NON-HOPF REAL HYPERSURFACES WITH CONSTANT PRINCIPAL CURVATURES IN COMPLEX SPACE FORMS

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Abstract. We classify real hypersurfaces in complex space forms with constant principal curvatures and whose Hopf vector field has two nontrivial projections onto the principal curvature spaces.

In complex projective spaces such real hypersurfaces do not exist. In complex hyperbolic spaces these are holomorphically congruent to open parts of tubes around the ruled minimal submanifolds with totally real normal bundle introduced by Berndt and Brück. In particular, they are open parts of homogenous ones.

1. Introduction

A homogeneous submanifold of a Riemannian manifold is an orbit of the action of a closed subgroup of the isometry group of the ambient manifold. One of the aims of submanifold geometry is to classify homogeneous submanifolds of a given manifold and to characterize them in terms of geometric data. Of particular interest are homogeneous hypersurfaces, which arise as principal orbits of cohomogeneity one actions. Obviously, homogeneous hypersurfaces have constant principal curvatures, that is, the eigenvalues of their shape operator are constant. It is an outstanding problem to determine under which conditions hypersurfaces with constant principal curvatures are open parts of homogeneous ones.

In spaces of constant curvature, a hypersurface has constant principal curvatures if and only if it is isoparametric. The classification of isoparametric hypersurfaces was achieved by Segre [20] in Euclidean spaces and by Cartan [9] in real hyperbolic spaces. They all are open parts of homogeneous ones. The situation is more involved in spheres. Cartan classified hypersurfaces with $g \in \{1, 2, 3\}$ constant principal curvatures in spheres. Subsequently, Hsiang and Lawson [12] classified homogeneous hypersurfaces in spheres; they have $g \in \{1, 2, 3, 4, 6\}$ principal curvatures. Then, Münzner [18] showed that $g \in \{1, 2, 3, 4, 6\}$ for isoparametric hypersurfaces in general. Surprisingly, for $g = 4$ there are isoparametric hypersurfaces that are not homogeneous [13]. Recently, Cecil, Chi and Jensen [10], and Immervoll [14] showed that, with a few possible exceptions, hypersurfaces with $g = 4$ constant principal curvatures are among the known homogeneous and inhomogeneous examples. Some progress has been made for $g = 6$ by Abresch [1] and Dorfmeister and Neher [11], but the problem remains open in full generality. See [24] for a survey.

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The problem is even more difficult in complex space forms. See [19] for a survey on this and related topics. By $c \neq 0$ we denote the constant holomorphic sectional curvature of a complex space form; thus, if $c > 0$ (resp. $c < 0$) we have a complex projective (resp. hyperbolic) space $\mathbb{C}P^n(c)$ (resp. $\mathbb{C}H^n(c)$). We denote by J its Kähler structure. Let M be a real hypersurface of a complex space form and ξ a (local) unit normal vector field. Then, $J\xi$ is tangent to M and is called the Hopf vector field of M. The hypersurface M is said to be Hopf if $J\xi$ is a principal curvature vector field. The motivation for our work is to address the classification of real hypersurfaces with constant principal curvatures in complex space forms. We briefly summarize the current state of the problem.

Assume M is a real hypersurface of a complex space form with g distinct constant principal curvatures. For $p \in M$ denote by $h(p)$ the number of nontrivial projections of $J\xi_p$ onto the principal curvature spaces of M. Clearly, this function is integer-valued and M is Hopf if and only if $h = 1$. The classification of homogeneous real hypersurfaces in complex projective spaces $\mathbb{C}P^n(c)$ was derived by Takagi [21]. It follows from this classification that $g \in \{2, 3, 5\}$. A remarkable feature of homogeneous real hypersurfaces in $\mathbb{C}P^n(c)$ is that they are Hopf. Subsequently, Takagi classified real hypersurfaces with $g \in \{2, 3\}$ constant principal curvatures [22], [23] ([25] for $n = 2$, $g = 3$). It follows from his work that they all are Hopf and open parts of homogeneous ones. Kimura [15] classified Hopf real hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$ and showed that these are open parts of homogeneous ones. No examples are known of real hypersurfaces with constant principal curvatures in $\mathbb{C}P^n(c)$ with $h > 1$. Surprisingly, in $\mathbb{C}H^n(c)$ there are non-Hopf homogeneous real hypersurfaces. The first example was discovered by Lohnherr [16] and further examples were given by Berndt and Brück [3], [4]. We refer to §2.2 for a brief introduction and to [7] for a deeper study of their geometry. Berndt and Tamaru obtained in [8] the classification of cohomogeneity one actions on $\mathbb{C}H^n(c)$. The number of principal curvatures of the homogeneous examples is $g \in \{2, 3, 4, 5\}$. Montiel [17] classified real hypersurfaces with $g = 2$ constant principal curvatures in $\mathbb{C}H^n(c)$ ($n \geq 3$). Berndt and the first author solved the case $g = 3$ and $g = 2$, $n = 2$ [5], [6]. It follows from [17] that $h = 1$ when $g = 2$, and from [5] and [6] we get $h \leq 2$ if $g = 3$. Hopf real hypersurfaces with constant principal curvatures in $\mathbb{C}H^n(c)$ were classified by Berndt [2] and they all are open parts of homogeneous ones. To our knowledge, [5] and [6] are the first classifications of this kind involving non-Hopf real hypersurfaces. Nothing is known about h if $g \geq 4$.

Our aim in this paper is to carry out the next natural step after Berndt and Kimura's classification of Hopf real hypersurfaces with constant principal curvatures in $\mathbb{C}P^n(c)$ and $\mathbb{C}H^n(c)$. Thus, we classify real hypersurfaces with constant principal curvatures whose Hopf vector field $J\xi$ has $h = 2$ nontrivial projections onto the principal curvature spaces.

Main Theorem. *We have:*

(a) *There are no real hypersurfaces with constant principal curvatures in $\mathbb{C}P^n(c)$, $n \geq 2$, whose Hopf vector field has $h = 2$ nontrivial projections onto the principal curvature spaces.*

(b) *Let M be a connected real hypersurface in $\mathbb{C}H^n(c)$, $n \geq 2$, with constant principal curvatures and whose Hopf vector field has $h = 2$ nontrivial projections onto the principal* *curvature spaces of M. Then, M has $g \in \{3,4\}$ principal curvatures and is holomorphically congruent to an open part of:*

(i) *a ruled minimal real hypersurface $W^{2n-1} \subset \mathbb{C}H^n(c)$ or one of the equidistant hypersurfaces to W^{2n-1}, or*

(ii) *a tube around a ruled minimal Berndt-Brück submanifold $W^{2n-k} \subset \mathbb{C}H^n(c)$ with totally real normal bundle, for some $k \in \{2,\dots,n-1\}$.*

In particular, M is an open part of a homogeneous real hypersurface of $\mathbb{C}H^n(c)$.

The ruled minimal submanifolds $W^{2n-k} \subset \mathbb{C}H^n(c)$ are homogeneous and have totally real normal bundle of rank $k \in \{1,\dots,n-1\}$. Actually, W^{2n-1} was discovered by Lohnherr [16]. Then, Berndt studied the geometry of the equidistant hypersurfaces to W^{2n-1} [3]. This construction was generalized by Berndt and Brück in [4]. Both W^{2n-1} and any of its equidistant hypersurfaces have $g = 3$ principal curvatures. The tubes around W^{2n-k}, $k \in \{2,\dots,n-1\}$ have $g = 4$ principal curvatures if $r \neq (1/\sqrt{-c})\log(2+\sqrt{3})$ and $g = 3$ principal curvatures if $r = (1/\sqrt{-c})\log(2+\sqrt{3})$. See [7] for a detailed description.

The proof is as follows. First we use the Gauss and Codazzi equations to derive some algebraic properties of the eigenvalue structure of the shape operator. The methods used for this are similar to those of [5], although a bit more general. We would like to emphasize that whenever we use a method similar to one in [5] we explicitly point it out and skip the details as much as possible. On the other hand, we focus on the new techniques and results, especially on Subsection 3.4. The most crucial step of the proof is to show that the number g of constant principal curvatures satisfies $g \leq 4$. For this we use a novel approach based on the study of some inequalities satisfied by the principal curvatures. Using standard Jacobi field theory one can deduce the geometry of the focal submanifolds of these hypersurfaces and then the result follows from a rigidity result in [7].

The paper is organized as follows. In Section 2 we introduce the basic elements of our paper. Subsection 2.1 is devoted to present the equations of submanifold geometry that we will use in the rest of the paper. In §2.2 we briefly describe the ruled minimal Berndt-Brück submanifolds W^{2n-k}. We prove our Main Theorem in Section 3. The proof is divided in several steps. Some vector fields and functions arise naturally in our proof (§3.1 and §3.2). We get some of their properties in Subsection 3.3. In §3.4 we show that the number g of principal curvatures satisfies $g \in \{3,4\}$. We summarize all the eigenvalue structure in §3.5. In Subsection 3.6 we use standard Jacobi field theory to finish the proof of the Main Theorem.

2. Preliminaries

In this section we introduce the basic notation of this paper. We write down the Gauss and Codazzi equations of a hypersurface in a complex space form and derive some basic consequences. Then, we briefly mention how the examples of the Main Theorem are constructed.

2.1. The equations of a hypersurface.

Let $\bar{M}(c)$ be a complex space form of constant holomorphic sectional curvature $c \neq 0$ and complex dimension n. If $c > 0$ then $\bar{M}(c)$ is a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c. Analogously, if $c < 0$ then $\bar{M}(c)$ is a complex hyperbolic space $\mathbb{C}H^n(c)$. We denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ its inner product, by J its Kähler structure, and by $\bar{\nabla}$ its Levi-Civita connection. The curvature tensor is defined by $\bar{R}(X,Y) = [\bar{\nabla}_X, \bar{\nabla}_Y] - \bar{\nabla}_{[X,Y]}$, so in this case we have

$$\bar{R}(X,Y)Z = \frac{c}{4}\left(\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY - 2\langle JX,Y\rangle JZ\right).$$

Let M be a connected submanifold of $\bar{M}(c)$. We denote by ∇ and R its Levi-Civita connection and its curvature tensor respectively. By TM and νM we denote the tangent and normal bundles of M. We use the symbol $\Gamma(\cdot)$ to refer to the smooth sections of any vector bundle. Let $X, Y, Z, W \in \Gamma(TM)$ and $\xi \in \Gamma(\nu M)$.

The second fundamental form $I\!I$ of M is defined by the Gauss formula as $\bar{\nabla}_X Y = \nabla_X Y + I\!I(X,Y)$. The Weingarten formula is then written as $\bar{\nabla}_X \xi = -S_\xi X + \nabla^\perp_X \xi$, where S_ξ is the shape operator with respect to ξ and $\nabla^\perp$ is the induced normal connection on νM. The second fundamental form and the shape operator are related by $\langle S_\xi X, Y\rangle = \langle I\!I(X,Y), \xi\rangle$.

Now let M be a connected real hypersurface of $\bar{M}(c)$. The word 'real' emphasizes the fact that the *real* codimension is one. Fix $\xi \in \Gamma(\nu M)$ a (local) unit normal vector field. We write S instead of S_ξ. The Gauss formula can be rewritten as

$$\bar{\nabla}_X Y = \nabla_X Y + \langle SX, Y\rangle \xi,$$

and hence, the Weingarten formula is $SX = -\bar{\nabla}_X \xi$. Moreover, the Gauss and Codazzi equations for a hypersurface are

$$\begin{aligned} \langle \bar{R}(X,Y)Z, W\rangle &= \langle R(X,Y)Z, W\rangle - \langle SY, Z\rangle\langle SX, W\rangle + \langle SX, Z\rangle\langle SY, W\rangle, \text{ and} \\ \langle \bar{R}(X,Y)Z, \xi\rangle &= \langle (\nabla_X S)Y - (\nabla_Y S)X, Z\rangle. \end{aligned}$$

We assume from now on that M has constant principal curvatures, that is, the eigenvalues of the shape operator S are constant. For each principal curvature λ of M we denote by T_λ the distribution on M formed by the principal curvature spaces of λ along M.

The Codazzi equation implies (see [5, Section 2] for a proof)

Lemma 2.1. (i) *Let $p \in M$. If the orthogonal projection of $J\xi_p$ onto $T_\alpha(p)$ is nonzero, then $T_\alpha(p)$ is a real subspace of $T_p\bar{M}(c)$, that is, $JT_\alpha(p)$ is orthogonal to $T_\alpha(p)$.*

(ii) *Let $X, Y \in \Gamma(T_\alpha)$ and $Z \in \Gamma(T_\beta)$ with $\alpha \neq \beta$. Then*

$$\langle \nabla_X Y, Z\rangle = \frac{c}{4(\alpha-\beta)}\left(\langle JY, Z\rangle\langle X, J\xi\rangle + \langle JX, Y\rangle\langle Z, J\xi\rangle + 2\langle JX, Z\rangle\langle Y, J\xi\rangle\right).$$

(iii) *Let $X \in \Gamma(T_\alpha)$, $Y \in \Gamma(T_\beta)$ and $Z \in \Gamma(T_\gamma)$. Then*

$$\langle \bar{R}(X,Y)Z, \xi\rangle = (\beta-\gamma)\langle \nabla_X Y, Z\rangle - (\alpha-\gamma)\langle \nabla_Y X, Z\rangle.$$

The Gauss equation implies (again, see [5, Lemma 4] for a proof)

Lemma 2.2. *Let $X \in \Gamma(T_\alpha)$ and $Y \in \Gamma(T_\beta)$, with $\alpha \neq \beta$, be unit vector fields. Then*

$$\begin{aligned} 0 &= (\beta-\alpha)(-c-4\alpha\beta-2c\langle JX,Y\rangle^2+8\langle \nabla_X Y, \nabla_Y X\rangle - 4\langle \nabla_X X, \nabla_Y Y\rangle) \\ &\quad -4c\langle JX,Y\rangle(X\langle Y,J\xi\rangle + Y\langle X,J\xi\rangle) \\ &\quad -c\langle X,J\xi\rangle(3Y\langle JX,Y\rangle + \langle \nabla_Y X, JY\rangle - 2\langle \nabla_X Y, JY\rangle) \\ &\quad -c\langle Y,J\xi\rangle(3X\langle JX,Y\rangle - \langle \nabla_X Y, JX\rangle + 2\langle \nabla_Y X, JX\rangle). \end{aligned}$$

2.2. **Discussion of examples.** Part (a) of the Main Theorem states that there are no examples of real hypersurfaces with constant principal curvatures in $\mathbb{C}P^n(c)$ whose Hopf vector field has $h = 2$ nontrivial projections onto the principal curvature spaces of M. Thus, we will focus on describing briefly the examples of part (b) of the Main Theorem. These examples where first constructed in [4] and their geometry was studied in [7].

The connected simple Lie group $G = SU(1,n)$ acts transitively on $\mathbb{C}H^n(c)$. Fix a point $o \in \mathbb{C}H^n(c)$ and let K be the isotropy group of G at o. The subgroup K of G is isomorphic to $S(U(1)U(n))$. Furthermore, (G,K) is a symmetric pair and $\mathbb{C}H^n(c)$ may be identified with the quotient G/K. Write $\mathfrak{g}$ for the Lie algebra of G and $\mathfrak{k}$ for the Lie algebra of K. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to $o \in \mathbb{C}H^n(c)$. We choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$; then, $\dim \mathfrak{a} = 1$ since $\mathbb{C}H^n(c)$ has rank one. Let $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ be the restricted root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$ and assume that α is a positive root. Then, $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ is a 2-step nilpotent subalgebra of $\mathfrak{g}$ isomorphic to the $(2n-1)$-dimensional Heisenberg algebra. Furthermore, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is an Iwasawa decomposition of $\mathfrak{g}$. If A and N denote the connected subgroups of G whose Lie algebras are $\mathfrak{a}$ and $\mathfrak{n}$, then $G = KAN$ is an Iwasawa decomposition of G. The solvable group AN is simply connected and acts simply transitively on $\mathbb{C}H^n(c)$. Thus, we can identify $\mathfrak{a} \oplus \mathfrak{n}$ with $T_o\mathbb{C}H^n(c)$. The Riemannian metric of $\mathbb{C}H^n(c)$ induces a left-invariant metric on AN which makes AN isometric to $\mathbb{C}H^n(c)$. Similarly, the complex structure J on $T_o\mathbb{C}H^n(c)$ induces a complex structure on $\mathfrak{a} \oplus \mathfrak{n}$ which we also denote by J. We have $J\mathfrak{a} = \mathfrak{g}_{2\alpha}$, and $\mathfrak{g}_\alpha$ is J-invariant. Let $B \in \mathfrak{a}$ be a unit vector and define $Z = JB \in \mathfrak{g}_{2\alpha}$.

Let $\mathfrak{w}$ be a linear subspace of $\mathfrak{g}_\alpha$ such that the orthogonal complement $\mathfrak{w}^\perp = \mathfrak{g}_\alpha \ominus \mathfrak{w}$ of $\mathfrak{w}$ in $\mathfrak{g}_\alpha$ has constant Kähler angle $\varphi \in (0, \pi/2]$, that is, the angle between Jv and $\mathfrak{w}^\perp$ is φ for all nonzero $v \in \mathfrak{w}^\perp$. Then, $\varphi = \pi/2$ if and only if $\mathfrak{w}^\perp$ is real, or equivalently, if and only if $J\mathfrak{w}^\perp$ is orthogonal to $\mathfrak{w}^\perp$. Let k be the dimension of $\mathfrak{w}^\perp$. Then, $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ is a subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$. Let S be the connected simply connected subgroup of AN whose Lie algebra is $\mathfrak{s}$. We define the Berndt-Brück submanifolds as [4] (see [16] for $k = 1$)

$$W^{2n-k}_\varphi = S \cdot o, \qquad \text{and} \qquad W^{2n-k} = W^{2n-k}_{\pi/2}.$$

The Berndt-Brück submanifolds W^{2n-k}_φ are homogeneous, have normal bundle of rank k and constant Kähler angle $\varphi \in (0, \pi/2]$, and their second fundamental form II is given by the trivial symmetric bilinear extension of $II(Z, P\xi) = (\sin(\varphi)\sqrt{-c}/2)\xi$ for all $\xi \in \mathfrak{w}^\perp$, where $P\xi$ is the orthogonal projection of $J\xi$ onto TW^{2n-k}_φ. In particular, the submanifolds W^{2n-k}_φ are minimal, and ruled by the totally geodesic complex hyperbolic subspaces determined by their maximal holomorphic distribution. If $\varphi = \pi/2$ then $P = J$ and the Berndt-Brück submanifolds have totally real normal bundle. Conversely [7, Theorem 1]

Theorem 2.3. *Let M be a $(2n-k)$-dimensional connected submanifold in $\mathbb{C}H^n(c)$, $n \geq 2$, with normal bundle νM of constant Kähler angle $\varphi \in (0, \pi/2]$. Assume that there exists a unit vector field Z tangent to the maximal holomorphic subbundle of TM such that the second fundamental form II of M is given by the trivial symmetric bilinear extension of*

$$II(Z, P\xi) = \sin(\varphi) \frac{\sqrt{-c}}{2} \xi$$

for all $\xi \in \Gamma(\nu M)$. Then M is holomorphically congruent to an open part of the ruled minimal submanifold W_φ^{2n-k}.

In particular, the Berndt-Brück submanifolds W^{2n-k} are determined by the equation $II(Z, J\xi) = (\sqrt{-c}/2)\xi$ and the fact that their normal bundle is totally real. Geometrically, they are constructed in the following way. Fix a horosphere $\mathcal{H}$ in a totally geodesic real hyperbolic space $\mathbb{R}H^{k+1}(c) \subset \mathbb{C}H^n(c)$. Attach at each point the totally geodesic $\mathbb{C}H^{n-k}(c)$ which is tangent to the orthogonal complement of the complex span of the tangent space of $\mathcal{H}$ at p. The resulting submanifold is congruent to W^{2n-k}.

Let $N_K^0(S)$ denote the connected component of the identity transformation of the normalizer of S in K. Then, $N_K^0(S)S$ acts on $\mathbb{C}H^n(c)$ with cohomogeneity one and $W_\varphi^{2n-k} = N_K^0(S)S \cdot o$. If $k > 1$, then the principal orbits of $N_K^0(S)S$ are tubes around W_φ^{2n-k}. If $k = 1$, then $\varphi = \pi/2$, the action of $N_K^0(S)S$ is orbit equivalent to the action of S, and its orbits form a homogeneous foliation on $\mathbb{C}H^n(c)$ that was first studied in [3].

Let M be a principal orbit of $N_K^0(S)S$. If $\varphi \in (0, \pi/2)$ then the Hopf vector field of M has $h = 3$ nontrivial projections onto the principal curvature spaces of M. If $\varphi = \pi/2$, then the Hopf vector field of M has $h = 2$ nontrivial projections onto the principal curvature spaces of M. The objective of part (b) of the Main Theorem is to give a geometric characterization of the tubes around W^{2n-k}, $k \in \{2, \dots, n-1\}$, and the equidistant hypersurfaces to W^{2n-1}.

3. Proof of the Main Theorem

In this section we prove the Main Theorem. Our main goal is to describe accurately the eigenvalue structure of a real hypersurface in the conditions of the Main Theorem (Theorem 3.12). Then we finish the proof using standard Jacobi field theory (§3.6).

3.1. **Notation and setup.** Let M be a connected real hypersurface with $g > 1$ distinct constant principal curvatures in a complex space form $\bar{M}(c)$. Since the calculations that follow are local we may assume that we have a globally defined unit normal vector field ξ. We denote by $\lambda_1, \dots, \lambda_g$ the principal curvatures of M.

By assumption, the number of nontrivial projections of $J\xi$ onto the principal curvature distributions T_{λ_i}, $i \in \{1, \dots, g\}$, is $h = 2$. By relabeling the indices we may also assume that $J\xi$ has nontrivial projection onto T_{λ_1} and T_{λ_2}. Hence, there exist unit vectors fields $U_i \in \Gamma(T_{\lambda_i})$, $i \in \{1, 2\}$, and positive smooth functions $b_i : M \to \mathbb{R}$, $i \in \{1, 2\}$, such that

$$J\xi = b_1 U_1 + b_2 U_2.$$

Obviously, $b_1^2 + b_2^2 = 1$. Moreover,

Lemma 3.1. *We have* $g \geq 3$, $\langle JU_1, U_2\rangle = 0$ *and there exists a unit vector field* $A \in \Gamma(\oplus_{k=3}^{g} T_{\lambda_k})$ *such that*

$$\begin{aligned} JU_i &= (-1)^i b_j A - b_i\xi, \quad (i,j \in \{1,2\}, i \neq j), \\ JA &= b_2U_1 - b_1U_2. \end{aligned}$$

Proof. The proof is similar to that of [5, Lemma 7], so we just sketch it. We will assume in what follows $i, j \in \{1,2\}$, $i \neq j$, and $k \in \{3,\dots,g\}$.

Since T_{λ_i}, $i \in \{1,2\}$, is real by Lemma 2.1 (i) we can write $JU_i = \langle JU_i, U_j\rangle U_j + W_{ij} + \sum_{k=3}^{g} W_{ik} - b_i\xi$, where $W_{ij} \in \Gamma(T_{\lambda_j} \ominus \mathbb{R}U_j)$ and $W_{ik} \in \Gamma(T_{\lambda_k})$. (Here and henceforth, the symbol $\ominus$ is used to denote orthogonal complement.) From $J\xi = b_1U_1 + b_2U_2$ we get

$$-\xi = J^2\xi = b_2(\langle JU_2, U_1\rangle U_1 + W_{21}) + b_1(\langle JU_1, U_2\rangle U_2 + W_{12}) + \sum_{k=3}^{g}(b_1W_{1k} + b_2W_{2k}) - \xi.$$

Thus, $g \geq 3$, $\langle JU_1, U_2\rangle = 0$, $W_{12} = W_{21} = 0$, and $b_1W_{1k} + b_2W_{2k} = 0$ for all k. If we define $A \in \Gamma(\oplus_{k=3}^{g} T_{\lambda_k})$ by $\sum_{k=3}^{g} W_{ik} = (-1)^i b_j A$, then the last equality implies $\sum_{k=3}^{g} W_{jk} = (-1)^j b_i A$ (recall $i, j \in \{1,2\}$, $i \neq j$). This gives the desired expression for JU_i, $i \in \{1,2\}$. Finally, from $b_1^2 + b_2^2 = 1$ and $-U_1 = J(JU_1) = -b_2JA - b_1J\xi = -b_2JA - U_1 + b_2^2U_1 - b_1b_2U_2$ we obtain $JA = b_2U_1 - b_1U_2$. □

3.2. **The vector field** A. In view of Lemma 3.1 we may write

$$A = \sum_{k=3}^{g} A_k, \text{ with } A_k \in \Gamma(T_{\lambda_k}), k \in \{3,\dots g\}.$$

The aim of this subsection is to show that all but one A_k are zero and hence we can assume for example that $A \in \Gamma(T_{\lambda_3})$ (Proposition 3.3). The main difficulty here is the fact that g is not known. We start with the following

Lemma 3.2. *Let* $i, j \in \{1,2\}$ *with* $i \neq j$. *Then we have*

$$\nabla_{U_i}U_i = \sum_{k=3}^{g}(-1)^j \frac{3cb_1b_2}{4(\lambda_k - \lambda_i)}A_k, \qquad \nabla_{U_i}U_j = \sum_{k=3}^{g}(-1)^j\left(\lambda_i - \frac{3cb_i^2}{4(\lambda_k - \lambda_i)}\right)A_k.$$

Proof. Again, this is quite similar to [5, Lemma 8]. We assume $i, j \in \{1,2\}$, $i \neq j$, and $k \in \{3,\dots,g\}$. Let $W_i \in \Gamma(T_{\lambda_i} \ominus \mathbb{R}U_i)$ and $W_k \in \Gamma(T_{\lambda_k} \ominus \mathbb{R}A_k)$.

Since U_i has unit length we get $\langle \nabla_{U_i}U_i, U_i\rangle = 0$. Lemma 2.1 (ii) yields $\langle \nabla_{U_i}U_i, U_j\rangle = \langle \nabla_{U_i}U_i, W_j\rangle = \langle \nabla_{U_i}U_i, W_k\rangle = 0$, and $\langle \nabla_{U_i}U_i, A_k\rangle = 3(-1)^j cb_1b_2/(4(\lambda_k - \lambda_i))$. From $\bar{\nabla}J = 0$, the Weingarten formula, and Lemma 3.1, we obtain $\langle W_i, \bar{\nabla}_{U_i}J\xi\rangle = -\lambda_i\langle W_i, JU_i\rangle = 0$. Hence, using $J\xi = b_1U_1 + b_2U_2$, and Lemma 2.1 (ii), we get

$$0 = U_i\langle W_i, J\xi\rangle = \langle \nabla_{U_i}W_i, J\xi\rangle + \langle W_i, \bar{\nabla}_{U_i}J\xi\rangle = -b_i\langle \nabla_{U_i}U_i, W_i\rangle.$$

Since $b_i \neq 0$ the expression for $\nabla_{U_i}U_i$ follows.

As U_j has unit length, $\langle \nabla_{U_i}U_j, U_j\rangle = 0$. From Lemma 2.1 (ii) we obtain $\langle \nabla_{U_i}U_j, U_i\rangle = \langle \nabla_{U_i}U_j, W_i\rangle = 0$. Now, the Weingarten formula and Lemma 3.1 imply $\langle W_j, \bar{\nabla}_{U_i}J\xi\rangle =$

$-\lambda_i\langle W_j, JU_i\rangle = 0$, and thus, Lemma 2.1 (ii), yields

$$0 = U_i\langle W_j, J\xi\rangle = \langle\nabla_{U_i}W_j, J\xi\rangle + \langle W_j, \bar{\nabla}_{U_i}J\xi\rangle = b_j\langle\nabla_{U_i}W_j, U_j\rangle.$$

This implies $\langle\nabla_{U_i}W_j, U_j\rangle = 0$. A similar calculation gives $\langle\nabla_{U_i}W_k, U_j\rangle = 0$. Finally, by Lemma 2.1 (ii), and Lemma 3.1 we have

$$0 = U_i\langle A_k, J\xi\rangle = \langle\nabla_{U_i}A_k, J\xi\rangle + \langle A_k, \bar{\nabla}_{U_i}J\xi\rangle = (-1)^i\frac{3cb_i^2b_j}{4(\lambda_k-\lambda_i)} - b_j\langle\nabla_{U_i}U_j, A_k\rangle - (-1)^i\lambda_i b_j,$$

from where we get $\langle\nabla_{U_i}U_j, A_k\rangle$. Altogether this yields the formula for $\nabla_{U_i}U_j$. $\square$

Now we can prove the main result of this section.

Proposition 3.3. *$A \in \Gamma(T_{\lambda_k})$ for some $k \in \{3, \dots, g\}$.*

Proof. On the contrary, assume that there exists a point $p \in M$ and two distinct integers $r, s \in \{3, \dots, g\}$ such that $(A_r)_p, (A_s)_p \neq 0$. Hence, in a neighborhood of p we have $A_r, A_s \neq 0$ as well. We will work in that neighborhood from now on.

Applying Lemma 2.1 (iii) to the vector fields U_1, U_2, and A_k, $k \in \{r, s\}$, and using Lemma 3.2 we easily get

$$\text{(1)}\qquad \frac{3c(\lambda_2-\lambda_k)}{4(\lambda_1-\lambda_k)}b_1^2 + \frac{3c(\lambda_1-\lambda_k)}{4(\lambda_2-\lambda_k)}b_2^2 = -\frac{c}{4} - \lambda_1(\lambda_2-\lambda_k) - \lambda_2(\lambda_1-\lambda_k), \quad k \in \{r, s\}.$$

Together with $b_1^2 + b_2^2 = 1$, this yields a linear system of three equations with unknowns b_1^2 and b_2^2. This system must be compatible. We show it is determined (that is, it has a unique solution). If it were not, the rank of the system would, at most, be one. In particular,

$$\begin{vmatrix} \frac{3c(\lambda_2-\lambda_k)}{4(\lambda_1-\lambda_k)} & \frac{3c(\lambda_1-\lambda_k)}{4(\lambda_2-\lambda_k)} \\ 1 & 1 \end{vmatrix} = 3c\frac{(\lambda_2-\lambda_1)(\lambda_1+\lambda_2-2\lambda_k)}{4(\lambda_1-\lambda_k)(\lambda_2-\lambda_k)} = 0, \quad k \in \{r, s\},$$

which implies $\lambda_1 + \lambda_2 - 2\lambda_k = 0$, $k \in \{r, s\}$, and hence $\lambda_r = \lambda_s$, contradiction. We conclude that the above system is determined. Therefore, we can find an expression for b_1^2 and b_2^2 in terms of the principal curvatures and c. Since these are constant, it follows that b_1 and b_2 are constant.

We take $i, j \in \{1, 2\}$, $i \neq j$, and $k \in \{r, s\}$. Since b_i is constant and U_i has unit length, using $J\xi = b_1U_1 + b_2U_2$, the Weingarten formula, and Lemma 3.1 we get

$$0 = A_k(b_i) = A_k\langle U_i, J\xi\rangle = \langle\nabla_{A_k}U_i, J\xi\rangle + \langle U_i, \bar{\nabla}_{A_k}J\xi\rangle = b_j\langle\nabla_{A_k}U_i, U_j\rangle - (-1)^j b_j\lambda_k,$$

and thus, $\langle\nabla_{A_k}U_i, U_j\rangle = (-1)^j\lambda_k$. Taking this, Lemma 3.1, and Lemma 3.2 into account, Lemma 2.1 (iii) for A_k, U_1 and U_2 yields

$$\frac{c}{4}(2b_2^2 - b_1^2) = \langle\bar{R}(A_k, U_1)U_2, \xi\rangle = (\lambda_1-\lambda_2)\lambda_k + (\lambda_k-\lambda_2)\left(\lambda_1 - \frac{3cb_1^2}{4(\lambda_k-\lambda_1)}\right), \quad k \in \{r, s\}.$$

We can rearrange this as:

$$\text{(2)}\qquad \left(\frac{c}{4} - \frac{3c(\lambda_k-\lambda_2)}{4(\lambda_k-\lambda_1)}\right)b_1^2 - \frac{c}{2}b_2^2 = (\lambda_2-\lambda_1)\lambda_k + \lambda_1(\lambda_2-\lambda_k), \quad k \in \{r, s\}.$$

Hence, (1), (2), and $b_1^2+b_2^2=1$ give a linear system of five equations with unknowns b_1^2 and b_2^2. This system is compatible by assumption, so it has rank two. Then, all minors of order three of the augmented matrix of the system vanish. This implies (take (1), (2), and $b_1^2+b_2^2=1$, with $k\in\{r,s\}$, and then both equations in (2) and $b_1^2+b_2^2=1$):

$$\frac{3c(\lambda_1-\lambda_2)^2(-12\lambda_k^2+8\lambda_1\lambda_k+8\lambda_2\lambda_k+c-4\lambda_1\lambda_2)}{16(\lambda_1-\lambda_k)(\lambda_k-\lambda_2)} = 0,\ k\in\{r,s\}, \tag{3}$$

$$\frac{3c(\lambda_2-\lambda_1)(\lambda_r-\lambda_s)(4\lambda_1^2-4\lambda_r\lambda_1-4\lambda_s\lambda_1+c+2\lambda_2\lambda_r+2\lambda_2\lambda_s)}{8(\lambda_1-\lambda_r)(\lambda_1-\lambda_s)} = 0. \tag{4}$$

In particular, (3) implies $-12\lambda_k^2+8\lambda_1\lambda_k+8\lambda_2\lambda_k+c-4\lambda_1\lambda_2=0$. Putting $k=r$ and $k=s$, and subtracting, we get $4(2\lambda_1+2\lambda_2-3\lambda_r-3\lambda_s)(\lambda_r-\lambda_s)=0$, from where we obtain $\lambda_r+\lambda_s=2(\lambda_1+\lambda_2)/3$. Taking this into account, (4) gives $(4\lambda_1^2-4\lambda_1\lambda_2+4\lambda_2^2+3c)/3=0$. The discriminant of $-12\lambda_k^2+8\lambda_1\lambda_k+8\lambda_2\lambda_k+c-4\lambda_1\lambda_2=0$ as a quadratic equation in λ_k is precisely $16(4\lambda_1^2-4\lambda_1\lambda_2+4\lambda_2^2+3c)$, so this discriminant vanishes. As a consequence, this quadratic equation has a unique solution and hence $\lambda_r=\lambda_s$. This is a contradiction. Therefore, all but one A_k, $k\in\{3,\dots,g\}$, are zero for each p. The result follows by continuity. $\square$

3.3. **Some properties of the principal curvature spaces.** In view of Proposition 3.3, we may assume from now on that $A\in\Gamma(T_{\lambda_3})$. Moreover, we can choose an orientation on M and a relabeling of the indices so that

$$\lambda_1<\lambda_2, \quad \text{and} \quad \lambda_3\geq 0.$$

We will follow this convention from now on.

First we calculate some covariant derivatives.

Lemma 3.4. *Let $i,j\in\{1,2\}$ with $i\neq j$. Then we have*

$$\nabla_{U_i}U_i = (-1)^j\frac{3cb_1b_2}{4(\lambda_3-\lambda_i)}A, \tag{5}$$

$$\nabla_{U_i}U_j = (-1)^j\left(\lambda_i-\frac{3cb_i^2}{4(\lambda_3-\lambda_i)}\right)A, \tag{6}$$

$$\nabla_{U_i}A = (-1)^i\frac{3cb_1b_2}{4(\lambda_3-\lambda_i)}U_i+(-1)^i\left(\lambda_i-\frac{3cb_i^2}{4(\lambda_3-\lambda_i)}\right)U_j, \tag{7}$$

$$\nabla_A U_i = \frac{(-1)^j}{\lambda_i-\lambda_j}\left(\frac{c(2b_j^2-b_i^2)}{4}+(\lambda_j-\lambda_3)\left(\lambda_i-\frac{3cb_i^2}{4(\lambda_3-\lambda_i)}\right)\right)U_j, \tag{8}$$

$$\nabla_A A = 0. \tag{9}$$

Proof. The proof is similar to that of [5, Lemma 8]. Equations (5) and (6) are a direct consequence of Lemma 3.2 and Proposition 3.3. Assume $i,j\in\{1,2\}$, $i\neq j$, and $k\in\{4,\dots,g\}$. Let $W_i\in\Gamma(T_{\lambda_i}\ominus\mathbb{R}U_i)$, $W_3\in\Gamma(T_{\lambda_3}\ominus\mathbb{R}A)$ and $W_k\in\Gamma(T_{\lambda_k})$.

According to (5) and (6), in order to prove (7) we have to show $\langle \nabla_{U_i} A, A\rangle = 0$ (obvious because A is a unit vector field), and $\langle \nabla_{U_i} A, W_l\rangle = 0$ for all $l \in \{1, \dots, g\}$. The latter follows after using $\bar{\nabla} J = 0$, the Weingarten formula, Lemma 3.1, and (5), with

$$\begin{aligned} 0 &= U_i\langle JU_i, W_l\rangle = \langle \bar{\nabla}_{U_i} JU_i, W_l\rangle + \langle JU_i, \bar{\nabla}_{U_i} W_l\rangle \\ &= -\langle \nabla_{U_i} U_i, JW_l\rangle + (-1)^i b_j \langle A, \nabla_{U_i} W_l\rangle - b_i\langle \xi, \bar{\nabla}_{U_i} W_l\rangle = (-1)^j b_j \langle \nabla_{U_i} A, W_l\rangle. \end{aligned}$$

We now prove (8). Obviously, $\langle \nabla_A U_i, U_i\rangle = 0$, and $\langle \nabla_A U_i, A\rangle = 0$ by Lemma 2.1 (ii). Applying Lemma 2.1 (iii) to A, U_i and U_j, using Lemma 3.1 and (6), gives

$$\frac{c}{4}(-1)^i(b_i^2 - 2b_j^2) = (\lambda_i - \lambda_j)\langle \nabla_A U_i, U_j\rangle - (\lambda_3 - \lambda_j)(-1)^i \left(\lambda_i - \frac{3cb_i^2}{4(\lambda_3 - \lambda_i)}\right),$$

from where we get $\langle \nabla_A U_i, U_j\rangle$. For $l \in \{j, 3, \dots, g\}$, a similar argument with Lemma 2.1 (iii) applied to A, U_i, and W_l, taking Lemma 3.1 and (7) into account, yields $\langle \nabla_A U_i, W_l\rangle = 0$. Finally, the previous equality (interchanging i and j and putting $l = i$) gives

$$\begin{aligned} 0 &= A\langle W_i, J\xi\rangle = \langle \nabla_A W_i, J\xi\rangle + \langle W_i, \bar{\nabla}_A J\xi\rangle \\ &= b_i\langle \nabla_A W_i, U_i\rangle + b_j\langle \nabla_A W_i, U_j\rangle - \lambda_3\langle W_i, JA\rangle = -b_i\langle \nabla_A U_i, W_i\rangle. \end{aligned}$$

Altogether this proves (8).

We have $\langle \nabla_A A, A\rangle = 0$, and $\langle \nabla_A A, U_i\rangle = \langle \nabla_A A, W_l\rangle = 0$ for all $l \in \{1, 2, 4, \dots, g\}$ by Lemma 2.1 (ii). From $\bar{\nabla} J = 0$, (8), Lemma 3.1, and the Weingarten formula we get

$$\begin{aligned} 0 &= A\langle JU_i, W_3\rangle = \langle \bar{\nabla}_A JU_i, W_3\rangle + \langle JU_i, \bar{\nabla}_A W_3\rangle \\ &= -\langle \nabla_A U_i, JW_3\rangle + (-1)^i b_j\langle A, \nabla_A W_3\rangle - b_i\langle \xi, \bar{\nabla}_A W_3\rangle = (-1)^j b_j\langle \nabla_A A, W_3\rangle. \end{aligned}$$

from where (9) follows. ☐

Our main difficulty from now on is the fact that the number g of principal curvatures is not known. In fact, the aim of Subsection 3.4 is to obtain a bound on g. An important step in the proof is the following

Proposition 3.5. *The functions b_1 and b_2 are constant. In fact*

$$b_i^2 = \frac{4(\lambda_j - 2\lambda_3)(\lambda_i - \lambda_3)^2}{c(\lambda_i - \lambda_j)}, \quad (i, j \in \{1, 2\}, i \neq j).$$

Moreover, $c - 4\lambda_1\lambda_2 + 8(\lambda_1 + \lambda_2)\lambda_3 - 12\lambda_3^2 = 0$.

Proof. First we show that the functions b_1 and b_2 are constant.

We apply Lemma 2.2 to U_1 and U_2, using Lemma 3.1 and Lemma 3.4,

$$\begin{aligned} 0 = {} & (\lambda_2 - \lambda_1)(-c - 4\lambda_1\lambda_2 + 8\langle \nabla_{U_1} U_2, \nabla_{U_2} U_1\rangle - 4\langle \nabla_{U_1} U_1, \nabla_{U_2} U_2\rangle) \\ & -cb_1(\langle \nabla_{U_2} U_1, JU_2\rangle - 2\langle \nabla_{U_1} U_2, JU_2\rangle) - cb_2(-\langle \nabla_{U_1} U_2, JU_1\rangle + 2\langle \nabla_{U_2} U_1, JU_1\rangle) \\ = {} & -(\lambda_2 - \lambda_1)(c + 12\lambda_1\lambda_2) - \frac{3c^2}{2(\lambda_3 - \lambda_1)} b_1^4 + \frac{3c^2}{2(\lambda_3 - \lambda_2)} b_2^4 + \frac{3c^2(\lambda_1 - \lambda_2)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} b_1^2 b_2^2 \\ & + \frac{c(6\lambda_2^2 - 7\lambda_1\lambda_2 - 2\lambda_1^2 + 2\lambda_1\lambda_3 + \lambda_2\lambda_3)}{\lambda_3 - \lambda_1} b_1^2 - \frac{c(6\lambda_1^2 - 7\lambda_1\lambda_2 - 2\lambda_2^2 + 2\lambda_2\lambda_3 + \lambda_1\lambda_3)}{\lambda_3 - \lambda_2} b_2^2. \end{aligned}$$

Now we substitute b_2^2 by $1-b_1^2$ to get

$$0=\frac{9c^2(\lambda_2-\lambda_1)}{2(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}b_1^4+\Lambda_1 b_1^2+\Lambda_0,$$

where Λ_1 and Λ_0 are constants depending on c, λ_1, λ_2, and λ_3. This equation is a quadratic equation in b_1^2 and the coefficient of b_1^4 does not vanish. Hence, it has at most two real solutions depending on the constants c, λ_1, λ_2 and λ_3. Since M is connected it follows that b_1 and b_2 are constant.

From the argument above one might derive an explicit expression for b_i, $i\in\{1,2\}$. However, that expression would involve square roots that would make later calculations difficult. Instead, we use the constancy of these functions to give an alternative formula which is easier to handle. For $i\in\{1,2\}$, using lemmas 3.1 and 3.4, and the Weingarten formula, we get

$$\begin{aligned}0 &= A(b_i) = A\langle U_i,J\xi\rangle = \langle\nabla_A U_i,J\xi\rangle+\langle U_i,\bar\nabla_A J\xi\rangle = b_j\langle\nabla_A U_i,U_j\rangle-\lambda_3\langle U_i,JA\rangle\\ &= (-1)^i b_j\left(c\frac{-\lambda_i+3\lambda_j-2\lambda_3}{4(\lambda_i-\lambda_j)(\lambda_3-\lambda_i)}b_i^2-\frac{c}{2(\lambda_i-\lambda_j)}b_j^2+\frac{2\lambda_i\lambda_3-\lambda_j\lambda_3-\lambda_i\lambda_j}{\lambda_i-\lambda_j}\right).\end{aligned}$$

Together with $b_1^2+b_2^2=1$, this gives a linear system of three equations with unknowns b_1^2 and b_2^2. Since this system is compatible by hypothesis, its rank is two and hence the determinant of its augmented matrix is zero. This implies

$$\frac{3c}{16(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}(c-4\lambda_1\lambda_2+8\lambda_3(\lambda_1+\lambda_2)-12\lambda_3^2)=0.$$

Solving the above system is only a matter of linear algebra. After some calculations we get $b_i^2=4(\lambda_j-2\lambda_3)(\lambda_i-\lambda_3)^2/(c(\lambda_i-\lambda_j))$ from where the result follows. □

We are now able to derive an important relation among λ_1, λ_2 and λ_3.

Proposition 3.6. *We have $c<0$. In this case, we get*

$$\lambda_i=\frac{1}{2}\left(3\lambda_3+(-1)^i\sqrt{-c-3\lambda_3^2}\right),\quad (i,j\in\{1,2\},i\neq j).$$

In particular, $\lambda_1<\lambda_3<\lambda_2$. Moreover, $c+4\lambda_3^2<0$, or equivalently, $0\le\lambda_3<\sqrt{-c}/2$.

Proof. Let $i,j\in\{1,2\}$ with $i\neq j$. Using Lemma 3.1, the constancy of b_i, and then Lemma 3.4, we get by Lemma 2.2 applied to U_i and A

$$\begin{aligned}0 &= (\lambda_3-\lambda_i)(-c-4\lambda_i\lambda_3-2cb_j^2+8\langle\nabla_{U_i}A,\nabla_A U_i\rangle)\\ &\quad -cb_i((-1)^i b_i\langle\nabla_A U_i,U_j\rangle-2(-1)^j b_j\langle\nabla_{U_i}A,U_i\rangle-2(-1)^i b_i\langle\nabla_{U_i}A,U_j\rangle)\\ &= \frac{c^2(\lambda_i-15\lambda_j+14\lambda_3)}{4(\lambda_3-\lambda_i)(\lambda_i-\lambda_j)}b_i^4+\frac{c^2(-10\lambda_i+3\lambda_j+7\lambda_3)}{2(\lambda_3-\lambda_i)(\lambda_i-\lambda_j)}b_i^2b_j^2-\frac{11c\lambda_i(\lambda_3-\lambda_j)}{\lambda_i-\lambda_j}b_i^2\\ &\quad -\frac{2c(\lambda_3-\lambda_i)(3\lambda_i-\lambda_j)}{\lambda_i-\lambda_j}b_j^2-\frac{(\lambda_3-\lambda_i)(c\lambda_i-c\lambda_j+8\lambda_i^2\lambda_j-4\lambda_3\lambda_i\lambda_j-4\lambda_3\lambda_i^2)}{\lambda_i-\lambda_j}.\end{aligned}$$

Now substituting b_i^2, $i \in \{1,2\}$, by the expressions given in Proposition 3.5, after multiplying by $(\lambda_j - \lambda_i)/(\lambda_i - \lambda_3)$ and some long calculations we get

$$72\lambda_3^3 - 48\lambda_i\lambda_3^2 - 108\lambda_j\lambda_3^2 + 4\lambda_i^2\lambda_3 + 32\lambda_j^2\lambda_3 + 72\lambda_i\lambda_j\lambda_3 - 16\lambda_i\lambda_j^2 - c\lambda_i - 8\lambda_i^2\lambda_j + c\lambda_j = 0.$$

Subtracting the above equation for $i = 2$ from the one with $i = 1$ we get $2(\lambda_1 - \lambda_2)(c - 4\lambda_1\lambda_2 + 14\lambda_3(\lambda_1 + \lambda_2) - 30\lambda_3^2) = 0$. Combining this with $c - 4\lambda_1\lambda_2 + 8(\lambda_1 + \lambda_2)\lambda_3 - 12\lambda_3^2 = 0$ (Proposition 3.5), we get $6\lambda_3(\lambda_1 + \lambda_2 - 3\lambda_3) = 0$. If $\lambda_3 = 0$, then the equation above gives $c(\lambda_i - \lambda_j) + 8\lambda_i^2\lambda_j + 16\lambda_i\lambda_j^2 = 0$, which combined with Proposition 3.5 yields $3c(\lambda_1 + \lambda_2) = 0$. This implies $\lambda_1 + \lambda_2 - 3\lambda_3 = \lambda_1 + \lambda_2 = 0$, so it suffices to deal with the case $\lambda_1 + \lambda_2 - 3\lambda_3 = 0$. In this situation we substitute λ_1 by $-\lambda_2 + 3\lambda_3$ in the equation in Proposition 3.5, thus obtaining $c + 4\lambda_2^2 - 12\lambda_2\lambda_3 + 12\lambda_3^2 = 0$. This is a quadratic equation with unknown λ_2 and discriminant $-c - 3\lambda_3^2$. So that this discriminant is nonnegative we already need $c < 0$, proving the first claim of this proposition. The solution to this equation is one of

$$\frac{1}{2}\left(3\lambda_3 \pm \sqrt{-c - 3\lambda_3^2}\right).$$

On the other hand, λ_1 is also one of the two values above. Since $\lambda_1 < \lambda_2$ by hypothesis, we get $c + 3\lambda_3^2 < 0$ and $\lambda_i = \frac{1}{2}\left(3\lambda_3 + (-1)^i\sqrt{-c - 3\lambda_3^2}\right)$.

Finally, we show that $0 \le \lambda_3 < \sqrt{-c}/2$. We already know that $0 \le \lambda_3 < \sqrt{-c/3}$. Substituting the above expression for λ_i, $i \in \{1,2\}$, in Proposition 3.5 we get

$$b_i^2 = -\frac{\left((-1)^i\lambda_3 + \sqrt{-c - 3\lambda_3^2}\right)^3}{2c\sqrt{-c - 3\lambda_3^2}}, \quad i \in \{1,2\}.$$

If $\sqrt{-c}/2 \le \lambda_3 < \sqrt{-c/3}$, then $-c - 4\lambda_3^2 \le 0$, and hence $-\lambda_3 + \sqrt{-c - 3\lambda_3^2} \le 0$. This implies $b_1^2 \le 0$, a contradiction. Therefore $0 \le \lambda_3 < \sqrt{-c}/2$ and the result follows. ☐

Proposition 3.6 already implies that there are no hypersurfaces with constant principal curvatures in $\mathbb{C}P^n(c)$, $n \ge 2$, whose Hopf vector field has $h = 2$ nontrivial projections onto the principal curvature spaces. From now on we can assume $c < 0$.

Corollary 3.7. *The distribution T_{λ_k} is totally real for all $k \in \{4, \dots, g\}$.*

Proof. Let $k \in \{4, \dots, g\}$ and take unit vector fields $V_k, W_k \in \Gamma(T_{\lambda_k})$. Using the Weingarten equation, Lemma 2.1 (ii), Proposition 3.5, and $\lambda_1 + \lambda_2 - 3\lambda_3 = 0$ (by Proposition 3.6) we get

$$\begin{aligned} 0 &= V_k\langle W_k, J\xi\rangle = \langle \nabla_{V_k} W_k, b_1U_1 + b_2U_2\rangle + \langle W_k, \bar\nabla_{V_k} J\xi\rangle \\ &= \left(\frac{cb_1^2}{4(\lambda_k - \lambda_1)} + \frac{cb_2^2}{4(\lambda_k - \lambda_2)} - \lambda_k\right)\langle JV_k, W_k\rangle = \frac{(\lambda_3 - \lambda_k)^3}{(\lambda_k - \lambda_1)(\lambda_k - \lambda_2)}\langle JV_k, W_k\rangle. \end{aligned}$$

Since $\lambda_k \ne \lambda_3$, we get $\langle JV_k, W_k\rangle = 0$. As V_k and W_k are arbitrary, the result follows. ☐

3.4. **A bound on the number of principal curvatures.** In this section we show, using the Gauss equation and some inequalities involving the principal curvatures, that the number g of distinct principal curvatures satisfies $g \in \{3,4\}$. This allows us to obtain further properties of the principal curvature spaces (see Proposition 3.11). We start with the Gauss equation.

Lemma 3.8. *Let us denote by $(\cdot)_i$, $i \in \{1,2\}$, the orthogonal projection onto the distribution $T_{\lambda_i} \ominus \mathbb{R}U_i$, and by $(\cdot)_k$, $k \in \{4,\dots,g\}$, the orthogonal projection onto T_{λ_k}. By $\|\cdot\|$ we denote the norm of a vector. Then we have:*

(i) *Let $i \in \{1,2\}$ and $W_i \in \Gamma(T_{\lambda_i} \ominus \mathbb{R}U_i)$ be a unit vector field. If $j \in \{1,2\}$ and $j \neq i$ then*

$$0 = -(c + 4\lambda_3\lambda_i) + 8\frac{\lambda_i - \lambda_j}{\lambda_3 - \lambda_j}\|(\nabla_A W_i)_j\|^2 + 8\sum_{k=4}^{g}\frac{\lambda_i - \lambda_k}{\lambda_3 - \lambda_k}\|(\nabla_A W_i)_k\|^2.$$

(ii) *Let $k \in \{4,\dots,g\}$ and $W_k \in \Gamma(T_{\lambda_k})$ be a unit vector field. Then*

$$\begin{aligned} 0 &= -(c + 4\lambda_3\lambda_k) + 8\frac{\lambda_k - \lambda_1}{\lambda_3 - \lambda_1}\|(\nabla_A W_k)_1\|^2 + 8\frac{\lambda_k - \lambda_2}{\lambda_3 - \lambda_2}\|(\nabla_A W_k)_2\|^2 \\ &\quad + 8\sum_{l=4,\, l\neq k}^{g}\frac{\lambda_k - \lambda_l}{\lambda_3 - \lambda_l}\|(\nabla_A W_k)_l\|^2. \end{aligned}$$

Proof. As usual, let $i, j \in \{1,2\}$ with $i \neq j$, and $k \in \{4,\dots,g\}$.

Let $W_i \in \Gamma(T_{\lambda_i}) \ominus \mathbb{R}U_i$ be a unit vector field. Applying Lemma 2.2 to W_i and A we get

$$-c - 4\lambda_3\lambda_i + 8\langle \nabla_{W_i}A, \nabla_A W_i\rangle = 0. \tag{10}$$

If $W_3 \in \Gamma(T_{\lambda_3})$, we get from Lemma 2.1 (ii) that $\langle \nabla_A W_i, W_3\rangle = 0$. This and Lemma 3.4 yield $\nabla_A W_i \in \Gamma((T_{\lambda_1} \ominus \mathbb{R}U_1) \oplus (T_{\lambda_2} \ominus \mathbb{R}U_2) \oplus T_{\lambda_4} \oplus \cdots \oplus T_{\lambda_g})$. Similarly, Lemma 2.1 (ii) implies $\nabla_{W_i}A \in \Gamma(T_{\lambda_j} \oplus (T_{\lambda_3} \ominus \mathbb{R}A) \oplus T_{\lambda_4} \oplus \cdots \oplus T_{\lambda_g})$. Hence $\langle \nabla_{W_i}A, \nabla_A W_i\rangle = \langle \nabla_{W_i}A, (\nabla_A W_i)_j\rangle + \sum_{k=4}^{g}\langle \nabla_{W_i}A, (\nabla_A W_i)_k\rangle$. For each addend of this sum we apply Lemma 2.1 (iii). Since $\langle \bar{R}(W_i, A)(\nabla_A W_i)_l, \xi\rangle = 0$ for all $l \in \{j, 4, \dots, g\}$ we get

$$\langle \nabla_{W_i}A, \nabla_A W_i\rangle = \frac{\lambda_i - \lambda_j}{\lambda_3 - \lambda_j}\langle \nabla_A W_i, (\nabla_A W_i)_j\rangle + \sum_{k=4}^{g}\frac{\lambda_i - \lambda_k}{\lambda_3 - \lambda_k}\langle \nabla_A W_i, (\nabla_A W_i)_k\rangle.$$

Now, part (i) follows by substituting the previous expression in (10).

Part (ii) follows in a similar way by applying Lemma 2.2 to W_k and A. □

We will use the following technical lemma several times in what follows.

Lemma 3.9. *Assume $g \geq 4$ and let $k \in \{4,\dots,g\}$. Assume that one of the following statements is true:*

(i) $\dim T_{\lambda_1} = \dim T_{\lambda_2} = 1$, *or*
(ii) $\dim T_{\lambda_1} = 1$ *and* $\lambda_k < \lambda_2$, *or*
(iii) $\lambda_1 < \lambda_k < \lambda_2$.

Then, $c + 4\lambda_3\lambda_k \geq 0$.

Proof. On the contrary, assume $c + 4\lambda_3\lambda_k < 0$. Let $W_k \in \Gamma(T_{\lambda_k})$ be a (local) unit vector field. When we apply Lemma 3.8 (ii) to W_k, any of the assumptions ensures that the first three addends of the equation given in Lemma 3.8 (ii) are nonnegative with the first one strictly positive. This already implies $g > 4$. In this case, it follows that there exists $r \in \{4, \dots, g\}$, $r \neq k$, such that $(\lambda_k - \lambda_r)/(\lambda_3 - \lambda_r) < 0$. We may choose λ_r to be the principal curvature that minimizes $|\lambda_3 - \lambda_l|$ among all λ_l, $l \in \{4, \dots, g\}$, $l \neq k$, with $(\lambda_k - \lambda_l)/(\lambda_3 - \lambda_l) < 0$. In particular we have

$$\text{either} \quad \lambda_k < \lambda_r < \lambda_3 \quad \text{or} \quad \lambda_3 < \lambda_r < \lambda_k. \tag{11}$$

It follows that λ_r satisfies the same assumption as λ_k: this is obvious for (i) and a consequence of (11) and $\lambda_1 < \lambda_3 < \lambda_2$ for (ii) and (iii). Using (11), $\lambda_3 \geq 0$, $c + 4\lambda_3^2 < 0$ (Proposition 3.6), and $c + 4\lambda_3\lambda_k < 0$, we also get $c + 4\lambda_3\lambda_r \leq c + 4\lambda_3 \max\{\lambda_3, \lambda_k\} < 0$. Thus we may apply Lemma 3.8 (ii) to a unit vector field $W_r \in \Gamma(T_{\lambda_r})$, from where it follows, as before, that there exists $s \in \{4, \dots, g\}$, $s \neq r$, such that $(\lambda_r - \lambda_s)/(\lambda_3 - \lambda_s) < 0$. This implies either $\lambda_r < \lambda_s < \lambda_3$ or $\lambda_3 < \lambda_s < \lambda_r$, and taking (11) into account we easily obtain

$$\text{either} \quad \lambda_k < \lambda_r < \lambda_s < \lambda_3 \quad \text{or} \quad \lambda_3 < \lambda_s < \lambda_r < \lambda_k. \tag{12}$$

In both cases (12) yields $s \neq k$, $(\lambda_k - \lambda_s)/(\lambda_3 - \lambda_s) < 0$, and $|\lambda_3 - \lambda_s| < |\lambda_3 - \lambda_r|$. This contradicts the definition of λ_r. Therefore, $c + 4\lambda_3\lambda_k \geq 0$. ☐

From the previous lemma we easily derive the first important consequence.

Proposition 3.10. *We have* $\dim T_{\lambda_1} = 1$.

Proof. On the contrary, assume $\dim T_{\lambda_1} > 1$ and let $W_1 \in \Gamma(T_{\lambda_1} \ominus \mathbb{R}U_1)$ be a (local) unit vector field. Since $c + 4\lambda_1\lambda_3 \leq c + 4\lambda_3^2 < 0$ by Proposition 3.6, from Lemma 3.8 (i) we deduce the existence of $k \in \{4, \dots, g\}$ such that $(\lambda_1 - \lambda_k)/(\lambda_3 - \lambda_k) < 0$. Since $\lambda_1 < \lambda_3$ we get $\lambda_1 < \lambda_k < \lambda_3 < \lambda_2$ and hence Lemma 3.9 (iii) yields $c + 4\lambda_3\lambda_k \geq 0$. This contradicts $c + 4\lambda_3\lambda_k \leq c + 4\lambda_3^2 < 0$. Therefore $\dim T_{\lambda_1} = 1$. ☐

This is the most crucial step of the proof.

Proposition 3.11. *We have*

(i) $g \in \{3, 4\}$.

(ii) *If* $g = 3$ *and* $\dim T_{\lambda_2} > 1$ *then* $\lambda_1 = 0$, $\lambda_2 = \frac{\sqrt{-3c}}{2}$, *and* $\lambda_3 = \frac{\sqrt{-c}}{2\sqrt{3}}$.

(iii) *If* $g = 4$ *then* $\dim T_{\lambda_2} = 1$, $0 \neq \lambda_3 \neq \frac{\sqrt{-c}}{2\sqrt{3}}$, *and* $\lambda_4 = -\frac{c}{4\lambda_3}$.

Proof. If $g = 3$ and $\dim T_{\lambda_2} > 1$, take a (local) unit $W_2 \in \Gamma(T_{\lambda_2} \ominus \mathbb{R}U_2)$ and apply Lemma 3.8 (i). Note that the last two addends vanish since $\dim T_{\lambda_1} = 1$ and $g = 3$. Then, $c + 4\lambda_2\lambda_3 = 0$, and from Proposition 3.6 we get $\lambda_1 = 0$, $\lambda_2 = \sqrt{-3c}/2$, and $\lambda_3 = \sqrt{-c}/(2\sqrt{3})$. This implies (ii).

Assume $g \geq 4$. We first have $\lambda_3 < \lambda_k$ for all $k \in \{4, \dots, g\}$; otherwise, if $\lambda_k < \lambda_3 < \lambda_2$ we would get $c + 4\lambda_3\lambda_k \leq c + 4\lambda_3^2 < 0$ contradicting Lemma 3.9 (ii) (by Proposition 3.10).

We show that $\dim T_{\lambda_2} = 1$. On the contrary, assume $\dim T_{\lambda_2} > 1$ and let $W_2 \in \Gamma(T_{\lambda_2} \ominus \mathbb{R}U_2)$ be a (local) unit vector field. If $c + 4\lambda_2\lambda_3 < 0$, then Lemma 3.8 (i) applied to W_2

(and taking Proposition 3.10 into account) implies that there exists $k \in \{4, \dots, g\}$ such that $(\lambda_2 - \lambda_k)/(\lambda_3 - \lambda_k) < 0$. Then, $\lambda_3 < \lambda_k < \lambda_2$, and thus $c + 4\lambda_3\lambda_k \leq c + 4\lambda_3\lambda_2 < 0$, which contradicts Lemma 3.9 (ii). Hence we can assume from now on that $c + 4\lambda_2\lambda_3 \geq 0$. This inequality does not hold if $\lambda_3 = 0$ so we already get $\lambda_3 > 0$.

We claim that there exists $r \in \{4, \dots, g\}$ such that $\lambda_2 < \lambda_r$. If $c + 4\lambda_2\lambda_3 = 0$, then the assertion is true for all $k \geq 4$; otherwise, if $\lambda_k < \lambda_2$, we would get $c+4\lambda_3\lambda_k < c+4\lambda_3\lambda_2 = 0$, contradicting Lemma 3.9 (ii). Hence, we have to prove our claim for the case $c+4\lambda_2\lambda_3 > 0$. In this case we apply Lemma 3.8 (i) to W_2. Then, there exists $r \in \{4, \dots, g\}$ such that $(\lambda_2 - \lambda_r)/(\lambda_3 - \lambda_r) > 0$. Since $\lambda_3 < \lambda_r$ this implies $\lambda_2 < \lambda_r$ as claimed.

In any case, there exists $r \in \{4, \dots, g\}$ such that $\lambda_2 < \lambda_r$. In fact, we may assume that λ_r is the largest principal curvature. Now, we have $c + 4\lambda_3\lambda_r > c + 4\lambda_3\lambda_2 \geq 0$, and hence Lemma 3.8 (ii) applied to a unit vector field $W_r \in \Gamma(T_{\lambda_r})$ implies the existence of $l \in \{4, \dots, g\}$, $l \neq r$, such that $(\lambda_r - \lambda_l)/(\lambda_3 - \lambda_l) > 0$. Since $\lambda_3 < \lambda_l$, we get $\lambda_r < \lambda_l$ which contradicts the fact that λ_r is the largest principal curvature. Altogether this implies $\dim T_{\lambda_2} = 1$.

From Lemma 3.9 (i) we obtain $c + 4\lambda_3\lambda_k \geq 0$ for all $k \geq 4$. In particular this implies $\lambda_3 > 0$. Assume that for some $r \in \{4, \dots, g\}$ we have strict inequality $c + 4\lambda_3\lambda_r > 0$ and let λ_r, $r \in \{4, \dots, g\}$, be the largest principal curvature satisfying this condition. Applying Lemma 3.8 (ii) once more to a unit $W_r \in \Gamma(T_{\lambda_r})$ (note that the second addend now vanishes) yields the existence of $l \in \{4, \dots, g\}$, $l \neq r$, such that $(\lambda_r - \lambda_l)/(\lambda_3 - \lambda_l) > 0$. Since $\lambda_3 < \lambda_l$ we get $\lambda_r < \lambda_l$. Obviously, $c + 4\lambda_3\lambda_l > c + 4\lambda_3\lambda_r > 0$, which contradicts the fact that λ_r is the largest principal curvature satisfying this condition.

As a consequence, $c + 4\lambda_3\lambda_k = 0$ for all $k \geq 4$. Since $\lambda_3 \neq 0$ and the principal curvatures are different, this immediately implies $g = 4$ and $\lambda_4 = -c/(4\lambda_3)$. Eventually, this also yields $c + 4\lambda_3\lambda_2 \neq 0$ and thus, by Proposition 3.6, $\lambda_3 \neq \sqrt{-c}/(2\sqrt{3})$ (otherwise the principal curvatures would not be different). This concludes the proof of (i) and (iii). □

Part (ii) of Proposition 3.11 had already been obtained in [5] by different methods. We have included a proof here as it is almost effortless to do so.

3.5. **The eigenvalue structure of the shape operator.** We summarize the results obtained so far:

Theorem 3.12. *We have:*

(a) *There are no real hypersurfaces with constant principal curvatures in $\mathbb{C}P^n(c)$, $n \geq 2$, whose Hopf vector field has $h = 2$ nontrivial projections onto the principal curvature spaces.*

(b) *Let M be a connected real hypersurface with g distinct constant principal curvatures $\lambda_1, \dots, \lambda_g$ in $\mathbb{C}H^n(c)$, $n \geq 2$, such that the number of nontrivial projections of its Hopf vector field $J\xi$ onto the principal curvature spaces of M is $h = 2$. Then, $g \in \{3, 4\}$ and, with a suitable labeling of the principal curvatures and a suitable choice of the normal vector field ξ, we have:*

(i) *The Hopf vector field can be written as* $J\xi = b_1 U_1 + b_2 U_2$*, where* $U_i \in \Gamma(T_{\lambda_i})$*,* $i \in \{1, 2\}$*, are unit vector fields, and* b_1 *and* b_2 *are positive constants satisfying*

$$b_i^2 = \frac{4(\lambda_j - 2\lambda_3)(\lambda_i - \lambda_3)^2}{c(\lambda_i - \lambda_j)}, \quad (i, j \in \{1, 2\}, i \neq j).$$

(ii) *There exists a unit vector field* $A \in \Gamma(T_{\lambda_3})$ *such that*

$$JU_i = (-1)^i b_j A - b_i \xi, \quad (i, j \in \{1, 2\}, i \neq j), \quad \text{and} \quad JA = b_2 U_1 - b_1 U_1.$$

(iii) *We have* $0 \leq \lambda_3 < \frac{1}{2}\sqrt{-c}$*, and*

$$\lambda_i = \frac{1}{2}\left(3\lambda_3 + (-1)^i \sqrt{-c - 3\lambda_3^2}\right), \quad (i, j \in \{1, 2\}, i \neq j).$$

(iv) $\dim T_{\lambda_1} = 1$.

(v) *If* $g = 4$ *then* $\dim T_{\lambda_2} = 1$*. We define* $k = \dim T_{\lambda_4} + 1$*, and thus,* $k \in \{2, \ldots, n-1\}$*. The distribution* T_{λ_4} *is totally real with* $JT_{\lambda_4} \subset T_{\lambda_3} \ominus \mathbb{R}A$*,*

$$0 \neq \lambda_3 \neq \frac{\sqrt{-c}}{2\sqrt{3}}, \quad \text{and} \quad \lambda_4 = -\frac{c}{4\lambda_3}.$$

(vi) *If* $g = 3$ *there are two possibilities:*
   (A) $\dim T_{\lambda_2} = 1$*; in this case we define* $k = 1$*.*
   (B) $\dim T_{\lambda_2} > 1$*; in this case we define* $k = \dim T_{\lambda_2} \in \{2, \ldots, n-1\}$ *and we have that* $T_{\lambda_2} \ominus \mathbb{R}U_2$ *is a real distribution with* $J(T_{\lambda_2} \ominus \mathbb{R}U_2) \subset T_{\lambda_3} \ominus \mathbb{R}A$*, and*

$$\lambda_1 = 0, \quad \lambda_2 = \frac{\sqrt{-3c}}{2}, \quad \lambda_3 = \frac{\sqrt{-c}}{2\sqrt{3}}.$$

Remark 3.13. Part (a) of Theorem 3.12 already provides a proof for part (a) of the Main Theorem.

We know that $\mathbb{R}U_1 \oplus \mathbb{R}U_2 \oplus \mathbb{R}A \oplus \mathbb{R}\xi$ is a complex subbundle on M by Lemma 3.1. Thus, in part (bv) of Theorem 3.12, the fact that T_{λ_4} is real (Corollary 3.7) implies $JT_{\lambda_4} \subset T_{\lambda_3} \ominus \mathbb{R}A$ as claimed. Similarly, in Theorem 3.12 b(vi)B, the assertion $J(T_{\lambda_2} \ominus \mathbb{R}U_2) \subset T_{\lambda_3} \ominus \mathbb{R}A$ follows from the fact that T_{λ_2} is real by Lemma 2.1 (i).

The definition of k above might seem a bit artificial at the moment, but it will be useful in the next section where we conclude the proof of the Main Theorem ($k - 1$ will be the dimension of the kernel of the differential of the map $\Phi^r : M \to \mathbb{C}H^n(c)$, $p \mapsto \exp_p(r\xi_p)$).

If we examine the proof of our theorem, so far we have actually shown that for any point $p \in M$ there exists a neighborhood of p where the conclusion of Theorem 3.12 is satisfied. However, by the connectedness of M and a continuity argument, it can be easily shown that M is orientable and that the conclusion of Theorem 3.12 is satisfied globally.

3.6. **Jacobi field theory and rigidity of focal submanifolds.** In this last section we finish the proof of part (b) of the Main Theorem. Since we use standard Jacobi field theory, we provide the reader just with the fundamental details and skip the long calculations. According to [5] we just have to take care of the case $g = 4$. However, it is not much overload to deal with the two cases simultaneously, so for the sake of completeness we will do so in what follows.

Let M be a real hypersurface of $\mathbb{C}H^n(c)$ in the conditions of Theorem 3.12 (b). For $r \in \mathbb{R}$ we define the map $\Phi^r : M \to \mathbb{C}H^n(c)$, $p \mapsto \exp_p(r\xi_p)$, where $\exp_p$ is the Riemannian exponential map of $\mathbb{C}H^n(c)$ at p. Then, $\Phi^r(M)$ is obtained by moving M a distance r along its normal direction. The singularities of Φ^r are the focal points of M. We will find a particular distance r for which Φ^r_* has constant rank, where Φ^r_* denotes the differential of Φ^r. Then we will apply Theorem 2.3 to $\Phi^r(M)$ for this choice of r . This way, $\Phi^r(M)$ will be an open part of the ruled minimal Berndt-Brück submanifold W^{2n-k}, $k \in \{1, \dots, n-1\}$, and hence M will be an open part of a tube around this ruled minimal submanifold W^{2n-k}. (If $k = 1$ then M will be an equidistant hypersurface to the ruled minimal hypersurface W^{2n-1} at distance r.)

Let $p \in M$ and denote by γ_p the geodesic determined by the initial conditions $\gamma_p(0) = p$ and $\dot\gamma_p(0) = \xi_p$. For any $v \in T_pM$ let B_v be the parallel vector field along the geodesic γ_p such that $B_v(0) = v$, and let ζ_v be the Jacobi field along γ_p with initial conditions $\zeta_v(0) = v$ and $\zeta'(0) = -S_pv$. Here $'$ denotes covariant derivative along γ_p. Since ζ_v is a solution to the differential equation $4\zeta_v'' - c\zeta_v - 3c\langle\zeta_v, J\dot\gamma_p\rangle J\dot\gamma_p = 0$, if $v \in T_{\lambda_i}(p)$ then

$$\zeta_v(t) = f_i(t)B_v(t) + \langle v, J\xi\rangle g_i(t)J\dot\gamma_p(t),$$

where

$$\begin{aligned} f_i(t) &= \cosh\left(\frac{t\sqrt{-c}}{2}\right) - \frac{2\lambda_i}{\sqrt{-c}}\sinh\left(\frac{t\sqrt{-c}}{2}\right), \\ g_i(t) &= \left(\cosh\left(\frac{t\sqrt{-c}}{2}\right) - 1\right)\left(1 + 2\cosh\left(\frac{t\sqrt{-c}}{2}\right) - \frac{2\lambda_i}{\sqrt{-c}}\sinh\left(\frac{t\sqrt{-c}}{2}\right)\right). \end{aligned}$$

We also define the smooth vector field η^r along Φ^r by $\eta^r_p = \dot\gamma_p(r)$. It is known that $\zeta_v(r) = \Phi^r_*v$ and $\zeta_v'(r) = \bar\nabla_{\Phi^r_*v}\eta^r$.

We now determine the value of r. Since $0 \le \lambda_3 < \sqrt{-c}/2$ we can find a real number $r \ge 0$ such that

$$\lambda_3 = \frac{\sqrt{-c}}{2}\tanh\left(\frac{r\sqrt{-c}}{2}\right).$$

Let $p \in M$. We define $u_i = (U_i)_p$, $i \in \{1, 2\}$. Let $v_2 \in T_{\lambda_2}(p) \ominus \mathbb{R}u_2$ and $v_k \in T_{\lambda_k}(p)$ for $3 \le k \le g$ (whenever these spaces are nontrivial). The explicit solution to the Jacobi equation above implies

$$(\Phi^r_*u_1, \Phi^r_*u_2) = (B_{u_1}(r), B_{u_2}(r))D(r),$$

$$\Phi^r_*v_2 = 0, \qquad \Phi^r_*v_3 = \operatorname{sech}\left(\frac{t\sqrt{-c}}{2}\right)B_{v_3}(r), \qquad \Phi^r_*v_4 = 0,$$

where

$$D(t) = \begin{pmatrix} f_1(t) + b_1^2 g_1(t) & b_1b_2g_2(t) \\ b_1b_2g_1(t) & f_2(t) + b_2^2g_2(t) \end{pmatrix}.$$

Since $\det(D(r)) = \operatorname{sech}^3\left(r\sqrt{-c}/2\right)$ we conclude that Φ^r_* has constant rank $2n-k$ (see Theorem 3.12 (bv)-(bvi) for the definition of k). Then, for each point $p \in M$ there exists an open neighborhood $\mathcal{V}$ of p such that $\mathcal{W} = \Phi^r(\mathcal{V})$ is an embedded submanifold of $\mathbb{C}H^n(c)$ and $\Phi^r : \mathcal{V} \to \mathcal{W}$ is a submersion. (If $k=1$, then Φ^r is actually a local diffeomorphism.)

Let $q = \Phi^r(p) \in \mathcal{W}$. The expression above for Φ^r_* shows that the tangent space $T_q\mathcal{W}$ of $\mathcal{W}$ at q is obtained by parallel translation of $\mathbb{R}u_1 \oplus \mathbb{R}u_2 \oplus T_{\lambda_3}(p)$ along the geodesic γ_p from $p = \gamma_p(0)$ to $q = \gamma_p(r)$. Therefore, the normal space $\nu_q\mathcal{W}$ of $\mathcal{W}$ at q is obtained by parallel translation of $(\ker \Phi^r_{*p}) \oplus \mathbb{R}\xi_p$ along γ_p from $p = \gamma_p(0)$ to $q = \gamma_p(r)$. The latter is $(T_{\lambda_2} \ominus \mathbb{R}u_2) \oplus \mathbb{R}\xi_p$ if $g = 3$ (see Theorem 3.12 (bvi)), or $T_{\lambda_4}(p) \oplus \mathbb{R}\xi_p$ if $g = 4$ (see Theorem 3.12 (bv)). In any case, by Theorem 3.12 (bv)-(bvi) it follows that $\mathcal{W}$ has totally real normal bundle of rank k.

We have that $\eta^r_p = B_{\xi_p}(r)$ is a unit normal vector of $\mathcal{W}$ at q. If S^r denotes the shape operator of $\mathcal{W}$, then it is known that $S^r_{\eta^r_p}\Phi^r_* v = -(\zeta_v'(r))^\top$, where $(\cdot)^\top$ denotes orthogonal projection onto the tangent space of $\mathcal{W}$. Using the explicit expression for ζ_v above, we get

$$(S^r_{\eta^r_p}B_{u_1}(r), S^r_{\eta^r_p}B_{u_2}(r)) = (B_{u_1}(r), B_{u_2}(r))C(r), \quad \text{and} \quad S^r_{\eta^r_p}B_{v_3}(r) = 0 \text{ for all } v_3 \in T_{\lambda_3}(p),$$

where $C(r) = -D'(r)D(r)^{-1}$. A lengthy and tedious calculation shows that

$$C(r) = \frac{\sqrt{-c}}{2}\begin{pmatrix} -2b_1b_2 & b_1^2 - b_2^2 \\ b_1^2 - b_2^2 & 2b_1b_2 \end{pmatrix}.$$

Since $J\eta^r_p = B_{J\xi_p}(r) = b_1B_{u_1}(r) + b_2B_{u_2}(r)$, and $B_{JA_p}(r) = b_2B_{u_1}(r) - b_1B_{u_2}(r)$, the above expression for $C(r)$ implies

$$S^r_{\eta^r_p}B_{JA_p}(r) = -\frac{\sqrt{-c}}{2}J\eta^r_p, \qquad S^r_{\eta^r_p}J\eta^r_p = -\frac{\sqrt{-c}}{2}B_{JA_p}(r),$$

and $S^r_{\eta^r_p}$ vanishes on the orthogonal complement of $\mathbb{R}J\eta^r_p \oplus \mathbb{R}B_{JA_p}(r)$ in $T_q\mathcal{W}$.

We have that $J(\nu_q\mathcal{W} \ominus \mathbb{R}\eta^r_p)$ is contained in the parallel translation along γ_p of $T_{\lambda_3}(p)$. This follows from Theorem 3.12 (bv)-(bvi) and the fact that $\nu_q\mathcal{W} \ominus \mathbb{R}\eta^r_p$ is the parallel translation along γ_p from $\gamma_p(0) = p$ to $\gamma_p(r) = q$ of $T_{\lambda_2}(p) \ominus \mathbb{R}u_2$ if $g = 3$, and of $T_{\lambda_4}(p)$ if $g = 4$. The linearity of $S^r_{\eta^p_r}$ implies

$$S^r_{\eta^r_p}J\tilde{\eta} = -\frac{\sqrt{-c}}{2}\langle \eta^r_p, \tilde{\eta}\rangle B_{JA_p}(r), \text{ for all } \tilde{\eta} \in \nu_q\mathcal{W}. \tag{13}$$

It follows from the Gauss formula and $\bar{\nabla}J = 0$ that $S^r_{\tilde{\eta}}J\eta^r_p = S^r_{\eta^r_p}J\tilde{\eta}$, and hence, $S^r_{\tilde{\eta}}J\eta^r_p = 0$ for all $\tilde{\eta} \in \nu_q\mathcal{W} \ominus \mathbb{R}\eta^r_p$. Let α be a curve in $(\Phi^r)^{-1}(\{q\}) \cap \mathcal{V}$ with $\alpha(0) = p$. Since η^r_p and $\eta^r_{\alpha(t)} - \langle \eta^r_{\alpha(t)}, \eta^r_p\rangle\eta^r_p$ are perpendicular, $S^r_{\tilde{\eta}}J\eta^r_p = 0$, and the linearity of $\eta \mapsto S^r_\eta$ imply

$$0 = S^r_{\eta^r_{\alpha(t)} - \langle \eta^r_{\alpha(t)}, \eta^r_p\rangle\eta^r_p}J\eta^r_p = S^r_{\eta^r_{\alpha(t)}}J\eta^r_p + \frac{\sqrt{-c}}{2}\langle \eta^r_{\alpha(t)}, \eta^r_p\rangle B_{JA_p}(r),$$

which together with (13) (with $\alpha(t)$ instead of p) yields

$$-\frac{\sqrt{-c}}{2}\langle \eta^r_{\alpha(t)}, \eta^r_p\rangle B_{JA_p}(r) = S^r_{\eta^r_{\alpha(t)}}J\eta^r_p = -\frac{\sqrt{-c}}{2}\langle \eta^r_{\alpha(t)}, \eta^r_p\rangle B_{JA_{\alpha(t)}}(r).$$

Since α is arbitrary we get that the map $\tilde{p} \mapsto B_{JA_{\tilde{p}}}(r)$ is constant in the connected component $\mathcal{V}_0$ of $(\Phi^r)^{-1}(\{q\}) \cap \mathcal{V}$ containing p. Thus it makes sense to define the unit vector $z = -B_{JA_{\tilde{p}}}(r) \in T_q\mathcal{W}$ for any $\tilde{p} \in \mathcal{V}_0$.

We may consider η^r as a map from $\mathcal{V}_0$ to the unit sphere of $\nu_q\mathcal{W}$. The tangent space of $\mathcal{V}_0$ at p is given by the kernel of Φ^r_{*p}. If $v \in \ker \Phi^r_{*p}$, then $\eta^r_{*p}v = \zeta'_v(r)$. If $g = 3$, then $v \in \ker \Phi^r_{*p} = T_{\lambda_2}(p) \ominus \mathbb{R}u_2$, and $\eta^r_{*p} = -\sqrt{-c/2}\, B_v(r)$. If $g = 4$, then $v \in \ker \Phi^r_{*p} = T_{\lambda_4}(p)$, and $\eta^r_{*p}v = -\operatorname{csch}(r\sqrt{-c}/2)B_v(r)$. In any case, we get that η^r is a local diffeomorphism from $\mathcal{V}_0$ into the unit sphere of $\nu_q\mathcal{W}$ (note that this is trivial if $g = 3$ and $k = 1$). Hence, $\eta^r(\mathcal{V}_0)$ is an open subset of the unit sphere of $\nu_q\mathcal{W}$. But since $\eta \mapsto S^r_\eta$ depends analytically on η we conclude

$$S^r_\eta J\eta = \frac{\sqrt{-c}}{2}z, \qquad S^r_\eta z = \frac{\sqrt{-c}}{2}J\eta, \qquad S^r_\eta v = 0,$$

for all unit $\eta \in \nu_q\mathcal{W}$, and $v \in T_q\mathcal{W} \ominus (\mathbb{R}J\eta \oplus \mathbb{R}z)$. Therefore, the second fundamental form $I\!I^r$ of $\mathcal{W}$ at q is given by the trivial symmetric bilinear extension of $I\!I^r(z, J\eta) = (\sqrt{-c}/2)\eta$ for all $\eta \in \nu_q\mathcal{W}$. By construction, z depends smoothly on the point $q \in \mathcal{W}$ and hence gives rise to a vector field Z which is tangent to the maximal holomorphic distribution of $\mathcal{W}$. The relation $S^r_\eta J\eta = (\sqrt{-c}/2)Z$ ensures that Z can actually be defined on $\Phi^r(M)$, and hence, the second fundamental form of $\Phi^r(M)$ is given by the trivial symmetric bilinear extension of $I\!I^r(Z, J\eta) = (\sqrt{-c}/2)\eta$ for all $\eta \in \Gamma(\nu\, \Phi^r(M))$. Since $\Phi^r(M)$ has totally real normal bundle of rank k we conclude from Theorem 2.3, and the remark that follows, that $\Phi^r(M)$ is holomorphically congruent to an open part of the ruled minimal Berndt-Brück submanifold W^{2n-k}. This readily implies that M is an open part of a tube (an equidistant hypersurface if $g = 3$ and $k = 1$) of radius r around the ruled minimal Berndt-Brück submanifold W^{2n-k}.

Finally, let us point out that if $g = 3$ and $\lambda_3 = 0$, then $r = 0$ and M is an open part of the ruled minimal hypersurface W^{2n-1}. Also, if $g = 3$ and $k > 1$ then $\lambda_3 = \sqrt{-c}/(2\sqrt{3})$ according to Theorem 3.12 b(vi)B, and hence $r = (1/\sqrt{-c})\log(2+\sqrt{3})$. The tube around the ruled minimal submanifold W^{2n-k}, $k > 1$, of radius $r = (1/\sqrt{-c})\log(2+\sqrt{3})$ has $g = 3$ principal curvatures whereas if $r \neq (1/\sqrt{-c})\log(2+\sqrt{3})$ the tube of radius r around the ruled minimal submanifold W^{2n-k}, $k > 1$, has $g = 4$ principal curvatures.

This finishes the proof of the Main Theorem.

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