+1)\lambda^\nu.$$

We again emphasize that this construction does not use the fact that M is Lorentzian, so it is true for all semi–Riemanian manifold.

Let W be a simple Weyl invariant. We define

$$\mathcal{W}(\xi, r) = \int_{S^\xi_m(r)} \hat{W},$$

which we will call the *total scalar curvature of the geodesic celestial sphere* $S^\xi(r)$ *associated to the simple Weyl invariant* W. Here, $\hat{W}$ denotes the corresponding simple Weyl invariant in the geodesic celestial sphere $S^\xi(r)$.

The aim of this paper is to use the total scalar curvatures of geodesic celestial spheres to characterize the Lorentzian manifolds of constant sectional curvature. In order to achieve such an objective we will proceed as follows. A geodesic celestial sphere of the manifold M at m is a geodesic sphere of the Riemannian submanifold $\widetilde{M} = exp_m(\mathfrak{U} \cap \xi^\perp)$, where $\mathfrak{U}$ is a sufficiently small neighborhood of $0 \in T_mM$. It is possible to obtain a power series expansion for a total scalar curvature of a geodesic sphere in terms of the scalar curvature invariants of the ambient Riemannian manifold. Thus, it suffices to write the scalar curvature invariants of $\widetilde{M}$ in terms of the curvature tensors of M if we want to express the total scalar curvatures of geodesic celestial spheres in terms of geometric objects of the ambient manifold M. This is essentially what the subsequent theorems describe in more detail.

We have the following result [5]:

Proposition 2.2 *Let* (M^{n+1}, g) *be a Lorentzian manifold and* W *a Weyl invariant. Denote by* $\mathcal{W}(\xi, r)$ *the corresponding total scalar curvature of* $S^\xi(r)$ *associated to the Weyl invariant* W *of order* 2ν*. Then, we have the following series expansion:*

$$\begin{aligned}\mathcal{W}(\xi,r) = c_{n-1} r^{n-1-2\nu} \Big\{ & (n-1)(n-2)A_W(n-1) \\ & - \frac{(n-2)(n-2\nu-1)}{6\,n} A_W(n-1)\widetilde{\tau}(m) r^2 \\ & + \frac{r^4}{n(n+2)} \Big(B^1_W(n)\|\widetilde{R}\|^2 + B^2_W(n)\|\widetilde{\rho}\|^2 + B^3_W(n)\widetilde{\tau}^2 \\ & \quad - \frac{(n-2)(n-2\nu-1)}{20} A_W(n-1)\widetilde{\Delta}\widetilde{\tau}\Big)(m) + O\left(r^6\right)\Big\},\end{aligned} \tag{3}$$

where A_W *is the polynomial defined in Remark 2.1 and* B^1_W*,* B^2_W *and* B^3_W *are polynomials verifying*

$$\begin{aligned}& 2\,B^1_W(n) + (n-1)\,B^2_W(n) + n(n-1)\,B^3_W(n) \\ & \qquad = \frac{(n-2)\,(n+2)\,(n-1-2\,\nu)\,(5n-10\,\nu-7)}{360} A_W(n-1).\end{aligned} \tag{4}$$

Thus, the above theorem gives us a power series expansion for the total scalar curvature of a geodesic celestial sphere associated to a simple Weyl invariant in terms of scalar curvature invariants of the submanifold $\widetilde{M}$ at the point m. So, our aim now is to express the scalar curvature invariants of $\widetilde{M}$ in terms of the the curvature tensor of M at m. This is achieved by means of the following theorem [4]:

Proposition 2.3 *Let (M^{n+1}, g) be a Lorentzian manifold. Let us denote by $\widetilde{M} = \exp_m(\mathfrak{U} \cap \xi^\perp)$ where $\mathfrak{U}$ is a sufficiently small neighborhood of $0 \in T_m M$ so that $\widetilde{M}$ is an embedded submanifold of M. Then, we have the following relations at m:*

$$\begin{aligned}
\|\widetilde{R}\|^2 &= \|R\|^2 + 4 \sum_{i,j,k=1}^{n} R^2_{\xi ijk} - 4 \sum_{i,j=1}^{n} R^2_{\xi i \xi j} \\
\|\widetilde{\rho}\|^2 &= \|\rho\|^2 + 2 \sum_{i=1}^{n} \rho^2_{\xi i} - \rho^2_{\xi\xi} + \sum_{i,j=1}^{n} R^2_{\xi i \xi j} + 2 \sum_{i,j=1}^{n} \rho_{ij} R_{\xi i \xi j} \\
\widetilde{\tau} &= \tau + 2\rho_{\xi\xi} \\
\widetilde{\Delta}\widetilde{\tau} &= \Delta\tau + 2\Delta\rho_{\xi\xi} + \nabla^2_{\xi\xi}\tau + 2\nabla^2_{\xi\xi}\rho_{\xi\xi} + \frac{4}{9} \sum_{i=1}^{n} \rho^2_{\xi i} + \frac{2}{3} \sum_{i,j,k=1}^{n} R^2_{\xi ijk}.
\end{aligned}$$

Now using theorems 2.2 and 2.3 we get:

Theorem 2.4 *Let (M^{n+1}, g) be a Lorentzian manifold and let W be a simple Weyl invariant. Denote by $\mathcal{W}(\xi, r)$ the corresponding total scalar curvature of $S^\xi(r)$ associated to the Weyl invariant W of order 2ν. Then, we have the* *following series expansion:*

$$
\begin{aligned}
\mathcal{W}(\xi,r) = c_{n-1} r^{n-1-2\nu} \Big\{ & (n-1)(n-2)A_W(n-1) \\
& -\frac{(n-2)(n-2\nu-1)}{6n} A_W(n-1)\left(\tau+2\rho_{\xi\xi}\right)(m)\, r^2 \\
& +\frac{r^4}{n(n+2)}\Big(B_W^1(n)\|R\|^2 + B_W^2(n)\|\rho\|^2 + B_W^3(n)\tau^2 \\
& -\frac{(n-2)(n-2\nu-1)}{20} A_W(n-1)\Delta\tau \\
& +\Big(4B_W^1(n) - \frac{(n-2)(n-2\nu-1)}{30} A_W(n-1)\Big) \sum_{i,j,k=1}^{n} R_{\xi ijk}^2 \\
& +\Big(-4B_W^1(n) + B_W^2(n)\Big) \sum_{i,j=1}^{n} R_{\xi i\xi j}^2 \\
& +\left(2B_W^2(n) - \frac{(n-2)(n-2\nu-1)}{45} A_W(n-1)\right) \sum_{i=1}^{n} \rho_{\xi i}^2 \\
& +\Big(-B_W^2(n) + 4B_W^3(n)\Big)\rho_{\xi\xi}^2 \; + \; 4B_W^3(n)\tau\rho_{\xi\xi} \\
& -\frac{(n-2)(n-2\nu-1)}{10} A_W(n-1)\Delta\rho_{\xi\xi} \\
& -\frac{(n-1)(n-2\nu-1)}{20} \nabla_{\xi\xi}^2\tau + 2B_W^2(n) \sum_{i,j=1}^{n} \rho_{ij} R_{\xi i\xi j} \\
& -\frac{(n-2)(n-2\nu-1)}{10} \nabla_{\xi\xi}^2 \rho_{\xi\xi} \Big)(m) + O\left(r^6\right) \Big\},
\end{aligned} \tag{5}
$$

where A_W is the polynomial defined in Remark 2.1 and B_W^1, B_W^2 and B_W^3 are polynomials satisfying Equation (4)*.*

Of particular interest is the case $W = 1$, where we consider W as a simple Weyl invariant of order 0. Then $\mathcal{W}(\xi, r)$ is nothing but the volume of a geodesic celestial sphere [4].

Theorem 2.5 *Let $\left(M^{n+1}, g\right)$ be a Lorentzian manifold and $\xi \in T_mM$ an instantaneous observer. The $(n-1)$–dimensional volume of the geodesic celestial* *spheres associated to $\xi \in T_m M$ satisfies*

$$\begin{aligned} vol_{n-1}\left(S_m^{\xi}(r)\right) = c_{n-1} r^{n-1} \Big\{ 1 - \frac{r^2}{6\,n}(\tau + 2\rho_{\xi\xi})(m) \\ + \frac{r^4}{n(n+2)} \Big(-\frac{1}{120}\|R\|^2 + \frac{1}{45}\|\rho\|^2 + \frac{1}{72}\tau^2 - \frac{1}{20}\Delta\tau \\ - \frac{1}{15}\sum_{i,j,k=1}^{n} R_{\xi ijk}^2 + \frac{1}{18}\sum_{i,j=1}^{n} R_{\xi i\xi j}^2 + \frac{2}{45}\sum_{i,j=1}^{n} \rho_{ij} R_{\xi i\xi j} \\ + \frac{1}{45}\sum_{i=1}^{n} \rho_{\xi i}^2 + \frac{1}{30}\rho_{\xi\xi}^2 + \frac{1}{18}\tau\rho_{\xi\xi} - \frac{1}{10}\Delta\rho_{\xi\xi} \\ - \frac{1}{20}\nabla_{\xi\xi}^2\tau - \frac{1}{10}\nabla_{\xi\xi}^2\rho_{\xi\xi} \Big)(m) + O\left(r^6\right) \Big\}. \end{aligned}$$

3 Characterization of constant curvature Lorentzian manifolds

The most trivial simple Weyl invariant one think of is $W = 1$, which has order 0. As we have seen, the total scalar curvature of geodesic celestial spheres associated to this invariant is nothing but the the $(n-1)$–dimensional volume of those geodesic celestial spheres. In [4] several results were obtained for this volume. We briefly state here the main theorems of that work. The proof is not difficult to obtain from the power series expansion of Theorem 2.5. Afterwards, we will see how these results can be generalized for total scalar curvatures associated to simple Weyl invariants of arbitrary order.

First, we fix some notation. We will denote by $vol_{n-1}^M\left(S^{\xi}(r)\right)$ the $(n-1)$–dimensional volume of the geodesic celestial sphere $S^{\xi}(r)$. For the special case when M is isotropic we can simply denote by $vol_{n-1}^M\left(S(r)\right)$ the $(n-1)$–dimensional volume of any geodesic celestial sphere $S^{\xi}(r)$ with ξ unit timelike.

Theorem 3.1 *Let (M^{n+1}, g) be a Lorentzian manifold and $N^{n+1}(\lambda)$ a Lorentzian manifold of constant sectional curvature λ. The following statements hold:*

(i) If $R(x,y,x,y) \geq \lambda\left(g(x,x)g(y,y) - g(x,y)^2\right)$ for all spacelike $x, y \in TM$ then

$$vol_{n-1}^M\left(S^{\xi}(r)\right) \leq vol_{n-1}^{N(\lambda)}\left(S(r)\right)$$

for all sufficiently small r and all instantaneous observer $\xi \in T_m M$.

(ii) If $R(x,y,x,y) \leq \lambda\left(g(x,x)g(y,y) - g(x,y)^2\right)$ for all spacelike $x, y \in TM$ then

$$vol_{n-1}^M\left(S^{\xi}(r)\right) \geq vol_{n-1}^{N(\lambda)}\left(S(r)\right)$$

for all sufficiently small r and all instantaneous observer $\xi \in T_m M$.

Moreover, the equality holds at (i) or (ii) for all unit timelike ξ if and only if M has constant sectional curvature λ at m.

We also have a Gromov like theorem.

Theorem 3.2 *Let (M^{n+1}, g) be a Lorentzian manifold and $N^{n+1}(\lambda)$ a Lorentzian manifold of constant sectional curvature λ.*

(i) If $R(x,y,x,y) \geq \lambda\left(g(x,x)g(y,y) - g(x,y)^2\right)$ for all spacelike $x, y \in TM$ then

$$\frac{vol_{n-1}^{M}\left(S_m^{\xi}(r)\right)}{vol_{n-1}^{N(\lambda)}\left(S(r)\right)}$$

is nonincreasing for sufficiently small r and all instantaneous observer $\xi \in T_mM$.

(ii) If $R(x,y,x,y) \leq \lambda\left(g(x,x)g(y,y) - g(x,y)^2\right)$ for all spacelike $x, y \in TM$ then

$$\frac{vol_{n-1}^{M}\left(S_m^{\xi}(r)\right)}{vol_{n-1}^{N(\lambda)}\left(S(r)\right)}$$

is nondecreasing for sufficiently small r and all instantaneous observer $\xi \in T_mM$.

We also have a rigidity result which is the one we would like to generalize for total scalar curvatures of geodesic celestial spheres. First, we introduce the concept of isotropy.

A Lorentzian manifold is said to be locally isotropic if for any point $m \in M$ and vectors $X, Y \in T_mM$ with $g(X,X) = g(Y,Y)$ there exists a local isometry of (M, g) fixing m and sending X to Y. A locally isotropic Lorentzian manifold is always locally homogeneous. Obviously, in this case, $\mathcal{W}(\xi, r)$ is independent of $\xi \in TM$. The following theorem shows that local isotropy can be recovered from the properties of the volume of geodesic celestial spheres.

Theorem 3.3 *Let (M^{n+1}, g) be a Lorentzian manifold. If the volume of the geodesic celestial spheres $S_m^{\xi}(r)$ is independent of the observer field $\xi \in TM$, then M has constant sectional curvature.*

In what follows we will generalize the above theorem. Then we will get some consequences paying special attention to simple Weyl invariants of low order.

The main theorem of this paper is the following:

Theorem 3.4 *Let (M^{n+1}, g) be a Lorentzian manifold and W a simple Weyl invariant of order 2ν. Let us denote by $\mathcal{W}(\xi, r)$ the total scalar curvature of $S^{\xi}(r)$ associated to W. Suppose $\mathcal{W}(\xi, r)$ is independent of the infinitesimal observer $\xi \in TM$ and*

$$\begin{aligned} n &\neq 2\nu + 1, \\ A_W(n-1) &\neq 0 \\ 4B_W^1(n) + B_W^2(n) - \frac{(n-2)(n-1-2\nu)}{15} A_W(n-1) &\neq 0, \\ 4(n+2)B_W^1(n) - 3B_W^2(n) - \frac{(n-2)(n-1)(n-1-2\nu)}{30} A_W(n-1) &\neq 0. \end{aligned} \tag{6}$$

Then M has constant sectional curvature.

Proof. As $\mathcal{W}(\xi, r)$ is independent of ξ it is clear that the coefficient of r^2 in the power series expansion (3) of Theorem 2.2 must be constant, which means, taking into account our hypotesis that

$$\tau + 2\rho_{\xi\xi} = k,$$

with k constant. This implies $-\tau\, g_{\xi\xi} + 2\rho_{\xi\xi} + k\, g_{\xi\xi} = 0$ for all timelike vector ξ. Therefore, for sufficiently small ϵ, $\xi + \epsilon\, v$ is timelike if ξ is timelike. Then,

$$\begin{aligned} &-\tau\, g_{\xi\xi} + 2\rho_{\xi\xi} + k\, g_{\xi\xi} \\ &+2\epsilon(-\tau\, g_{\xi v} + 2\rho_{\xi v} + k\, g\xi v) \\ &+\epsilon^2(-\tau\, g_{vv} + 2\rho_{vv} + k\, g_{vv}) \quad = \quad 0. \end{aligned}$$

As ϵ is arbitrary small, we finally get

$$-\tau\, g_{vv} + 2\rho_{vv} + k\, g_{vv} = 0$$

for all $v \in TM$. Taking traces in the above equality shows that τ is constant, and hence the manifold is Einstein. As a consequence, the power series expansion (3) becomes

$$\begin{aligned} \mathcal{W}(\xi, r) = c_{n-1} r^{n-1-2\nu} \Big\{ &(n-1)(n-2) A_W(n-1) \\ &- \frac{(n-2)(n-2\nu-1)(n-1)}{6\, n(n+1)} A_W(n-1)\tau\, r^2 \\ &+ \frac{r^4}{n(n+2)} \Big(B_W^1(n) \|R\|^2(m) \\ &\quad + \Big(4B_W^1(n) - \frac{(n-2)(n-2\nu-1)}{30} A_W(n-1) \Big) \sum_{i,j,k=1}^{n} R_{\xi ijk}^2 \\ &\quad + \Big(-4B_W^1(n) + B_W^2(n) \Big) \sum_{i,j=1}^{n} R_{\xi i\xi j}^2 \\ &\quad + \frac{\tau^2}{(n+1)^2} \Big[(n-2) B_W^2(n) + (n-1)^2 B_W^3(n) \Big] \Big) + O\left(r^6\right) \Big\}, \end{aligned}$$

So, as the coefficient of r^4 must also be constant we get

$$\begin{aligned} &B_W^1(n)\|R\|^2 \\ &+ \Big(4B_W^1(n) - \frac{(n-2)(n-2\nu-1)}{30} A_W(n-1) \Big) \sum_{i,j,k=1}^{n} R_{\xi ijk}^2 \\ &+ \Big(-4B_W^1(n) + B_W^2(n) \Big) \sum_{i,j=1}^{n} R_{\xi i\xi j}^2 \quad = \quad constant \end{aligned}$$

Using a similar argument as before it can be shown that the manifold is 2–stein (see [4]) provided that

$$\begin{aligned} 4B^1_W(n) + B^2_W(n) - \frac{(n-2)(n-1-2\nu)}{15} A_W(n-1) &\neq 0, \\ 4(n+2)B^1_W(n) - 3B^2_W(n) - \frac{(n-2)(n-1)(n-1-2\nu)}{30} A_W(n-1) &\neq 0. \end{aligned}$$

But 2–stein Lorentzian manifolds have constant sectional curvature [4], [8], so the result follows. □

Corollary 3.5 *Let (M^{n+1}, g) be a Lorentzian manifold and W a simple Weyl invariant. If for each small radius r and each $\xi \in TM$, $\mathcal{W}(\xi, r)$ is the same as in a $(n+1)$–Lorentzian manifold of constant sectional curvature λ and the conditions in (6) hold, then M is a Lorentzian manifold of constant sectional curvature λ.*

Proof. A Lorentzian manifold of constant sectional curvature is locally isotropic, so the total scalar curvatures of geodesic celestial spheres do not depend on the infinitesimal observer ξ. Thus, from Theorem 3.4 it follows that M has also constant sectional curvature. Moreover, it is easy to see that the power series expansion (5) of $\mathcal{W}(\xi, r)$ for a Lorentzian manifold of constant sectional curvature becomes:

$$\begin{aligned} \mathcal{W}(\xi, r) &= c_{n-1}\,(n-1)(n-2)A_W(n-1)\,r^{n-1-2\nu}\Big\{1 - \frac{(n-2\nu-1)}{6}\,\lambda\,r^2 \\ &\quad + \frac{(n-1-2\nu)(5n-10\nu-7)}{360}\,\lambda^2\,r^4 + O\left(r^6\right)\Big\}. \end{aligned}$$

Comparing the coefficient of r^2 in (5) with the corresponding one in the equation above gives $\tau = n(n+1)\lambda$, and hence the curvature is exactly λ. □

As an example of Theorem 3.4 we will show how the low order simple Weyl invariants can be used for characterizing the Lorentzian manifolds of constant sectional curvature.

Corollary 3.6 *Let (M^{n+1}, g) be a Lorentzian manifold with $n > 3$ such that $\int_{S^\xi(r)} \hat{\tau}$ only depends on the radius. Then, M has constant sectional curvature.*

Proof. Just use Theorem 3.4 taking into account that [3]:

$$\begin{aligned} A_\tau(n-1) &= 1, \\ B^1_\tau(n) &= -\frac{(n+2)(n+3)}{120}, \\ B^2_\tau(n) &= \frac{n^2+5n+21}{45}, \\ B^3_\tau(n) &= \frac{n^2-7n-6}{72}. \end{aligned}$$

As a matter of fact we note here that when $n = 2$, geodesic celestial spheres are flat, and hence all scalar curvature invariants vanish. When $n = 3$ geodesic celestial spheres are 2–dimensional Riemannian manifolds. Therefore, by Gauss–Bonnet Theorem $\int_{S^{\xi}(r)} \hat{\tau} = 8\pi$, which makes τ useless for the purpose of characterizing Lorentzian manifolds by means of total scalar curvatures. □

Corollary 3.7 *Let (M^{n+1}, g) be a Lorentzian manifold with $n \neq 5$. The following statements are equivalent:*

(i) $\int_{S^{\xi}(r)} \|\hat{R}\|^2$ *depends only on the radius;*

(ii) $\int_{S^{\xi}(r)} \|\hat{\rho}\|^2$ *depends only on the radius;*

(iii) $\int_{S^{\xi}(r)} \hat{\tau}^2$ *depends only on the radius;*

(iv) *M has constant sectional curvature.*

Proof. Using the results in [3] we get (we do not include $B^3_W(n)$ in the following table as it can be obtained form (4)):

W	$A_W(n-1)$	$B^1_W(n)$	$B^2_W(n)$
$\|R\|^2$	2	$\frac{59n^2-93n-10}{60}$	$\frac{2(n^2-37n+60)}{15}$
$\|\rho\|^2$	$n-2$	$-\frac{n^3-9n^2+16n-20}{120}$	$\frac{n^3+31n^2-16n-120}{}$
τ^2	$(n-1)(n-2)$	$-\frac{(n-2)(n-1)(n^2+13n+10)}{120}$	$\frac{n^4-14n^3+29n^2-60n-188}{72}$

Now the result follows from Theorem 3.4. □

for $n > 3$, $n \neq 7$

W	$A_W(n-1)$	$B^1_W(n)$	$B^2_W(n)$
τ^3	$(n-2)^2(n-1)^2$	$-\frac{(n-2)^2(n-1)^2(n^2+21n+14)}{120}$	$\frac{(n-2)(n-1)(n^4+18n^3+118n^2+105n)}{45}$
$\tau\|\rho\|^2$	$(n-2)^2(n-1)$	$-\frac{(n-2)(n-1)(n^3-n^2-28n-28)}{120}$	$\frac{(n-2)(n^4+38n^3+28n^2+15n+238)}{45}$
$\tau\|R\|^2$		$\overline{120}$	$\overline{45}$
$\check{\rho}$		$\overline{120}$	$\overline{45}$
$\langle \rho \otimes \rho, \bar{R} \rangle$		$\overline{120}$	$\overline{45}$
$\langle \rho, \dot{R} \rangle$		$\overline{120}$	$\overline{45}$
$\check{R}$		$\overline{120}$	$\overline{45}$
$\check{\check{R}}$		$\overline{120}$	$\overline{45}$

For the L^2–norma of the super–Ricci tensor η (call it however you want) everything works:

$$
\begin{aligned}
A_{\|\eta\|^2}(n-1) &= 4(n-2), \\
B^1_{\|\eta\|^2}(n) &= \frac{119n^3 - 347n^2 + 360n + 36}{30}, \\
B^2_{\|\eta\|^2}(n) &= \frac{4(n^3 + 7n^2 + 140n - 396)}{45}, \\
B^3_{\|\eta\|^2}(n) &= \frac{n^3 - 21n^2 - 56n + 60}{18}.
\end{aligned}
$$

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