arXiv:0807.3517v2 [math.DG] 25 Nov 2008

HYPERPOLAR HOMOGENEOUS FOLIATIONS ON SYMMETRIC SPACES OF NONCOMPACT TYPE

JÜRGEN BERNDT, JOSÉ CARLOS DÍAZ-RAMOS, AND HIROSHI TAMARU

Abstract. A foliation $\mathcal{F}$ on a Riemannian manifold M is hyperpolar if it admits a flat section, that is, a connected closed flat submanifold of M that intersects each leaf of $\mathcal{F}$ orthogonally. In this article we classify the hyperpolar homogeneous foliations on every Riemannian symmetric space M of noncompact type.

These foliations are constructed as follows. Let Φ be an orthogonal subset of a set of simple roots associated with the symmetric space M. Then Φ determines a horospherical decomposition $M = F^s_\Phi \times \mathbb{E}^{\operatorname{rank} M - |\Phi|} \times N_\Phi$, where F^s_Φ is the Riemannian product of $|\Phi|$ symmetric spaces of rank one. Every hyperpolar homogeneous foliation on M is isometrically congruent to the product of the following objects: a particular homogeneous codimension one foliation on each symmetric space of rank one in F^s_Φ, a foliation by parallel affine subspaces on the Euclidean space $\mathbb{E}^{\operatorname{rank} M - |\Phi|}$, and the horocycle subgroup N_Φ of the parabolic subgroup of the isometry group of M determined by Φ.

1. Introduction

Let M be a connected complete Riemannian manifold and H a connected closed subgroup of the isometry group $I(M)$ of M. Then each orbit $H \cdot p = \{h(p) : h \in H\}$, $p \in M$, is a connected closed submanifold of M. A connected complete submanifold $\mathcal{S}$ of M that meets each orbit of the H-action and intersects the orbit $H \cdot p$ perpendicularly at each point $p \in \mathcal{S}$ is called a section of the action. A section $\mathcal{S}$ is always a totally geodesic submanifold of M (see e.g. [11]). In general, actions do not admit a section. The action of H on M is called polar if it has a section, and it is called hyperpolar if it has a flat section. For motivation and classification of polar and hyperpolar actions on Euclidean spaces and symmetric spaces of compact type we refer to the papers by Dadok [7], Podestà and Thorbergsson [23], and Kollross [18], [19]. If all orbits of H are principal, then the orbits form a homogeneous foliation $\mathcal{F}$ on M. In general, a foliation $\mathcal{F}$ on M is called homogeneous if the subgroup of $I(M)$ consisting of all isometries preserving $\mathcal{F}$ acts transitively on each leaf of $\mathcal{F}$. Homogeneous foliations are basic examples of metric foliations. A homogeneous

2000 *Mathematics Subject Classification.* Primary 53C12, 53C35; Secondary 53S20, 22E25.

Key words and phrases. Riemannian symmetric spaces of noncompact type, homogeneous foliations, hyperpolar foliations.

The second author has been supported by a Marie-Curie Intra-European Fellowship (MEIF-CT-2006-038754) and by project PGIDIT06PXIB207054PR (Spain). The third author has been supported in part by Grant-in-Aid for Young Scientists (B) 20740040, The Ministry of Education, Culture, Sports, Science and Technology, Japan.

foliation is called polar resp. hyperpolar if its leaves coincide with the orbits of a polar resp. hyperpolar action.

An action of the Euclidean space $\mathbb{E}^n$ is polar if and only if it is hyperpolar. An example of a polar homogeneous foliation on $\mathbb{E}^n$ is the foliation given by the Euclidean subspace $\mathbb{E}^k$, $0 < k < n$, and its parallel affine subspaces. A corresponding section is given by the Euclidean space $\mathbb{E}^{n-k}$ which is perpendicular to $\mathbb{E}^k$ at the origin 0. In fact, every polar homogeneous foliation on $\mathbb{E}^n$ is isometrically congruent to one of these foliations. The main result of this paper is the classification of all hyperpolar homogeneous foliations on Riemannian symmetric spaces of noncompact type. For codimension one foliations this was already achieved by the first and third author in [4]. We mention that on symmetric spaces of compact type every hyperpolar action has a singular orbit, and there is no relation between such actions using duality between symmetric spaces of compact and noncompact type. The methodology for the classification presented in this paper is significantly different from the known methodologies in the compact case. Our methodology is conceptual and based on structure theory of parabolic subalgebras of real semisimple Lie algebras which is irrelevant in the compact case.

We will see that these foliations can be constructed from rather elementary foliations on Euclidean spaces and the hyperbolic spaces over normed real division algebras. We first describe these elementary foliations. Each Riemannian symmetric spaces of rank one is a hyperbolic space $\mathbb{F}H^n$ over a normed real division algebra $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, where $n \geq 2$, and $n = 2$ if $\mathbb{F} = \mathbb{O}$. It was proved in [5] that on each hyperbolic space $\mathbb{F}H^n$ there exist exactly two isometric congruency classes of homogeneous codimension one foliations. One of these two classes is determined by the horosphere foliation on $\mathbb{F}H^n$. We denote by $\mathcal{F}^n_{\mathbb{F}}$ a representative of the other congruency class, and refer to Section 4 for an explicit description. If $M = \mathbb{F}_1 H^{n_1} \times \ldots \times \mathbb{F}_k H^{n_k}$ is the Riemannian product of k Riemannian symmetric spaces of rank one, then $\mathcal{F}^{n_1}_{\mathbb{F}_1} \times \ldots \times \mathcal{F}^{n_k}_{\mathbb{F}_k}$ is a hyperpolar homogeneous foliation on M. If V is a linear subspace of $\mathbb{E}^m$, we denote by $\mathcal{F}^m_V$ the homogeneous foliation on $\mathbb{E}^m$ whose leaves are the affine subspaces of $\mathbb{E}^m$ which are parallel to V. We will now explain how these particular foliations lead to the classification of hyperpolar homogeneous foliations on Riemannian symmetric spaces of noncompact type.

Let $M = G/K$ be a Riemannian symmetric space of noncompact type, where G is the connected component of the isometry group of M containing the identity transformation. We denote by r the rank of M. The Lie algebra $\mathfrak{g}$ of G is a semisimple real Lie algebra. Let $\mathfrak{k}$ be the Lie algebra of K, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and $\mathfrak{g}_0 \oplus \left(\bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda\right)$ be the corresponding restricted root space decomposition of $\mathfrak{g}$. The set Σ denotes the corresponding set of restricted roots. We choose a subset $\Lambda \subset \Sigma$ of simple roots and denote by Σ^+ the resulting set of positive restricted roots in Σ. It is well known that there is a one-to-one correspondence between the subsets Φ of Λ and the conjugacy classes of parabolic subalgebras $\mathfrak{q}_\Phi$ of $\mathfrak{g}$. Let Φ be a subset of Λ and consider the Langlands decomposition $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ of the corresponding parabolic subalgebra $\mathfrak{q}_\Phi$ of $\mathfrak{g}$. This determines a corresponding Langlands decomposition $Q_\Phi = M_\Phi A_\Phi N_\Phi$ of the parabolic subgroup Q_Φ of G with Lie algebra $\mathfrak{q}_\Phi$ and a horospherical decomposition

$M = F^s_\Phi \times \mathbb{E}^{r-r_\Phi} \times N_\Phi$ of the symmetric space M. Here, r_Φ is equal to the cardinality $|\Phi|$ of the set Φ, $F^s_\Phi = M_\Phi \cdot o$ is a semisimple Riemannian symmetric space of noncompact type with rank equal to r_Φ embedded as a totally geodesic submanifold in M, and $\mathbb{E}^{r-r_\Phi} = A_\Phi \cdot o$ is an $(r - r_\Phi)$-dimensional Euclidean space embedded as a totally geodesic submanifold in M. Now assume that Φ is a subset of Λ with the property that any two roots in Φ are not connected in the Dynkin diagram of the restricted root system associated with Λ. We call such a subset Φ an orthogonal subset of Λ. Each simple root $\alpha \in \Phi$ determines a hyperbolic space $\mathbb{F}_\alpha H^{n_\alpha}$ embedded in M as a totally geodesic submanifold, and F^s_Φ is isometric to the Riemannian product of r_Φ Riemannian symmetric spaces of rank one,

$$F^s_\Phi \cong \prod_{\alpha \in \Phi} \mathbb{F}_\alpha H^{n_\alpha}.$$

We denote by $\mathcal{F}_\Phi$ the hyperpolar homogeneous foliation on this product of hyperbolic spaces as described above, that is,

$$\mathcal{F}_\Phi = \prod_{\alpha \in \Phi} \mathcal{F}^{n_\alpha}_{\mathbb{F}_\alpha}.$$

We are now in a position to state the main result of this paper.

Main Theorem. *Let M be a connected Riemannian symmetric space of noncompact type.*

(i) *Let Φ be an orthogonal subset of Λ and V be a linear subspace of $\mathbb{E}^{r-r_\Phi}$. Then*

$$\mathcal{F}_{\Phi,V} = \mathcal{F}_\Phi \times \mathcal{F}^{r-r_\Phi}_V \times N_\Phi \subset F^s_\Phi \times \mathbb{E}^{r-r_\Phi} \times N_\Phi = M$$

is a hyperpolar homogeneous foliation on M.

(ii) *Every hyperpolar homogeneous foliation on M is isometrically congruent to $\mathcal{F}_{\Phi,V}$ for some orthogonal subset Φ of Λ and some linear subspace V of $\mathbb{E}^{r-r_\Phi}$.*

For $\Phi = \emptyset$ the symmetric space F^s_Φ consists of a single point and we need to assume that $\dim V < r$ in this case to get a proper foliation. The foliation $\mathcal{F}_{\emptyset,\{0\}}$ is the horocycle foliation on M.

We briefly describe how to construct a subgroup of G whose orbits form the foliation $\mathcal{F}_{\Phi,V}$. Since A_Φ acts freely on M and $\mathbb{E}^{r-r_\Phi} = A_\Phi \cdot o$, there is a canonical identification of $\mathbb{E}^{r-r_\Phi}$ with the Lie algebra $\mathfrak{a}_\Phi \subset \mathfrak{a}$. We define a nilpotent subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ by $\mathfrak{n} = \mathfrak{n}_\emptyset$ and put $\mathfrak{a} = \mathfrak{a}_\emptyset$. Then the closed subgroup AN of G with Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$ acts simply transitively on M, and M is isometric to the solvable Lie group AN equipped with a suitable left-invariant Riemannian metric. Let ℓ_Φ be an r_Φ-dimensional linear subspace of $\mathfrak{n}$ such that $\dim(\ell_\Phi \cap \mathfrak{g}_\alpha) = 1$ for all $\alpha \in \Phi$. We denote by $\mathfrak{a}^\Phi$ the orthogonal complement of $\mathfrak{a}_\Phi$ in $\mathfrak{a}$ and by $\mathfrak{n} \ominus \ell_\Phi$ the orthogonal complement of ℓ_Φ in $\mathfrak{n}$. Here, the orthogonal complement is taken with respect to the standard positive definite inner product on $\mathfrak{g}$ given by the Killing form on $\mathfrak{g}$ and the Cartan involution on $\mathfrak{g}$ determined by $\mathfrak{k}$. Then

$$\mathfrak{s}_{\Phi,V} = (\mathfrak{a}^\Phi \oplus V) \oplus (\mathfrak{n} \ominus \ell_\Phi) \subset \mathfrak{a} \oplus \mathfrak{n}$$

is a subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$. Denote by $S_{\Phi,V}$ the connected closed subgroup of AN with Lie algebra $\mathfrak{s}_{\Phi,V}$. Then the action of $S_{\Phi,V}$ on M is hyperpolar and the orbits of this action form the hyperpolar homogeneous foliation $\mathcal{F}_{\Phi,V}$ on M. We will see later in this paper that for a given set Φ different choices of ℓ_Φ lead to isometrically congruent foliations on M.

We now describe the contents of this paper in more detail. In Section 2 we show that all homogeneous foliations on Hadamard manifolds can be produced by isometric actions of solvable Lie groups all of whose orbits are principal. In Section 3 we present the aspects of the general theory of symmetric spaces of noncompact type and of parabolic subalgebras of real semisimple Lie algebras which are relevant for our paper. In Section 4 we prove a necessary and sufficient Lie algebraic criterion for an isometric Lie group action inducing a foliation on a symmetric space of noncompact type to be polar or hyperpolar. Using this criterion we present examples of polar and of hyperpolar actions on symmetric spaces of noncompact type. In this section we also prove part (i) of the main theorem, which is the easiest part of the proof. Section 5 constitutes the main part of this paper and contains the proof of part (ii) of the main theorem. Finally, in Section 6 we discuss aspects of the geometry of the leaves of the hyperpolar homogeneous foliations on symmetric spaces of noncompact type.

2. Homogeneous foliations on Hadamard manifolds

A simply connected complete Riemannian manifold with nonpositive sectional curvature is called a *Hadamard manifold*.

Proposition 2.1. *Let M be a Hadamard manifold and H be a connected closed subgroup of $I(M)$ whose orbits form a homogeneous foliation on M. Then each orbit of H is a principal orbit.*

Proof. Assume that there exists an exceptional orbit, that is, a non-principal orbit whose dimension coincides with the dimension of the principal orbits. Let K be a maximal compact subgroup of H. By Cartan's Fixed Point Theorem (see e.g. [9], p. 21), K has a fixed point $o \in M$. Since K is maximal compact, the orbit through o must be exceptional and $K = H_o$. Then $H \cdot o = H/K$ is diffeomorphic to $\mathbb{R}^k$, where k is the dimension of the foliation (see for example [22, p. 148, Theorem 3.4]). Since the orbit $H \cdot o$ is simply connected, the stabilizer K is connected. The cohomogeneity of the slice representation at o coincides with the cohomogeneity of the action of H on M, and since all the orbits of H have the same dimension it follows that the orbits of the slice representation at o are zero-dimensional. Since K is connected, it follows that the orbits of the slice representation at o are points. This means that K acts trivially on the normal space $\nu_o(H \cdot o)$ of $H \cdot o$ at o, which contradicts the assumption that the orbit $H \cdot o$ is exceptional. □

We will now use the previous result to show that every homogeneous foliation on a Hadamard manifold can be realized as the orbits of the action of a closed solvable group of isometries.

Proposition 2.2. *Let M be a Hadamard manifold and let H be a connected closed subgroup of $I(M)$ whose orbits form a homogeneous foliation $\mathcal{F}$ on M. Then there exists a connected closed solvable subgroup S of H such that the leaves of $\mathcal{F}$ coincide with the orbits of S.*

Proof. Consider a Levi-Malcev decomposition $\mathfrak{h} = \mathfrak{l} \Subset \mathfrak{r}$ (semidirect sum of Lie algebras) of the Lie algebra $\mathfrak{h}$ of H into the radical $\mathfrak{r}$ of $\mathfrak{h}$ and a Levi subalgebra $\mathfrak{l}$. The radical $\mathfrak{r}$ is the largest solvable ideal in $\mathfrak{h}$ and $\mathfrak{l}$ is a semisimple subalgebra. Let $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ (direct sum of vector spaces) be an Iwasawa decomposition of $\mathfrak{l}$. Then $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{l}$, $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{l}$, and $\mathfrak{d} = \mathfrak{a} \Subset \mathfrak{n}$ (semidirect sum of Lie algebras) is a solvable subalgebra of $\mathfrak{l}$. Since the semidirect sum of two solvable Lie algebras is again solvable, the subalgebra $\mathfrak{s} = \mathfrak{d} \Subset \mathfrak{r}$ (semidirect sum of Lie algebras) is a solvable subalgebra of $\mathfrak{h}$, and we have $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{s}$ (direct sum of vector spaces). Let S be the connected solvable subgroup of H with Lie algebra $\mathfrak{s}$ and let K be the connected subgroup of H with Lie algebra $\mathfrak{k}$. Since M is a Hadamard manifold, Cartan's Fixed Point Theorem implies that the compact group K has a fixed point $o \in M$. Since $H = SK$, it follows that the orbits $H \cdot o$ and $S \cdot o$ coincide.

By Proposition 2.1, the orbit $H \cdot o$ is a principal orbit of the H-action. Let p be a point in M which does not lie on the principal orbit $H \cdot o$. Since $H \cdot o$ is a closed subset of M, there exists a point $q \in H \cdot o$ such that the distance t between p and q minimizes the distance between p and $H \cdot o$. Since M is complete there exists a geodesic joining q and p, and a standard variational argument shows that this geodesic intersects the orbit $H \cdot o$ perpendicularly. This proves that every orbit of H is of the form $H \cdot p$ with $p = \exp_o(\xi)$ and $\xi \in \nu_o(H \cdot o)$. Since $H \cdot o$ is a principal orbit of the H-action on M and $S \subset H$, the slice representation at o of each of these two actions is trivial. This fact and $H \cdot o = S \cdot o$ imply

$$\begin{aligned} S \cdot p &= \{s(\exp_o(\xi)) : s \in S\} = \{\exp_{s(o)}(s_*\xi) : s \in S\} \\ &= \{\exp_{h(o)}(h_*\xi) : h \in H\} = \{h(\exp_o(\xi)) : h \in H\} = H \cdot p, \end{aligned}$$

which shows that the actions of S and H are orbit equivalent.

Since S is solvable, its closure $\bar{S}$ in $I(M)$ is a closed solvable subgroup of $I(M)$ (see e.g. [21], p. 54, Theorem 5.3). Since the actions of S and H are orbit equivalent, the orbits of S are closed, and hence by [8], the actions of S and $\bar{S}$ are orbit equivalent. This finishes the proof of the proposition. □

3. Riemannian symmetric spaces of noncompact type

In this section we present some material about Riemannian symmetric spaces of noncompact type. We follow [13] for the theory of symmetric spaces and [15] for the theory of semisimple Lie algebras.

Let M be a connected Riemannian symmetric space of noncompact type. We denote by n the dimension of M and by r the rank of M. It is well known that M is a Hadamard manifold and therefore diffeomorphic to $\mathbb{R}^n$. Let G be the connected component of the isometry group of M containing the identity transformation of M. We fix a point $o \in M$ and denote by K the isotropy subgroup of G at o. We identify M with the homogeneous space G/K in the usual way and denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively. Let B be the Killing form of $\mathfrak{g}$ and define $\mathfrak{p}$ as the orthogonal complement of $\mathfrak{k}$ with respect to B. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. If θ is the corresponding Cartan involution, we can define a positive definite inner product on $\mathfrak{g}$ by $\langle X, Y\rangle = -B(X, \theta Y)$ for all $X, Y \in \mathfrak{g}$. We identify $\mathfrak{p}$ with T_oM and we normalize the Riemannian metric on M so that its restriction to $T_oM \times T_oM = \mathfrak{p} \times \mathfrak{p}$ coincides with $\langle \cdot\,, \cdot\rangle$.

We now fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and denote by $\mathfrak{a}^*$ the dual space of $\mathfrak{a}$. For each $\lambda \in \mathfrak{a}^*$ we define $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : \operatorname{ad}(H)X = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}$. We say that $0 \neq \lambda \in \mathfrak{a}^*$ is a restricted root if $\mathfrak{g}_\lambda \neq \{0\}$, and we denote by Σ the set of all restricted roots. Since $\mathfrak{a}$ is abelian, $\operatorname{ad}(\mathfrak{a})$ is a commuting family of self-adjoint linear transformations of $\mathfrak{g}$. This implies that the subset $\Sigma \subset \mathfrak{a}^*$ of all restricted roots is nonempty, finite and $\mathfrak{g} = \mathfrak{g}_0 \oplus (\bigoplus_{\lambda\in\Sigma} \mathfrak{g}_\lambda)$ is an orthogonal direct sum called the restricted root space decomposition of $\mathfrak{g}$ determined by $\mathfrak{a}$. Here, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, where $\mathfrak{k}_0 = Z_{\mathfrak{k}}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. For each $\lambda \in \mathfrak{a}^*$ let $H_\lambda \in \mathfrak{a}$ denote the dual vector in $\mathfrak{a}$ with respect to the Killing form, that is, $\lambda(H) = \langle H_\lambda, H\rangle$ for all $H \in \mathfrak{a}$. This also defines an inner product on $\mathfrak{a}^*$ by setting $\langle \lambda, \mu\rangle = \langle H_\lambda, H_\mu\rangle$ for all $\lambda, \mu \in \mathfrak{a}^*$.

We now introduce an ordering in Σ and denote by Σ^+ the resulting set of positive roots. We denote by $\Lambda = \{\alpha_1, \dots, \alpha_r\}$ the set of simple roots of Σ^+ in line with the notation used in [15]. By $\{H^1, \dots, H^r\} \subset \mathfrak{a}$ we denote the dual basis of $\{\alpha_1, \dots, \alpha_r\}$, that is, $\alpha_i(H^j) = \delta_i^j$, where δ is the Kronecker delta. Then each root $\lambda \in \Sigma$ can be written as $\lambda = \sum_{i=1}^r c_i\alpha_i$ where all the c_i are integers, and they are all nonpositive or nonnegative depending on whether the root is negative or positive. The sum $\sum_{i=1}^r c_i$ is called the level of the root.

The subspace $\mathfrak{n} = \bigoplus_{\lambda\in\Sigma^+} \mathfrak{g}_\lambda$ of $\mathfrak{g}$ is a nilpotent subalgebra of $\mathfrak{g}$. Moreover, $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable subalgebra of $\mathfrak{g}$ with $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$. We can write $\mathfrak{g}$ as the direct sum of vector subspaces $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, the so-called Iwasawa decomposition of $\mathfrak{g}$. Let A, N and AN be the connected subgroups of G with Lie algebra $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{a} \oplus \mathfrak{n}$, respectively. All these subgroups are simply connected and G is diffeomorphic to the product $K \times A \times N$. Moreover, the solvable Lie group AN acts simply transitively on M. Hence M is isometric to the connected, simply connected solvable Lie group AN equipped with the left-invariant Riemannian metric that is induced from the inner product $\langle \cdot\,, \cdot\rangle$. Consider $X, Y, Z \in \mathfrak{a} \oplus \mathfrak{n}$ as left-invariant vector fields on M. If ∇ denotes the Levi-Civita covariant derivative of $M = AN$, the equality $\langle \operatorname{ad}(X)Y, Z\rangle = -\langle \operatorname{ad}(\theta X)Z, Y\rangle$ implies that the Koszul formula can be written as

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y] + (1 - \theta)[\theta X, Y], Z\rangle.$$

We will now associate to each subset Φ of Λ a parabolic subalgebra $\mathfrak{q}_\Phi$ of $\mathfrak{g}$. Let Φ be a subset of Λ. We denote by Σ_Φ the root subsystem of Σ generated by Φ, that is, Σ_Φ is the intersection of Σ and the linear span of Φ, and put $\Sigma_\Phi^+ = \Sigma_\Phi \cap \Sigma^+$. Let

$$\mathfrak{l}_\Phi = \mathfrak{g}_0 \oplus \left(\bigoplus_{\lambda\in\Sigma_\Phi} \mathfrak{g}_\lambda \right) \text{ and } \mathfrak{n}_\Phi = \bigoplus_{\lambda\in\Sigma^+\setminus\Sigma_\Phi^+} \mathfrak{g}_\lambda.$$

Then $\mathfrak{l}_\Phi$ is a reductive subalgebra of $\mathfrak{g}$ and $\mathfrak{n}_\Phi$ is a nilpotent subalgebra of $\mathfrak{g}$. Let

$$\mathfrak{a}_\Phi = \bigcap_{\alpha\in\Phi} \ker \alpha \quad \text{and} \quad \mathfrak{a}^\Phi = \mathfrak{a} \ominus \mathfrak{a}_\Phi.$$

Then $\mathfrak{a}_\Phi$ is an abelian subalgebra of $\mathfrak{g}$ and $\mathfrak{l}_\Phi$ is the centralizer and the normalizer of $\mathfrak{a}_\Phi$ in $\mathfrak{g}$. The abelian subalgebra $\mathfrak{a}_\Phi$ is also known as the split component of the reductive Lie algebra $\mathfrak{l}_\Phi$. Since $[\mathfrak{l}_\Phi, \mathfrak{n}_\Phi] \subset \mathfrak{n}_\Phi$,

$$\mathfrak{q}_\Phi = \mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi$$

is a subalgebra of $\mathfrak{g}$, the so-called parabolic subalgebra of $\mathfrak{g}$ associated with the subset Φ of Λ. The subalgebra $\mathfrak{l}_\Phi = \mathfrak{q}_\Phi \cap \theta(\mathfrak{q}_\Phi)$ is a reductive Levi subalgebra of $\mathfrak{q}_\Phi$ and $\mathfrak{n}_\Phi$ is the unipotent radical of $\mathfrak{q}_\Phi$, and therefore the decomposition $\mathfrak{q}_\Phi = \mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi$ is a semidirect sum of the Lie algebras $\mathfrak{l}_\Phi$ and $\mathfrak{n}_\Phi$. The decomposition $\mathfrak{q}_\Phi = \mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi$ is known as the Chevalley decomposition of the parabolic subalgebra $\mathfrak{q}_\Phi$.

We now define a reductive subalgebra $\mathfrak{m}_\Phi$ of $\mathfrak{g}$ by

$$\mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi = \mathfrak{k}_0 \oplus \mathfrak{a}^\Phi \oplus \left(\bigoplus_{\lambda \in \Sigma_\Phi} \mathfrak{g}_\lambda \right).$$

The subalgebra $\mathfrak{m}_\Phi$ normalizes $\mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$, and

$$\mathfrak{g}_\Phi = [\mathfrak{m}_\Phi, \mathfrak{m}_\Phi] = [\mathfrak{l}_\Phi, \mathfrak{l}_\Phi]$$

is a semisimple subalgebra of $\mathfrak{g}$. The center $\mathfrak{z}_\Phi$ of $\mathfrak{m}_\Phi$ is contained in $\mathfrak{k}_0$ and induces the direct sum decomposition $\mathfrak{m}_\Phi = \mathfrak{z}_\Phi \oplus \mathfrak{g}_\Phi$. The decomposition

$$\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$$

is known as the Langlands decomposition of the parabolic subalgebra $\mathfrak{q}_\Phi$.

For $\Phi = \emptyset$ we obtain $\mathfrak{l}_\emptyset = \mathfrak{g}_0$, $\mathfrak{m}_\emptyset = \mathfrak{k}_0$, $\mathfrak{a}_\emptyset = \mathfrak{a}$ and $\mathfrak{n}_\emptyset = \mathfrak{n}$. In this case $\mathfrak{q}_\emptyset = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n} = \mathfrak{g}_0 \oplus \mathfrak{n}$ is a minimal parabolic subalgebra of $\mathfrak{g}$. For $\Phi = \Lambda$ we obtain $\mathfrak{l}_\Lambda = \mathfrak{m}_\Lambda = \mathfrak{g}$ and $\mathfrak{a}_\Lambda = \mathfrak{n}_\Lambda = \{0\}$. Each parabolic subalgebra of $\mathfrak{g}$ is conjugate in $\mathfrak{g}$ to $\mathfrak{q}_\Phi$ for some subset Φ of Λ. The set of conjugacy classes of parabolic subalgebras of $\mathfrak{g}$ therefore has 2^r elements. Two parabolic subalgebras $\mathfrak{q}_{\Phi_1}$ and $\mathfrak{q}_{\Phi_2}$ of $\mathfrak{g}$ are conjugate in the full automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$ if and only if there exists an automorphism F of the Dynkin diagram associated to Λ with $F(\Phi_1) = \Phi_2$. Every parabolic subalgebra contains a minimal parabolic subalgebra.

Each parabolic subalgebra $\mathfrak{q}_\Phi$ determines a gradation of $\mathfrak{g}$. For this we define $H^\Phi = \sum_{\alpha_i \in \Lambda \setminus \Phi} H^i$ and put $\mathfrak{g}_\Phi^k = \bigoplus_{\lambda \in \Sigma \cup \{0\}, \lambda(H^\Phi) = k} \mathfrak{g}_\lambda$. Then $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_\Phi^k$ is a gradation of $\mathfrak{g}$ with $\mathfrak{g}_\Phi^0 = \mathfrak{l}_\Phi$, $\bigoplus_{k>0} \mathfrak{g}_\Phi^k = \mathfrak{n}_\Phi$ and $\bigoplus_{k \geq 0} \mathfrak{g}_\Phi^k = \mathfrak{q}_\Phi$. The vector $H^\Phi \in \mathfrak{a}$ is called the characteristic element of the gradation. The Cartan involution θ acts grade-reversing on the gradation, that is, we have $\theta \mathfrak{g}_\Phi^k = \mathfrak{g}_\Phi^{-k}$ for all $k \in \mathbb{Z}$. Moreover, this gradation is of type α_0, that is, $\mathfrak{g}_\Phi^{k+1} = [\mathfrak{g}_\Phi^1, \mathfrak{g}_\Phi^k]$ and $\mathfrak{g}_\Phi^{-k-1} = [\mathfrak{g}_\Phi^{-1}, \mathfrak{g}_\Phi^{-k}]$ holds for all $k > 0$ (see e.g. [14]). If λ is the highest root in Σ and $m_\Phi = \lambda(H^\Phi)$, we have $\mathfrak{g}_\Phi^{m_\Phi} \neq \{0\}$ and $\mathfrak{g}_\Phi^k = \{0\}$ for all $k > m_\Phi$. For $\Phi = \emptyset$ we have $\mathfrak{n}_\emptyset = \mathfrak{n}$, and we also use the notation $\mathfrak{n}^k = \mathfrak{g}_\emptyset^k$ for all $k > 0$. Thus we have a gradation $\mathfrak{n} = \bigoplus_{k=1}^{m_\emptyset} \mathfrak{n}^k$ of $\mathfrak{n}$ which is generated by $\mathfrak{n}^1$. Note that $m_\emptyset$ is the level of the highest root in Σ^+. For each $k > 0$ we define

$$\mathfrak{p}^k = \mathfrak{p} \cap \left(\mathfrak{g}_\emptyset^k \oplus \mathfrak{g}_\emptyset^{-k} \right),$$

which gives a direct sum decomposition $\mathfrak{p} = \mathfrak{a} \oplus \left(\bigoplus_{k=1}^{m_\emptyset} \mathfrak{p}^k \right)$.

For each $\lambda \in \Sigma$ we define

$$\mathfrak{k}_\lambda = \mathfrak{k} \cap (\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}) \text{ and } \mathfrak{p}_\lambda = \mathfrak{p} \cap (\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}).$$

Then we have $\mathfrak{p}_\lambda = \mathfrak{p}_{-\lambda}$, $\mathfrak{k}_\lambda = \mathfrak{k}_{-\lambda}$ and $\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda = \mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}$ for all $\lambda \in \Sigma$. It is easy to see that the subspaces

$$\mathfrak{p}_\Phi = \mathfrak{l}_\Phi \cap \mathfrak{p} = \mathfrak{a} \oplus \left(\bigoplus_{\lambda \in \Sigma_\Phi} \mathfrak{p}_\lambda \right) \text{ and } \mathfrak{p}^s_\Phi = \mathfrak{m}_\Phi \cap \mathfrak{p} = \mathfrak{g}_\Phi \cap \mathfrak{p} = \mathfrak{a}^\Phi \oplus \left(\bigoplus_{\lambda \in \Sigma_\Phi} \mathfrak{p}_\lambda \right)$$

are Lie triple systems in $\mathfrak{p}$. We define a subalgebra $\mathfrak{k}_\Phi$ of $\mathfrak{k}$ by

$$\mathfrak{k}_\Phi = \mathfrak{q}_\Phi \cap \mathfrak{k} = \mathfrak{l}_\Phi \cap \mathfrak{k} = \mathfrak{m}_\Phi \cap \mathfrak{k} = \mathfrak{k}_0 \oplus \left(\bigoplus_{\lambda \in \Sigma_\Phi} \mathfrak{k}_\lambda \right).$$

Then $\mathfrak{g}_\Phi = (\mathfrak{g}_\Phi \cap \mathfrak{k}_\Phi) \oplus \mathfrak{p}^s_\Phi$ is a Cartan decomposition of the semisimple subalgebra $\mathfrak{g}_\Phi$ of $\mathfrak{g}$ and $\mathfrak{a}^\Phi$ is a maximal abelian subspace of $\mathfrak{p}^s_\Phi$. If we define $(\mathfrak{g}_\Phi)_0 = (\mathfrak{g}_\Phi \cap \mathfrak{k}_0) \oplus \mathfrak{a}^\Phi$, then $\mathfrak{g}_\Phi = (\mathfrak{g}_\Phi)_0 \oplus \left(\bigoplus_{\lambda \in \Sigma_\Phi} \mathfrak{g}_\lambda\right)$ is the restricted root space decomposition of $\mathfrak{g}_\Phi$ with respect to $\mathfrak{a}^\Phi$ and Φ is the corresponding set of simple roots. Since $\mathfrak{m}_\Phi = \mathfrak{z}_\Phi \oplus \mathfrak{g}_\Phi$ and $\mathfrak{z}_\Phi \subset \mathfrak{k}_0$, we see that $\mathfrak{g}_\Phi \cap \mathfrak{k}_0 = \mathfrak{k}_0 \ominus \mathfrak{z}_\Phi$.

We now relate these algebraic constructions to the geometry of the symmetric space M. Let Φ be a subset of Λ and $r_\Phi = |\Phi|$. We denote by A_Φ the connected abelian subgroup of G with Lie algebra $\mathfrak{a}_\Phi$ and by N_Φ the connected nilpotent subgroup of G with Lie algebra $\mathfrak{n}_\Phi$. The centralizer $L_\Phi = Z_G(\mathfrak{a}_\Phi)$ of $\mathfrak{a}_\Phi$ in G is a reductive subgroup of G with Lie algebra $\mathfrak{l}_\Phi$. The subgroup A_Φ is contained in the center of L_Φ. The subgroup L_Φ normalizes N_Φ and $Q_\Phi = L_\Phi N_\Phi$ is a subgroup of G with Lie algebra $\mathfrak{q}_\Phi$. The subgroup Q_Φ coincides with the normalizer $N_G(\mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi)$ of $\mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi$ in G, and hence Q_Φ is a closed subgroup of G. The subgroup Q_Φ is the parabolic subgroup of G associated with the subsystem Φ of Λ.

Let G_Φ be the connected subgroup of G with Lie algebra $\mathfrak{g}_\Phi$. Since $\mathfrak{g}_\Phi$ is semisimple, G_Φ is a semisimple subgroup of G. The intersection K_Φ of L_Φ and K, i.e. $K_\Phi = L_\Phi \cap K$, is a maximal compact subgroup of L_Φ and $\mathfrak{k}_\Phi$ is the Lie algebra of K_Φ. The adjoint group $\mathrm{Ad}(L_\Phi)$ normalizes $\mathfrak{g}_\Phi$, and consequently $M_\Phi = K_\Phi G_\Phi$ is a subgroup of L_Φ. One can show that M_Φ is a closed reductive subgroup of L_Φ, K_Φ is a maximal compact subgroup of M_Φ, and the center Z_Φ of M_Φ is a compact subgroup of K_Φ. The Lie algebra of M_Φ is $\mathfrak{m}_\Phi$ and L_Φ is isomorphic to the Lie group direct product $M_\Phi \times A_\Phi$, i.e. $L_\Phi = M_\Phi \times A_\Phi$. For this reason A_Φ is called the split component of L_Φ. The parabolic subgroup Q_Φ acts transitively on M and the isotropy subgroup at o is K_Φ, that is, $M = Q_\Phi / K_\Phi$.

Since $\mathfrak{g}_\Phi = (\mathfrak{g}_\Phi \cap \mathfrak{k}_\Phi) \oplus \mathfrak{p}^s_\Phi$ is a Cartan decomposition of the semisimple subalgebra $\mathfrak{g}_\Phi$, we have $[\mathfrak{p}^s_\Phi, \mathfrak{p}^s_\Phi] = \mathfrak{g}_\Phi \cap \mathfrak{k}_\Phi$. Thus G_Φ is the connected closed subgroup of G with Lie algebra $[\mathfrak{p}^s_\Phi, \mathfrak{p}^s_\Phi] \oplus \mathfrak{p}^s_\Phi$. Since $\mathfrak{p}^s_\Phi$ is a Lie triple system in $\mathfrak{p}$, the orbit $F^s_\Phi = G_\Phi \cdot o$ of the G_Φ-action on M containing o is a connected totally geodesic submanifold of M with $T_o F^s_\Phi = \mathfrak{p}^s_\Phi$. If $\Phi = \emptyset$, then $F^s_\emptyset = \{o\}$, otherwise F^s_Φ is a Riemannian symmetric space of noncompact type and $\mathrm{rank}(F^s_\Phi) = r_\Phi$, and

$$F^s_\Phi = G_\Phi \cdot o = G_\Phi / (G_\Phi \cap K_\Phi) = M_\Phi \cdot o = M_\Phi / K_\Phi.$$

The submanifold F^s_Φ is also known as a boundary component of M in the context of the maximal Satake compactification of M (see e.g. [6]).

Clearly, $\mathfrak{a}_\Phi$ is a Lie triple system as well, and the corresponding totally geodesic submanifold is a Euclidean space

$$\mathbb{E}^{r-r_\Phi} = A_\Phi \cdot o.$$

Since the action of A_Φ on M is free and A_Φ is simply connected, we can identify $\mathbb{E}^{r-r_\Phi}$, A_Φ and $\mathfrak{a}_\Phi$ canonically. This identification will be used throughout this paper.

Finally, $\mathfrak{p}_\Phi = \mathfrak{p}^s_\Phi \oplus \mathfrak{a}_\Phi$ is a Lie triple system, and the corresponding totally geodesic submanifold F_Φ is the symmetric space

$$F_\Phi = L_\Phi \cdot o = L_\Phi/K_\Phi = (M_\Phi \times A_\Phi)/K_\Phi = F^s_\Phi \times \mathbb{E}^{r-r_\Phi}.$$

The submanifolds F_Φ and F^s_Φ have a natural geometric interpretation. Denote by $\bar{C}^+(\Lambda) \subset \mathfrak{a}$ the closed positive Weyl chamber which is determined by the simple roots Λ. Let Z be nonzero vector in $\bar{C}^+(\Lambda)$ such that $\alpha(Z) = 0$ for all $\alpha \in \Phi$ and $\alpha(Z) > 0$ for all $\alpha \in \Lambda \setminus \Phi$, and consider the geodesic $\gamma_Z(t) = \operatorname{Exp}(tZ) \cdot o$ in M with $\gamma_Z(0) = o$ and $\dot{\gamma}_Z(0) = Z$. The totally geodesic submanifold F_Φ is the union of all geodesics in M which are parallel to γ_Z, and F^s_Φ is the semisimple part of F_Φ in the de Rham decomposition of F_Φ (see e.g. [9], Proposition 2.11.4 and Proposition 2.20.10).

The group Q_Φ is diffeomorphic to the product $M_\Phi \times A_\Phi \times N_\Phi$. This analytic diffeomorphism induces an analytic diffeomorphism between $F^s_\Phi \times \mathbb{E}^{r-r_\Phi} \times N_\Phi$ and M known as a horospherical decomposition of the symmetric space M.

4. Polar and hyperpolar foliations

We first prove an algebraic characterization of polar actions and of hyperpolar actions on Riemannian symmetric spaces of noncompact type (see also Proposition 4.1 in [19]), and then present some examples.

Theorem 4.1. *Let $M = G/K$ be a Riemannian symmetric space of noncompact type and H be a connected closed subgroup of G whose orbits form a homogeneous foliation $\mathcal{F}$ on M. Consider the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and define*

$$\mathfrak{h}^\perp_\mathfrak{p} = \{\xi \in \mathfrak{p} : \langle \xi, Y \rangle = 0 \textit{ for all } Y \in \mathfrak{h}\}.$$

Then the following statements hold:

(i) *The action of H on M is polar if and only if $\mathfrak{h}^\perp_\mathfrak{p}$ is a Lie triple system in $\mathfrak{p}$ and $\mathfrak{h}$ is orthogonal to the subalgebra $[\mathfrak{h}^\perp_\mathfrak{p}, \mathfrak{h}^\perp_\mathfrak{p}] \oplus \mathfrak{h}^\perp_\mathfrak{p}$ of $\mathfrak{g}$.*

(ii) *The action of H on M is hyperpolar if and only if $\mathfrak{h}^\perp_\mathfrak{p}$ is an abelian subspace of $\mathfrak{p}$.*

In both cases, let $H^\perp_\mathfrak{p}$ be the connected subgroup of G with Lie algebra $[\mathfrak{h}^\perp_\mathfrak{p}, \mathfrak{h}^\perp_\mathfrak{p}] \oplus \mathfrak{h}^\perp_\mathfrak{p}$. Then the orbit $\mathcal{S} = H^\perp_\mathfrak{p} \cdot o$ is a section of the H-action on M.

Proof. Statement (ii) is an obvious consequence of statement (i). So we proceed with proving (i).

If the action of H on M is polar, then $\mathfrak{h}^\perp_\mathfrak{p}$ is a Lie triple system by definition of a polar action. We now assume that $\mathfrak{h}^\perp_\mathfrak{p}$ is a Lie triple system. We have to show that the action of H on M is polar if and only if $\mathfrak{h}$ is orthogonal to $[\mathfrak{h}_{\mathfrak{p}}^\perp, \mathfrak{h}_{\mathfrak{p}}^\perp] \oplus \mathfrak{h}_{\mathfrak{p}}^\perp$. Since $\mathfrak{h}_{\mathfrak{p}}^\perp$ is a Lie triple system, the orbit $\mathcal{S} = H_{\mathfrak{p}}^\perp \cdot o$ is a connected complete totally geodesic submanifold of M. Let p be a point in M which does not lie on the orbit $H \cdot o$. Since $H \cdot o$ is a closed submanifold of M, there exists a point $q \in H \cdot o$ such that the distance between p and q is equal to the distance between p and $H \cdot o$. Since M is complete, there exists a geodesic in M from p to q such that the distance from p to q can be measured along this geodesic. A standard variational argument shows that this geodesic intersects $H \cdot o$ perpendicularly. It follows now easily that $\mathcal{S}$ intersects each orbit. Since H induces a foliation, it therefore remains to show that $T_p(H \cdot p)$ and $T_p\mathcal{S}$ are orthogonal for each $p \in \mathcal{S}$ if and only if $\mathfrak{h}$ is orthogonal to $[\mathfrak{h}_{\mathfrak{p}}^\perp, \mathfrak{h}_{\mathfrak{p}}^\perp] \oplus \mathfrak{h}_{\mathfrak{p}}^\perp$.

Let γ be the geodesic in $\mathcal{S}$ with $\gamma(0) = o$ and $\dot\gamma(0) = \xi \in \mathfrak{h}_{\mathfrak{p}}^\perp$, and assume that $\xi \neq 0$. For $X \in \mathfrak{h}$ and $\eta \in \mathfrak{h}_{\mathfrak{p}}^\perp$ we denote by X^* and η^* the Killing vector fields on M that are induced from X and η, respectively. Then we have

$$T_{\gamma(t)}(H \cdot \gamma(t)) = \{X^*_{\gamma(t)} : X \in \mathfrak{h}\}\ ,\ T_{\gamma(t)}\mathcal{S} = \{\eta^*_{\gamma(t)} : \eta \in \mathfrak{h}_{\mathfrak{p}}^\perp\}.$$

The restrictions of two such Killing vector fields X^* and η^* to γ satisfy the equation

$$\left.\frac{d}{dt}\right|_{t=0} \langle X^*_{\gamma(t)}, \eta^*_{\gamma(t)}\rangle = \langle [\xi^*, X^*]_o, \eta^*_o\rangle + \langle X^*_o, [\xi^*, \eta^*]_o\rangle = -2\langle [\xi, \eta], X\rangle,$$

using the facts that $[\xi^*, X^*] = -[\xi, X]^*$ and $[\xi^*, \eta^*] = -[\xi, \eta]^*$, and that $\mathrm{ad}(\xi)$ is a self-adjoint endomorphism on $\mathfrak{g}$. From this it easily follows that $\mathfrak{h}$ is orthogonal to $[\mathfrak{h}_{\mathfrak{p}}^\perp, \mathfrak{h}_{\mathfrak{p}}^\perp] \oplus \mathfrak{h}_{\mathfrak{p}}^\perp$ if $T_p(H \cdot p)$ and $T_p\mathcal{S}$ are orthogonal for each $p \in \mathcal{S}$. Conversely, assume that $\mathfrak{h}$ is orthogonal to $[\mathfrak{h}_{\mathfrak{p}}^\perp, \mathfrak{h}_{\mathfrak{p}}^\perp] \oplus \mathfrak{h}_{\mathfrak{p}}^\perp$. Then, for each $X \in \mathfrak{h}$, the restriction X^*_γ of the Killing vector field X^* to γ is the Jacobi vector field along γ with initial values $X^*_\gamma(0) = X^*_o = X_{\mathfrak{p}} \in \mathfrak{h}_{\mathfrak{p}}$ and $(X^*_\gamma)'(0) = [\xi^*, X^*]_o = -[\xi, X]^*_o = -[\xi, X]_{\mathfrak{p}} \in \mathfrak{h}_{\mathfrak{p}}$. Here the subscript indicates orthogonal projection onto $\mathfrak{p}$. Since both initial values are in $\mathfrak{h}_{\mathfrak{p}} = \nu_o\mathcal{S}$, it follows that X^*_γ takes values in the normal bundle of $\mathcal{S}$ along γ. This implies that $T_{\gamma(t)}(H \cdot \gamma(t))$ and $T_{\gamma(t)}\mathcal{S}$ are orthogonal for each $t \in \mathbb{R}$. Since this holds for each geodesic γ in $\mathcal{S}$ with $\gamma(0) = o$ and $\dot\gamma(0) = \xi \in \mathfrak{h}_{\mathfrak{p}}^\perp$, $\xi \neq 0$, we conclude that $T_p(H \cdot p)$ and $T_p\mathcal{S}$ are orthogonal for each $p \in \mathcal{S}$. □

We will use the previous result to show polarity and hyperpolarity of certain actions.

Proposition 4.2. *Let M be a Riemannian symmetric space of noncompact type and consider a horospherical decomposition $F^s_\Phi \times \mathbb{E}^{r-r_\Phi} \times N_\Phi$ of M. Let V be a linear subspace of $\mathbb{E}^{r-r_\Phi}$ and assume that $(\Phi, V) \neq (\emptyset, \mathbb{E}^r)$. Then the action of $V \times N_\Phi \subset A_\Phi \times N_\Phi$ on M is polar and $F^s_\Phi \times (\mathbb{E}^{r-r_\Phi} \ominus V)$ is a section of this action. Moreover, the action of $V \times N_\Phi$ on M is hyperpolar if and only if $\Phi = \emptyset$.*

Proof. The subspace $(V \oplus \mathfrak{n}_\Phi)^\perp_{\mathfrak{p}} = \mathfrak{p}_\Phi \ominus V$ of $\mathfrak{p}$ is a Lie triple system and $F^s_\Phi \times (\mathbb{E}^{r-r_\Phi} \ominus V)$ is the connected complete totally geodesic submanifold of M corresponding to $\mathfrak{p}_\Phi \ominus V$. Next, we have $[(V \oplus \mathfrak{n}_\Phi)^\perp_{\mathfrak{p}}, (V \oplus \mathfrak{n}_\Phi)^\perp_{\mathfrak{p}}] = [\mathfrak{p}_\Phi \ominus V, \mathfrak{p}_\Phi \ominus V] \subset \mathfrak{k}_\Phi \subset \mathfrak{m}_\Phi$, and since $\mathfrak{m}_\Phi$ is orthogonal to $\mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$, we see that $V \oplus \mathfrak{n}_\Phi$ is orthogonal to $[(V \oplus \mathfrak{n}_\Phi)^\perp_{\mathfrak{p}}, (V \oplus \mathfrak{n}_\Phi)^\perp_{\mathfrak{p}}] \oplus (V \oplus \mathfrak{n}_\Phi)^\perp_{\mathfrak{p}}$. Since $A_\Phi \times N_\Phi$ acts freely on M, it is clear that $V \times N_\Phi$ induces a foliation on M. From Theorem

4.1 we conclude that the action of $V \times N_\Phi$ on M is polar and that $F^s_\Phi \times (\mathbb{E}^{r-r_\Phi} \ominus V)$ is a section of the action. The statement about hyperpolarity follows from the fact that $F^s_\Phi \times (\mathbb{E}^{r-r_\Phi} \ominus V)$ is flat if and only if $\Phi = \emptyset$. $\square$

The previous result provides examples of polar actions which are not hyperpolar on each Riemannian symmetric space of noncompact type with rank ≥ 2. It is worthwhile to compare this with the results by Kollross [19] that in the compact case polar actions are in general hyperpolar. Special cases of these actions on Hermitian symmetric spaces of noncompact type have also been discussed by Kobayashi [16] in the context of strongly visible actions on complex manifolds.

Remark 4.3. Let M be a symmetric space of noncompact type with the property that its restricted root system contains two simple roots of the same length which are not connected in the Dynkin diagram. The following example illustrates that the condition in Theorem 4.1 (i) that $\mathfrak{h}$ is orthogonal to $[\mathfrak{h}^\perp_\mathfrak{p}, \mathfrak{h}^\perp_\mathfrak{p}] \oplus \mathfrak{h}^\perp_\mathfrak{p}$ is necessary for polarity. Let us consider $\mathfrak{h} = (\mathfrak{a} \ominus \mathbb{R}(H_\alpha - H_\beta)) \oplus (\mathfrak{n} \ominus \mathbb{R}(X_\alpha + X_\beta))$, with α and β two simple roots of the same length which are not connected in the Dynkin diagram, and $X_\alpha \in \mathfrak{g}_\alpha$, $X_\beta \in \mathfrak{g}_\beta$ unit vectors. In order to prove that this is indeed a subalgebra, by the properties of root systems it suffices to show that $[H, X_\alpha - X_\beta] \in \mathfrak{h}$ for any $H \in \mathfrak{a} \ominus \mathbb{R}(H_\alpha - H_\beta)$ (because $\langle X_\alpha + X_\beta, X_\alpha - X_\beta\rangle = 0$). However, if $H \in \mathfrak{a} \ominus \mathbb{R}(H_\alpha - H_\beta)$ we have $\alpha(H) = \beta(H)$, which implies $[H, X_\alpha - X_\beta] = \alpha(H)(X_\alpha - X_\beta) \in \mathfrak{h}$ as desired.

By construction we have $\mathfrak{h}^\perp_\mathfrak{p} = \mathbb{R}(H_\alpha - H_\beta) \oplus \mathbb{R}(1-\theta)(X_\alpha + X_\beta)$. A simple calculation using $\langle \alpha, \alpha\rangle = \langle \beta, \beta\rangle$ and $\langle \alpha, \beta\rangle = 0$ shows that

$$[H_\alpha - H_\beta, (1-\theta)(X_\alpha + X_\beta)] = \langle \alpha, \alpha\rangle(1+\theta)(X_\alpha - X_\beta).$$

This implies in particular that $\mathfrak{h}^\perp_\mathfrak{p}$ is not abelian. Using again $\langle \alpha, \alpha\rangle = \langle \beta, \beta\rangle$ and $\langle \alpha, \beta\rangle = 0$ we get

$$[H_\alpha - H_\beta, (1+\theta)(X_\alpha - X_\beta)] = \langle \alpha, \alpha\rangle(1-\theta)(X_\alpha + X_\beta),$$

and also using $[X_\alpha, X_\beta] = 0$ (because α and β are not connected in the Dynkin diagram) we obtain

$$[(1-\theta)(X_\alpha + X_\beta), (1+\theta)(X_\alpha - X_\beta)] = -2(H_\alpha - H_\beta).$$

All in all this means that $\mathfrak{h}^\perp_\mathfrak{p}$ is a non-abelian Lie triple system. However, $\mathfrak{h}$ cannot give rise to a polar action because $\mathfrak{h}$ is not perpendicular to $[\mathfrak{h}^\perp_\mathfrak{p}, \mathfrak{h}^\perp_\mathfrak{p}] = \mathbb{R}(1+\theta)(X_\alpha - X_\beta)$.

This action has the interesting feature that it gives a homogeneous foliation with the property that the normal bundle consists of Lie triple systems. It is easy to see that the totally geodesic submanifold of M generated by any of these Lie triple systems is a real hyperbolic plane. These real hyperbolic planes have the property that they do not intersect orthogonally the other orbits. It is interesting to observe that the normal bundle is not integrable, as otherwise the integral manifolds would provide sections and then the action would be polar.

Remark 4.4. The hypothesis in Theorem 4.1 that H induces a foliation is necessary. For example, in $\mathfrak{sl}_2(\mathbb{C})$ consider the usual Cartan decomposition $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{su}_2 \oplus \mathfrak{p}$, where $\mathfrak{p}$ denotes the real vector space of (2×2)-Hermitian matrices with trace zero. Let $\mathfrak{a}$ be the subspace of diagonal matrices in $\mathfrak{sl}_2(\mathbb{C})$ with real coefficients and $\mathfrak{t}$ the subspace of diagonal matrices in $\mathfrak{sl}_2(\mathbb{C})$ with purely imaginary coefficients. Also, denote by $\mathfrak{n}$ the set of strictly upper triangular matrices in $\mathfrak{sl}_2(\mathbb{C})$. Then, $\mathfrak{su}_2 \oplus \mathfrak{a} \oplus \mathfrak{n}$ is an Iwasawa decomposition of $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{C})$. Consider the vectors

$$B = \begin{pmatrix} 1 & i \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} -i & 1 \\ 0 & i \end{pmatrix}, \quad \xi = \begin{pmatrix} 1 & -2i \\ -2i & -1 \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 0 & i/2 \\ 0 & 0 \end{pmatrix}.$$

Let $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ spanned by B and X. Then $\mathfrak{h}^\perp_{\mathfrak{p}} = \mathbb{R}\xi$ is abelian because it is one-dimensional. The connected closed subgroup H of $SL_2(\mathbb{C})$ with Lie algebra $\mathfrak{h}$ acts hyperpolarly on the real hyperbolic space $\mathbb{R}H^3 = SL_2(\mathbb{C})/SU_2$ but does not give rise to a hyperpolar foliation. To see this let $g = \mathrm{Exp}(E)$. It is easy to verify that $\mathrm{Ad}(g)B \in \mathfrak{a}$ and $\mathrm{Ad}(g)X \in \mathfrak{t}$, and hence $\mathrm{Ad}(g)\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$. The corresponding connected subgroup of $SL_2(\mathbb{C})$ acts with cohomogeneity one on $\mathbb{R}H^3$. This action has one singular orbit, a totally geodesic $\mathbb{R}H^1 \subset \mathbb{R}H^3$, and the other orbits are the tubes around it. Obviously, the action of H is orbit equivalent to this one.

We continue with a discussion of some further hyperpolar actions on Riemannian symmetric spaces of nonpositive curvature.

Example 4.5. (*Polar and hyperpolar homogeneous foliations on Euclidean spaces.*) Let m be a positive integer. For each linear subspace V of the m-dimensional Euclidean space $\mathbb{E}^m$ we define a homogeneous hyperpolar foliation $\mathcal{F}^m_V$ on $\mathbb{E}^m$ by

$$(\mathcal{F}^m_V)_p = p + V = \{p + v \mid v \in V\}$$

for all $p \in \mathbb{E}^m$. Geometrically, the foliation $\mathcal{F}^m_V$ consists of the union of all affine subspaces of $\mathbb{E}^m$ which are parallel to V. It is obvious that $\mathcal{F}^m_V$ is a hyperpolar homogeneous foliation on $\mathbb{E}^m$ whenever $0 < \dim V < m$.

Indeed, any hyperpolar homogeneous foliation on a Euclidean space $\mathbb{E}^m$ is isometrically congruent to one of these examples. Assume that H acts isometrically on $\mathbb{E}^m$ and that its orbits form a hyperpolar homogeneous foliation. Since the action of H is isometric and gives a foliation on $\mathbb{E}^m$ it suffices to prove that each orbit of H is totally geodesic.

On the contrary, assume that the orbit of H through o is not totally geodesic. Then, there exist a nonzero vector $v \in T_o(H \cdot o)$ and a unit vector $\xi \in \nu_o(H \cdot o)$ such that $A_\xi v = cv$ with $c \neq 0$, where A_ξ denotes the shape operator of $H \cdot o$ with respect to ξ. Since the orbit through o is principal, ξ induces an equivariant normal vector field on $H \cdot o$ which we also denote by ξ. This vector field satisfies $\xi_{h(o)} = h_* \xi_o$ for all $h \in H$. Consider the point $p = \exp_o(\frac{1}{c}\xi_o)$. Since ξ is equivariant, the orbit of H through p is $H \cdot p = \{\exp_{h(o)}(\frac{1}{c}\xi_{h(o)}) : h \in H\}$. Hence we can define the map $F : H \cdot o \to H \cdot p$, $h(o) \mapsto \exp_{h(o)}(\frac{1}{c}\xi_{h(o)}) = h(o) + \frac{1}{c}\xi_{h(o)}$. Since the action of H is polar, the equivariant vector field ξ is parallel with respect to the normal connection (see e.g. [2], p. 44, Corollary 3.2.5), and thus we get $F_{*o}v = v - \frac{1}{c}A_\xi v = 0$, which contradicts the fact that H gives a foliation.

Example 4.6. (*Codimension one foliations on Riemannian manifolds.*) Let M be a connected complete Riemannian manifold and $\mathcal{F}$ be a homogeneous foliation on M with codimension one. Then $\mathcal{F}$ is hyperpolar. In fact, consider a geodesic $\gamma : \mathbb{R} \to M$ which is parametrized by arc length and for which $\dot\gamma(0)$ is perpendicular to $\mathcal{F}_{\gamma(0)}$. Since M is connected and complete, γ must intersect each leaf of $\mathcal{F}$, and since $\mathcal{F}$ is homogeneous, the geodesic intersects each leaf orthogonally. Therefore $\mathcal{S} = \gamma(\mathbb{R})$ is a section of $\mathcal{F}$. Clearly, $\mathcal{S}$ is a flat totally geodesic submanifold of M, and hence $\mathcal{F}$ is hyperpolar.

Example 4.7. (*Hyperpolar homogeneous foliations on hyperbolic spaces.*) Let M be a Riemannian symmetric space of rank one, that is, M is a hyperbolic space $\mathbb{F}H^n$ over a normed real division algebra $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. Here $n \geq 2$, and $n = 2$ if $\mathbb{F} = \mathbb{O}$. Using the notations introduced in the previous section, we have

$$\mathfrak{g} = \begin{cases} \mathfrak{so}_{1,n} & \text{if } \mathbb{F} = \mathbb{R}, \\ \mathfrak{su}_{1,n} & \text{if } \mathbb{F} = \mathbb{C}, \\ \mathfrak{sp}_{1,n} & \text{if } \mathbb{F} = \mathbb{H}, \\ \mathfrak{f}_4^{-20} & \text{if } \mathbb{F} = \mathbb{O}. \end{cases}$$

The restricted root space decomposition of $\mathfrak{g}$ is of the form

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha},$$

where $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha} = (n-1)\dim_{\mathbb{R}} \mathbb{F}$, $\dim \mathfrak{g}_{2\alpha} = \dim \mathfrak{g}_{-2\alpha} = \dim_{\mathbb{R}} \mathbb{F} - 1$. Moreover, we have $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$ with a one-dimensional subspace $\mathfrak{a} \subset \mathfrak{p}$ and

$$\mathfrak{k}_0 \cong \begin{cases} \mathfrak{so}_{n-1} & \text{if } \mathbb{F} = \mathbb{R}, \\ \mathfrak{u}_{n-1} & \text{if } \mathbb{F} = \mathbb{C}, \\ \mathfrak{sp}_{n-1} \oplus \mathfrak{sp}_1 & \text{if } \mathbb{F} = \mathbb{H}, \\ \mathfrak{so}_7 & \text{if } \mathbb{F} = \mathbb{O}. \end{cases}$$

Let ℓ be a one-dimensional linear subspace of $\mathfrak{g}_\alpha$ and define

$$\mathfrak{s}_\ell = \mathfrak{a} \oplus (\mathfrak{g}_\alpha \ominus \ell) \oplus \mathfrak{g}_{2\alpha} = \mathfrak{a} \oplus (\mathfrak{n} \ominus \ell).$$

The subspace $\mathfrak{s}_\ell$ is a subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$ of codimension one, and the corresponding connected closed subgroup S_ℓ of AN acts freely on $\mathbb{F}H^n$ with cohomogeneity one. The corresponding homogeneous foliation $\mathcal{F}_\ell$ on $\mathbb{F}H^n$ is hyperpolar according to the previous example. Since K_0 acts transitively on the unit sphere in $\mathfrak{g}_\alpha$ by means of the adjoint representation, $\mathcal{F}_\ell$ and $\mathcal{F}_{\ell'}$ are orbit equivalent for any two one-dimensional linear subspaces ℓ, ℓ' of $\mathfrak{g}_\alpha$. We denote by $\mathcal{F}_{\mathbb{F}}^n$ a representative of the set of hyperpolar homogeneous foliations of the form $\mathcal{F}_\ell$ on $\mathbb{F}H^n$. We mention that the leaf of $\mathcal{F}_{\mathbb{F}}^n$ containing $o \in \mathbb{F}H^n$ is a minimal hypersurface in $\mathbb{F}H^n$. If $\mathbb{F} = \mathbb{R}$, this leaf is a totally geodesic real hyperbolic hyperplane $\mathbb{R}H^{n-1} \subset \mathbb{R}H^n$. If $\mathbb{F} = \mathbb{C}$, this leaf is the minimal ruled real hypersurface in $\mathbb{C}H^n$ which is determined by a horocycle in a totally geodesic and totally real $\mathbb{R}H^2 \subset \mathbb{C}H^n$. For more details on these foliations we refer to [1] and [3]. It was shown in [4] that apart from this foliation and the horosphere foliation there are no other homogeneous hyperpolar foliations on Riemannian symmetric spaces of rank one.

Example 4.8. (*Hyperpolar homogeneous foliations on products of hyperbolic spaces.*) Let

$$M = \mathbb{F}_1 H^{n_1} \times \ldots \times \mathbb{F}_k H^{n_k}$$

be the Riemannian product of k Riemannian symmetric spaces of rank one, where k is a positive integer. Then

$$\mathcal{F}^{n_1}_{\mathbb{F}_1} \times \ldots \times \mathcal{F}^{n_k}_{\mathbb{F}_k}$$

is a hyperpolar homogeneous foliation on M. This is an elementary consequence of the previous example.

Example 4.9. (*Hyperpolar homogeneous foliations on products of hyperbolic spaces and Euclidean spaces.*) Let

$$M = \mathbb{F}_1 H^{n_1} \times \ldots \times \mathbb{F}_k H^{n_k} \times \mathbb{E}^m$$

be the Riemannian product of k Riemannian symmetric spaces of rank one and an m-dimensional Euclidean space, where k and m are positive integers. Moreover, let V be a linear subspace of $\mathbb{E}^m$. Then

$$\mathcal{F}^{n_1}_{\mathbb{F}_1} \times \ldots \times \mathcal{F}^{n_k}_{\mathbb{F}_k} \times \mathcal{F}^m_V$$

is a hyperpolar homogeneous foliation on M.

Example 4.10. (*Homogeneous foliations on symmetric spaces of noncompact type.*) Let M be a Riemannian symmetric space of noncompact type and Φ be a subset of Λ with the property that any two roots in Φ are not connected in the Dynkin diagram of the restricted root system associated with Λ. We call such a subset Φ an orthogonal subset of Λ. Each simple root $\alpha \in \Phi$ determines a totally geodesic hyperbolic space $\mathbb{F}_\alpha H^{n_\alpha} \subset M$. In fact, $\mathbb{F}_\alpha H^{n_\alpha} \subset M$ is the orbit of the connected subgroup of G with Lie algebra $\mathfrak{g}_{\{\alpha\}}$. Then F_Φ is isometric to the Riemannian product of r_Φ Riemannian symmetric spaces of rank one and an $(r - r_\Phi)$-dimensional Euclidean space,

$$F_\Phi = F^s_\Phi \times \mathbb{E}^{r-r_\Phi} \cong \left(\prod_{\alpha \in \Phi} \mathbb{F}_\alpha H^{n_\alpha} \right) \times \mathbb{E}^{r-r_\Phi}.$$

Note that $\mathbb{F}_\alpha = \mathbb{R}$ if α is reduced and $\mathbb{F}_\alpha \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ if α is non-reduced (i.e., if $2\alpha \in \Sigma$ as well). Then

$$\mathcal{F}_\Phi = \prod_{\alpha \in \Phi} \mathcal{F}^{n_\alpha}_{\mathbb{F}_\alpha}.$$

is a hyperpolar homogeneous foliation on $\mathcal{F}^s_\Phi$. Let V be a linear subspace of $\mathbb{E}^{r-r_\Phi}$. Then

$$\mathcal{F}_{\Phi,V} = \mathcal{F}_\Phi \times \mathcal{F}^{r-r_\Phi}_V \times N_\Phi \subset F^s_\Phi \times \mathbb{E}^{r-r_\Phi} \times N_\Phi = F_\Phi \times N_\Phi \cong M$$

is a homogeneous foliation on M. We will see below that it is hyperpolar.

Recall that each foliation $\mathcal{F}^{n_\alpha}_{\mathbb{F}_\alpha}$ on $\mathbb{F}_\alpha H^{n_\alpha}$ corresponds to a subalgebra of $\mathfrak{g}_{\{\alpha\}}$ of the form $\mathfrak{a}^{\{\alpha\}} \oplus (\mathfrak{g}_\alpha \ominus \ell_\alpha) \oplus \mathfrak{g}_{2\alpha}$ with some one-dimensional linear subspace ℓ_α of $\mathfrak{g}_\alpha$. Thus the foliation $\mathcal{F}_\Phi$ on F^s_Φ corresponds to the subalgebra $\mathfrak{a}^\Phi \oplus \left(\bigoplus_{\alpha \in \Phi} ((\mathfrak{g}_\alpha \ominus \ell_\alpha) \oplus \mathfrak{g}_{2\alpha}) \right) = \mathfrak{a}^\Phi \oplus (\mathfrak{n}_\Phi \ominus \ell_\Phi)$ of $\mathfrak{g}_\Phi$, where $\ell_\Phi = \bigoplus_{\alpha \in \Phi} \ell_\alpha$. Therefore the foliation $\mathcal{F}_{\Phi,V}$ on M corresponds to the subalgebra

$$\mathfrak{s}_{\Phi,V} = (\mathfrak{a}^\Phi \oplus V) \oplus (\mathfrak{n}_\Phi \ominus \ell_\Phi) = (\mathfrak{a}^\Phi \oplus V \oplus \mathfrak{n}_\Phi) \ominus \ell_\Phi \subset \mathfrak{a} \oplus \mathfrak{n}_\Phi$$

of $\mathfrak{q}_\Phi$, where we identify canonically $V \subset \mathbb{E}^{r-r_\Phi} = A_\Phi \cdot o$ with the corresponding subspace of $\mathfrak{a}_\Phi$.

It is easy to see from the arguments given above that different choices of ℓ_α and ℓ'_α in $\mathfrak{g}_\alpha$ lead to isometrically congruent foliations $\mathcal{F}_\Phi$ and $\mathcal{F}'_\Phi$ on F^s_Φ. However, it is not obvious that different choices of ℓ_α and ℓ'_α in $\mathfrak{g}_\alpha$ lead to isometrically congruent foliations $\mathcal{F}_{\Phi,V}$ and $\mathcal{F}'_{\Phi,V}$ on M. That this is in fact true follows from the following two facts. On a semisimple symmetric space the holonomy algebra is isomorphic to the Lie algebra of the isotropy subgroup of the isometry group. Moreover, on a simply connected symmetric space each element of the holonomy group at a point o induces an isometry of the symmetric space with fixed point o. Hence, different choices of ℓ_α and ℓ'_α in $\mathfrak{g}_\alpha$ lead to isometrically congruent foliations $\mathcal{F}_{\Phi,V}$ and $\mathcal{F}'_{\Phi,V}$ on M.

We note that these homogeneous foliations on symmetric spaces of noncompact type have also been discussed by Koike [17] in the context of his investigations about "complex hyperpolar actions".

We are now in the position to formulate the main result of this paper.

Theorem 4.11. *Let M be a connected Riemannian symmetric space of noncompact type.*

(i) *Let Φ be an orthogonal subset of Λ and V be a linear subspace of $\mathbb{E}^{r-r_\Phi}$. Then*

$$\mathcal{F}_{\Phi,V} = \mathcal{F}_\Phi \times \mathcal{F}_V^{r-r_\Phi} \times N_\Phi \subset F^s_\Phi \times \mathbb{E}^{r-r_\Phi} \times N_\Phi = M$$

is a hyperpolar homogeneous foliation on M.

(ii) *Every hyperpolar homogeneous foliation on M is isometrically congruent to $\mathcal{F}_{\Phi,V}$ for some orthogonal subset Φ of Λ and some linear subspace V of $\mathbb{E}^{r-r_\Phi}$.*

Proof. We prove part (i) of the theorem here. Section 5 is devoted to the proof of part (ii).

According to Theorem 4.1 we have to prove that $\mathfrak{s}_{\Phi,V}$ is a subalgebra and that $(\mathfrak{s}_{\Phi,V})^\perp_\mathfrak{p} = \{\xi \in \mathfrak{p} : \langle \xi, Y\rangle = 0 \text{ for all } Y \in \mathfrak{s}_{\Phi,V}\}$ is abelian. Assume that the one-dimensional linear space $\ell_\alpha = \ell_\Phi \cap \mathfrak{g}_\alpha$ is generated by the nonzero vector E_α.

The fact that $\mathfrak{s}_{\Phi,V}$ is a subalgebra follows from the elementary properties of root systems. It is easy to see that

$$(\mathfrak{s}_{\Phi,V})^\perp_\mathfrak{p} = (\mathfrak{a}_\Phi \ominus V) \oplus \left(\bigoplus_{\alpha\in\Phi} \mathbb{R}((1-\theta)E_\alpha) \right).$$

We now check that $(\mathfrak{s}_{\Phi,V})^\perp_\mathfrak{p}$ is abelian.

If $H, H' \in \mathfrak{a}_\Phi \ominus V$ we obviously have $[H, H'] = 0$. If $H \in \mathfrak{a}_\Phi \ominus V$ and $\alpha \in \Phi$ we have $[H, (1-\theta)E_\alpha] = \alpha(H)(1+\theta)E_\alpha = 0$ by definition of $\mathfrak{a}_\Phi$. If $\alpha, \beta \in \Phi$ with $\alpha \neq \beta$, then $[(1-\theta)E_\alpha, (1-\theta)E_\beta] = (1+\theta)[E_\alpha, E_\beta] - (1+\theta)[E_\alpha, \theta E_\beta]$. Now, $[E_\alpha, E_\beta] \in \mathfrak{g}_{\alpha+\beta} = 0$ because $\alpha + \beta$ is not a root (since α and β are not connected in the Dynkin diagram) and $[E_\alpha, \theta E_\beta] \in \mathfrak{g}_{\alpha-\beta} = 0$ as $\alpha - \beta$ is not a root (because both α and β are simple). ∎

5. Classification

In this section we prove Theorem 4.11 (ii), thus settling the main result of this paper.

A subalgebra $\mathfrak{b}$ of a Lie algebra $\mathfrak{g}$ is called a *Borel subalgebra* if $\mathfrak{b}$ is a maximal solvable subalgebra of $\mathfrak{g}$. Borel subalgebras of real semisimple Lie algebras have been described in [20]. Any such Borel subalgebra can be written as $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{n}$ is nilpotent. The subspace $\mathfrak{t}$ is called the toroidal part of $\mathfrak{h}$ and consists of all $X \in \mathfrak{h}$ for which the eigenvalues of $\mathrm{ad}(X)$ are purely imaginary. The subspace $\mathfrak{a}$ is called the vector part of $\mathfrak{h}$ and consists of all $X \in \mathfrak{h}$ for which the eigenvalues of $\mathrm{ad}(X)$ are real. There exists a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that $\mathfrak{t} \subset \mathfrak{k}$ and $\mathfrak{a} \subset \mathfrak{p}$. We say that $\mathfrak{h}$ or $\mathfrak{b}$ is *maximally noncompact* if $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$ and *maximally compact* if $\mathfrak{t}$ is maximal abelian in $\mathfrak{k}$. We use this description for the following

Proposition 5.1. *Let $M = G/K$ be a symmetric space of noncompact type. Let S be a closed subgroup of G which induces a hyperpolar foliation. Then the action of S is orbit equivalent to the action of a closed solvable subgroup whose Lie algebra is contained in a maximally noncompact Borel subalgebra.*

Proof. By means of Proposition 2.2 we may assume that S is solvable and closed in $I(M)$. The Lie algebra $\mathfrak{s}$ of S is contained in a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. As we explained above, there exists a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that $\mathfrak{b} = \mathfrak{t} \oplus \tilde{\mathfrak{a}} \oplus \tilde{\mathfrak{n}}$ with $\mathfrak{t} \subset \mathfrak{k}$ and $\tilde{\mathfrak{a}} \subset \mathfrak{p}$. Since $\tilde{\mathfrak{a}}$ is abelian we have the decomposition $\mathfrak{g} = \tilde{\mathfrak{g}}_0 \oplus \left(\bigoplus_{\tilde{\lambda} \in \tilde{\Sigma}} \tilde{\mathfrak{g}}_{\tilde{\lambda}}\right)$, where $\tilde{\Sigma}$ is the set of roots with respect to $\tilde{\mathfrak{a}}$ and $\tilde{\mathfrak{g}}_{\tilde{\lambda}} = \{X \in \mathfrak{g} : \mathrm{ad}(H)X = \tilde{\lambda}(H)X \text{ for all } H \in \tilde{\mathfrak{a}}\}$. We can choose an ordering in $\tilde{\mathfrak{a}}$ that induces a set of positive roots $\tilde{\Sigma}^+$ in such a way that $\tilde{\mathfrak{n}} = \bigoplus_{\tilde{\lambda} \in \tilde{\Sigma}^+} \tilde{\mathfrak{g}}_{\tilde{\lambda}}$. It remains to prove that this Borel subalgebra is maximally noncompact, that is, $\tilde{\mathfrak{a}}$ is maximal abelian in $\mathfrak{p}$.

On the contrary, assume that $\tilde{\mathfrak{a}}$ is not maximal abelian. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ containing $\tilde{\mathfrak{a}}$. Then we have the usual restricted root space decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda\right)$. We choose an ordering of $\mathfrak{a}$ compatible with that of $\tilde{\mathfrak{a}}$ and denote by Σ^+ the corresponding set of positive roots, and write $\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$. We have the relations

$$\tilde{\mathfrak{a}} = \bigcap_{\substack{\lambda \in \Sigma^+ \\ \lambda_{|\tilde{\mathfrak{a}}} = 0}} \operatorname{Ker} \lambda, \qquad \tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \oplus \left(\bigoplus_{\substack{\lambda \in \Sigma^+ \\ \lambda_{|\tilde{\mathfrak{a}}} = 0}} \mathfrak{g}_\lambda \right), \qquad \tilde{\mathfrak{g}}_{\tilde{\lambda}} = \bigoplus_{\substack{\lambda \in \Sigma^+ \\ \lambda_{|\tilde{\mathfrak{a}}} = \tilde{\lambda}}} \mathfrak{g}_\lambda.$$

Recall from Theorem 4.1 (ii) that S acts hyperpolarly on M if and only if $\mathfrak{s}_{\mathfrak{p}}^\perp = \{\xi \in \mathfrak{p} : \xi \perp \mathfrak{s}\}$ is abelian. Obviously, $\mathfrak{a} \ominus \tilde{\mathfrak{a}} \subset \mathfrak{p}$ and $\mathfrak{a} \ominus \tilde{\mathfrak{a}}$ is orthogonal to $\tilde{\mathfrak{n}}$, and so $\mathfrak{a} \ominus \tilde{\mathfrak{a}} \subset \mathfrak{s}_{\mathfrak{p}}^\perp$. On the other hand, $\bigoplus_{\lambda \in \Sigma^+, \lambda_{|\tilde{\mathfrak{a}}} = 0} \mathfrak{g}_\lambda \subset \tilde{\mathfrak{g}}_0 \subset \mathfrak{n} \ominus \tilde{\mathfrak{n}}$, and so $\bigoplus_{\lambda \in \Sigma^+, \lambda_{|\tilde{\mathfrak{a}}} = 0} \mathfrak{p}_\lambda \subset \mathfrak{s}_{\mathfrak{p}}^\perp$. Altogether this implies $(\mathfrak{a} \ominus \tilde{\mathfrak{a}}) \oplus \left(\bigoplus_{\lambda \in \Sigma^+, \lambda_{|\tilde{\mathfrak{a}}} = 0} \mathfrak{p}_\lambda\right) \subset \mathfrak{s}_{\mathfrak{p}}^\perp$. Now choose $\lambda \in \Sigma^+$ with $\lambda_{|\tilde{\mathfrak{a}}} = 0$. By the first relation above, we can choose $H \in \mathfrak{a} \ominus \tilde{\mathfrak{a}}$ with $\lambda(H) \neq 0$. If $X_\lambda \in \mathfrak{g}_\lambda$ is a nonzero vector then $[H, (1-\theta)X_\lambda] = (1+\theta)\lambda(H)X_\lambda \neq 0$, which contradicts the fact that $\mathfrak{s}_{\mathfrak{p}}^\perp$ is abelian. Hence, $\tilde{\mathfrak{a}}$ must be maximal abelian in $\mathfrak{p}$ and the theorem follows. ∎

We now prove that the foliations in Example 4.10 exhaust all the possibilities for hyperpolar homogeneous foliations up to orbit equivalence. Let S be a connected closed subgroup of the isometry group inducing a hyperpolar homogeneous foliation on M. From now on we fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and a maximally noncompact Borel subalgebra $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ with $\mathfrak{t} \subset \mathfrak{k}$ and $\mathfrak{a} \subset \mathfrak{p}$ maximal abelian. According to Proposition 5.1 we may assume that the Lie algebra $\mathfrak{s}$ of S is solvable and that $\mathfrak{s} \subset \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$. The proof goes as follows. First we classify the abelian subspaces of $\mathfrak{a} \oplus \mathfrak{p}^1$. A bit more work leads to a description of all subalgebras $\mathfrak{s}$ of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ for which $\mathfrak{s}_\mathfrak{p}^\perp$ is abelian and contained in $\mathfrak{a} \oplus \mathfrak{p}^1$. Hence, the problem reduces to prove that, if $\mathfrak{s}$ is a subalgebra of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ for which $\mathfrak{s}_\mathfrak{p}^\perp$ is abelian and the corresponding connected subgroup of G with Lie algebra $\mathfrak{s}$ induces a foliation on M, then $\mathfrak{s}_\mathfrak{p}^\perp \subset \mathfrak{a} \oplus \mathfrak{p}^1$. We will consider an auxiliary subalgebra $\tilde{\mathfrak{s}} = \mathfrak{s} + (\mathfrak{n} \ominus \mathfrak{n}^1)$. This subalgebra satisfies $\tilde{\mathfrak{s}}_\mathfrak{p}^\perp \subset \mathfrak{a} \oplus \mathfrak{p}^1$ and hence its projection onto $\mathfrak{a} \oplus \mathfrak{n}$ is one of the known examples. Then $\mathfrak{s}_\mathfrak{p}^\perp$ is contained in the centralizer of $\tilde{\mathfrak{s}}_\mathfrak{p}^\perp$ in $\mathfrak{p}$. A bit more work allows us to calculate $\mathfrak{s}_\mathfrak{p}^\perp$ explicitly using the fact that $\mathfrak{s}$ is a subalgebra. Then we will conclude that the projection of $\mathfrak{s}$ onto $\mathfrak{a} \oplus \mathfrak{n}$ is one of the known examples. The final step is to prove that $\mathfrak{s}$ induces the same orbits as its projection onto $\mathfrak{a} \oplus \mathfrak{n}$.

In what follows (until Lemma 5.13 inclusive) we will work in a context slightly more general than that described above. Let $\mathfrak{s}$ be a subalgebra of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ such that $\mathfrak{s}_\mathfrak{p}^\perp$ is abelian. Hence, it is not assumed that the orbits of the connected subgroup of G whose Lie algebra is $\mathfrak{s}$ form a foliation. Example 4.4 shows that this can happen. We first state a few basic lemmas.

From now on, if $\mathfrak{v}$ is a vector subspace of $\mathfrak{g}$, we denote by $\pi_\mathfrak{v}$ the orthogonal projection of $\mathfrak{g}$ onto $\mathfrak{v}$. Also, we denote by $\mathfrak{s}_n = \pi_{\mathfrak{a}\oplus\mathfrak{n}}(\mathfrak{s})$ the projection of $\mathfrak{s}$ onto $\mathfrak{a} \oplus \mathfrak{n}$, the noncompact part of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

We will first derive some elementary results.

Lemma 5.2. *Let $\lambda \in \Sigma$ and $X, Y \in \mathfrak{g}_\lambda$. Then $(1-\theta)[\theta X, Y] = 2\langle X, Y\rangle H_\lambda$.*

Proof. It follows from polarization of the identity $[\theta(X+Y), X+Y] = \langle X+Y, X+Y\rangle H_\lambda$. $\square$

Lemma 5.3. *Let α be a simple root and $\mathfrak{v} \subset \mathfrak{g}_\alpha$ be a linear subspace such that $[\mathfrak{v}, \mathfrak{v}] = \{0\}$. Then $[\mathfrak{v}, \theta\mathfrak{v}] \subset \mathfrak{a}$ if and only if $\mathfrak{v}$ is one-dimensional.*

Proof. If $\mathfrak{v} = \mathbb{R}X$ with nonzero $X \in \mathfrak{g}_\alpha$, then $[\theta X, X] = \langle X, X\rangle H_\alpha \in \mathfrak{a}$. For the converse, assume that $\mathfrak{v}$ has dimension greater than 1 and that $[\mathfrak{v}, \theta\mathfrak{v}] \subset \mathfrak{a}$. Let $X, Y \in \mathfrak{v}$ be two nonzero orthogonal vectors. By Lemma 5.2 and orthogonality of X and Y, $(1-\theta)[\theta X, Y] = 2\langle X, Y\rangle H_\alpha = 0$, so $[\theta X, Y] \in \mathfrak{k}_0 \cap \mathfrak{a} = \{0\}$. Now, $\langle [\theta X, Y], [\theta X, Y]\rangle = -\langle [X, [\theta X, Y]], Y\rangle$ and using the Jacobi identity and the fact that $[\mathfrak{v}, \mathfrak{v}] = \{0\}$ we get $[X, [\theta X, Y]] = -[Y, [X, \theta X]] = \langle X, X\rangle [Y, H_\alpha] = -\langle \alpha, \alpha\rangle\langle X, X\rangle Y$. Altogether this implies $\langle [\theta X, Y], [\theta X, Y]\rangle = \langle \alpha, \alpha\rangle\langle X, X\rangle\langle Y, Y\rangle > 0$, which gives a contradiction. $\square$

Lemma 5.4. *Let $\lambda, \mu \in \Sigma$ such that $\lambda - \mu \notin \Sigma$. Let $X \in \mathfrak{g}_\lambda$ and $Y \in \mathfrak{g}_\mu$ be nonzero vectors. If $[X, Y] = 0$ then $\lambda + \mu$ is not a root. In particular, if $\alpha, \beta \in \Lambda$ and $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$ are nonzero vectors, then $[X, Y] = 0$ implies that α and β are not connected in the Dynkin diagram.*

Proof. Assume that $[X,Y]=0$. Since $[\theta Y,X]\in\mathfrak{g}_{\lambda-\mu}=0$ we have, using the Jacobi identity, that

$$0=[[X,Y],\theta Y]=-[[Y,\theta Y],X]=\langle Y,Y\rangle[H_\mu,X]=\langle Y,Y\rangle\langle\lambda,\mu\rangle X.$$

Hence, $\langle\lambda,\mu\rangle=0$. Since $\lambda-\mu\notin\Sigma$ and the corresponding Cartan integer satisfies $A_{\mu\lambda}=0$, we get that $\lambda\pm\mu\notin\Sigma$. For the second part, just note that $\alpha-\beta$ is not a root. □

Lemma 5.5. *Let $\Psi\subset\Sigma^+$. For each $\lambda\in\Psi$ let $v_\lambda\subset\mathfrak{g}_\lambda$ be a one-dimensional linear subspace. Then the linear subspace $[\mathfrak{k}_0,\mathfrak{a}\oplus\left(\bigoplus_{\lambda\in\Psi}v_\lambda\right)]$ is orthogonal to $\mathfrak{a}\oplus\left(\bigoplus_{\lambda\in\Psi}(1-\theta)v_\lambda\right)\oplus\left(\bigoplus_{\lambda\in\Sigma^+\setminus\Psi}\mathfrak{p}_\lambda\right)$.*

Proof. Obviously, $[\mathfrak{k}_0,\mathfrak{a}]=0$, so there is nothing to prove in this case. Assume that each v_λ is spanned by a corresponding vector E_λ. Let $T\in\mathfrak{k}_0$. If $H\in\mathfrak{a}$ then $\langle[T,E_\lambda],H\rangle=-\langle E_\lambda,[T,H]\rangle=0$. Since $[\mathfrak{k}_0,\mathfrak{g}_\lambda]\subset\mathfrak{g}_\lambda$, for any $\mu\in\Sigma^+$ with $\mu\neq\lambda$ and any $\xi\in\mathfrak{p}_\mu$ we obviously have $\langle[T,E_\lambda],\xi\rangle=0$. Finally, since $\operatorname{ad}(T)$ is skewsymmetric, $\langle[T,E_\lambda],(1-\theta)E_\lambda\rangle=\langle[T,E_\lambda],E_\lambda\rangle=0$, from where the result follows. □

We now proceed with the first step of the proof, which is describing abelian subspaces of $\mathfrak{a}\oplus\mathfrak{p}^1$. Recall that $\mathfrak{p}^1=\mathfrak{p}\cap\left(\mathfrak{g}^1_\emptyset\oplus\mathfrak{g}^{-1}_\emptyset\right)$. First, we need the following lemma.

Lemma 5.6. *Let $\mathfrak{q}\subset\mathfrak{a}\oplus\mathfrak{p}^1$ be an abelian subspace and define $\Psi=\{\alpha\in\Lambda:\pi_{\mathfrak{g}_\alpha}(\mathfrak{q})\neq0\}$. Then $\dim\pi_{\mathfrak{g}_\alpha}(\mathfrak{q})=1$ for all $\alpha\in\Psi$.*

Proof. Assume the statement is not true. Any two vectors $\xi,\eta\in\mathfrak{q}$ can be written as $\xi=\xi_0+\sum_{\alpha\in\Psi}(1-\theta)\xi_\alpha$ and $\eta=\eta_0+\sum_{\alpha\in\Psi}(1-\theta)\eta_\alpha$, with $\xi_\alpha,\eta_\alpha\in\mathfrak{g}_\alpha$. We denote by $\Psi'\subset\Psi\subset\Lambda$ the subset of roots $\alpha\in\Lambda$ such that ξ_α and η_α are linearly independent. If the statement of this lemma is not true, we can find ξ and η such that the corresponding Ψ' is nonempty. An easy calculation taking into account that $\alpha-\beta$ is not a root if $\alpha,\beta\in\Lambda$ yields

$$0=[\xi,\eta]=\sum_\alpha(1+\theta)(\alpha(\xi_0)\eta_\alpha-\alpha(\eta_0)\xi_\alpha)-\sum_\alpha(1+\theta)[\xi_\alpha,\theta\eta_\alpha]+\sum_{\alpha,\beta}(1+\theta)[\xi_\alpha,\eta_\beta].$$

Then it follows in particular that $\sum_\alpha(1+\theta)[\xi_\alpha,\theta\eta_\alpha]=0$, $[\xi_\alpha,\eta_\alpha]=0$ for all $\alpha\in\Psi$ and $[\xi_\alpha,\eta_\beta]+[\xi_\beta,\eta_\alpha]=0$ for all $\alpha,\beta\in\Psi$ with $\alpha\neq\beta$.

By Lemma 5.2, $(1-\theta)[\theta\xi_\alpha,\eta_\alpha]=2\langle\xi_\alpha,\eta_\alpha\rangle H_\alpha$, which implies

$$0=\sum_{\alpha\in\Psi}(1+\theta)[\xi_\alpha,\theta\eta_\alpha]=2\sum_{\alpha\in\Psi}([\xi_\alpha,\theta\eta_\alpha]+\langle\xi_\alpha,\eta_\alpha\rangle H_\alpha)=2\sum_{\alpha\in\Psi'}([\xi_\alpha,\theta\eta_\alpha]+\langle\xi_\alpha,\eta_\alpha\rangle H_\alpha),$$

the last equality following from the fact that $[\theta\xi_\alpha,\xi_\alpha]=\langle\xi_\alpha,\xi_\alpha\rangle H_\alpha$.

For $\alpha\in\Psi'$, using $[\xi_\alpha,\eta_\alpha]=0$ and $[\theta\xi_\alpha,\xi_\alpha]=\langle\xi_\alpha,\xi_\alpha\rangle H_\alpha$, we get

$$[[\xi_\alpha,\theta\eta_\alpha],\theta\xi_\alpha]=-[[\theta\eta_\alpha,\theta\xi_\alpha],\xi_\alpha]-[[\theta\xi_\alpha,\xi_\alpha],\theta\eta_\alpha]=\langle\alpha,\alpha\rangle\langle\xi_\alpha,\xi_\alpha\rangle\theta\eta_\alpha.$$

Now choose $\alpha,\beta\in\Psi'$ with $\beta\neq\alpha$. Since $\beta-\alpha$ is not a root and $[\theta\eta_\beta,\theta\xi_\alpha]=\theta[\eta_\beta,\xi_\alpha]=\theta[\xi_\beta,\eta_\alpha]=[\theta\xi_\beta,\theta\eta_\alpha]$ we obtain

$$\begin{aligned}[[\xi_\beta,\theta\eta_\beta],\theta\xi_\alpha] &= -[[\theta\eta_\beta,\theta\xi_\alpha],\xi_\beta]-[[\theta\xi_\alpha,\xi_\beta],\theta\eta_\beta]] = -[[\theta\xi_\beta,\theta\eta_\alpha],\xi_\beta]\\ &= [[\theta\eta_\alpha,\xi_\beta],\theta\xi_\beta]+[[\xi_\beta,\theta\xi_\beta],\theta\eta_\alpha] = \langle\alpha,\beta\rangle\langle\xi_\beta,\xi_\beta\rangle\theta\eta_\alpha.\end{aligned}$$

Taking into account the last two displayed equations we conclude

$$[[\xi_\beta, \theta\eta_\beta], \theta\xi_\alpha] = \langle\alpha, \beta\rangle\langle\xi_\beta, \xi_\beta\rangle\theta\eta_\alpha, \text{ for all } \alpha, \beta \in \Psi'.$$

Therefore, for arbitrary $\alpha \in \Psi'$, the identity $\sum_{\beta\in\Psi'}([\xi_\beta, \theta\eta_\beta] + \langle\xi_\beta, \eta_\beta\rangle H_\beta) = 0$ yields

$$\begin{aligned} 0 &= \left[\sum_{\beta\in\Psi'}([\xi_\beta, \theta\eta_\beta] + \langle\xi_\beta, \eta_\beta\rangle H_\beta), \theta\xi_\alpha\right] = \sum_{\beta\in\Psi'}([[\xi_\beta, \theta\eta_\beta], \theta\xi_\alpha] + \langle\xi_\beta, \eta_\beta\rangle[H_\beta, \theta\xi_\alpha]) \\ &= \sum_{\beta\in\Psi'}(\langle\alpha, \beta\rangle\langle\xi_\beta, \xi_\beta\rangle\theta\eta_\alpha - \langle\alpha, \beta\rangle\langle\xi_\beta, \xi_\beta\rangle\theta\xi_\alpha) \\ &= \left(\sum_{\beta\in\Psi'}\langle\alpha, \beta\rangle\langle\xi_\beta, \xi_\beta\rangle\right)\theta\eta_\alpha - \left(\sum_{\beta\in\Psi'}\langle\alpha, \beta\rangle\langle\xi_\beta, \xi_\beta\rangle\right)\theta\xi_\alpha. \end{aligned}$$

Since $\alpha \in \Psi'$, $\theta\eta_\alpha$ and $\theta\xi_\alpha$ are linearly independent, the only way the above equality can hold is when the coefficients of $\theta\eta_\alpha$ and $\theta\xi_\alpha$ are simultaneously zero. In particular, $\sum_{\beta\in\Psi'}\langle\alpha, \beta\rangle\langle\xi_\beta, \xi_\beta\rangle = 0$. Hence, $\sum_{\beta\in\Psi'}\langle\xi_\beta, \xi_\beta\rangle\beta$ is orthogonal to span Ψ'. Since it is also a vector in span Ψ' and the simple roots are linearly independent, it follows that $\langle\xi_\beta, \xi_\beta\rangle = 0$ for all $\beta \in \Psi'$, contradiction. ☐

We say that a subset $\Phi \subset \Lambda$ is *connected* if the subdiagram of the Dynkin diagram determined by the roots of Φ is connected. We say that two subsets $\Phi, \Phi' \subset \Lambda$ are *disconnected* or *orthogonal* if for any $\alpha \in \Phi$ and any $\beta \in \Phi'$, α and β are not connected in the Dynkin diagram (that is, $\alpha + \beta$ is not a root).

Proposition 5.7. *Let $\mathfrak{q} \subset \mathfrak{a} \oplus \mathfrak{p}^1$ be an abelian subspace and $\Psi = \{\alpha \in \Lambda : \pi_{\mathfrak{g}_\alpha}(\mathfrak{q}) \neq 0\}$. This set can be decomposed as $\Psi = \bigcup_{i=1}^k \Psi_i$ with $\Psi_i \subset \Lambda$ connected, and Ψ_i and Ψ_j disconnected whenever $i \neq j$. Then there exists a map $c : \Psi \to \mathbb{R}$, $\alpha \mapsto c_\alpha$, and vectors $E_i \in \bigoplus_{\alpha\in\Psi_i}\mathfrak{g}_\alpha$, $i \in \{1, \ldots, k\}$, with $\pi_{\mathfrak{g}_\alpha}(E_i) \neq 0$ for all $\alpha \in \Psi_i$, such that $\mathfrak{q}$ is a linear subspace of*

$$\mathfrak{v}_\mathfrak{q} = \mathfrak{a}_\Psi \oplus \left(\bigoplus_{i=1}^k \mathbb{R}\left(\sum_{\alpha\in\Psi_i} c_\alpha H_\alpha + (1-\theta)E_i\right)\right).$$

Proof. Using the fact that the sets Ψ_i are disconnected, it is easy to see that a subalgebra $\mathfrak{v}_\mathfrak{q}$ as considered above is abelian. Hence, in order to prove the proposition, it suffices to take an abelian subalgebra $\mathfrak{q} \subset \mathfrak{a}\oplus\mathfrak{p}^1$ and prove that it can be realized as a subspace of one of the Lie subalgebras $\mathfrak{v}_\mathfrak{q}$ as defined above. For that, consider $\Psi = \{\alpha \in \Lambda : \pi_{\mathfrak{g}_\alpha}(\mathfrak{q}) \neq 0\}$ and write $\Psi = \bigcup_{i=1}^k \Psi_i$ with $\Psi_i \subset \Lambda$ connected, and Ψ_i and Ψ_j disconnected whenever $i \neq j$. Our first assertion is that for each $i \in \{1, \ldots, k\}$ there exists a nonzero vector $E_i \in \bigoplus_{\alpha\in\Psi_i}\mathfrak{g}_\alpha$ such that any vector $\xi \in \mathfrak{q}$ can be written as $\xi = \xi_0 + \sum_{i=1}^k x_i(1-\theta)E_i$ for certain $\xi_0 \in \mathfrak{a}$ and $x_i \in \mathbb{R}$.

We fix $i \in \{1, \ldots, k\}$. From Lemma 5.6 it follows that $\dim \pi_{\mathfrak{g}_\alpha}(\mathfrak{q}) = 1$ for all $\alpha \in \Psi$. Hence, for each $\alpha \in \Psi$ we can choose a nonzero vector $E_\alpha \in \mathfrak{g}_\alpha$ such that any vector $\xi \in \mathfrak{q}$ can be written as $\xi = \xi_0 + \sum_{\alpha\in\Psi} a_\alpha(1-\theta)E_\alpha$ for certain $\xi_0 \in \mathfrak{a}$ and $a_\alpha \in \mathbb{R}$. If, on the contrary, the previous assertion is not true, we can find $\xi = \xi_0 + \sum_\alpha a_\alpha(1-\theta)E_\alpha \in \mathfrak{q}$ and $\eta = \eta_0 + \sum_\alpha b_\alpha(1-\theta)E_\alpha \in \mathfrak{q}$ such that for some $\alpha, \beta \in \Psi_i$ connected in the Dynkin diagram the vectors $(a_\alpha, a_\beta), (b_\alpha, b_\beta) \in \mathbb{R}^2$ are linearly independent. Since $\mathfrak{q}$ is abelian, we have

$$0 = [\xi, \eta] = \sum_\alpha (\alpha(\xi_0)b_\alpha - \alpha(\eta_0)a_\alpha)(1+\theta)E_\alpha + \sum_{\alpha,\beta} a_\alpha b_\beta (1+\theta)[E_\alpha, E_\beta].$$

In particular, taking the $\mathfrak{g}_{\alpha+\beta}$ component we get $a_\alpha b_\beta - a_\beta b_\alpha = 0$ by Lemma 5.4, which contradicts the fact that (a_α, a_β) and (b_α, b_β) are linearly independent.

Therefore, we have proved our assertion, that is, any vector $\xi \in \mathfrak{q}$ can be written as $\xi = \xi_0 + \sum_{i=1}^k x_i(1-\theta)E_i$ for certain $\xi_0 \in \mathfrak{a}$, $x_i \in \mathbb{R}$ and $E_i \in \bigoplus_{\alpha\in\Psi_i} \mathfrak{g}_\alpha$. By the definition of Ψ, it is obvious that $\pi_{\mathfrak{g}_\alpha}(E_i) \neq 0$ for all $\alpha \in \Psi_i$, and indeed we can write $E_i = \sum_{\alpha\in\Psi_i} E_\alpha$ with suitable $E_\alpha \in \mathfrak{g}_\alpha$ (note that these might be different from the above E_α's). Since Ψ_i and Ψ_j are disconnected if $i \neq j$, it is clear that $[E_i, E_j] = 0$ for all $i, j \in \{1, \ldots, k\}$.

We choose such a vector $\xi = \xi_0 + \sum_{j=1}^k x_j(1-\theta)E_j$ and assume $x_i = 0$. By definition of Ψ there certainly exists $\eta = \eta_0 + \sum_{j=1}^k y_j(1-\theta)E_j \in \mathfrak{q}$ with $y_i \neq 0$. Hence,

$$0 = [\xi, \eta] = \sum_{j=1}^k (1+\theta)\left(\sum_{\alpha\in\Psi_j} (y_j\alpha(\xi_0) - x_j\alpha(\eta_0))\, E_\alpha\right),$$

and taking the $\mathfrak{g}_\alpha$-component for any $\alpha \in \Psi_i$ we get $\alpha(\xi_0) = 0$ because $x_i = 0$ and $y_i \neq 0$. This implies that for each $\xi \in \mathfrak{q}$ and each $\alpha \in \Psi_i$ we can write $\langle \xi, H_\alpha\rangle = c_\alpha(\xi)\langle\alpha,\alpha\rangle x_i$ (if $x_i = 0$ any $c_\alpha(\xi)$ will do). The next step is to prove that we can choose the same c_α for all $\xi \in \mathfrak{q}$.

Assume that we cannot choose the c_α's in such a way. Then there would be $\xi = \xi_0 + \sum_{j=1}^k x_j(1-\theta)E_j \in \mathfrak{q}$ and $\eta = \eta_0 + \sum_{j=1}^k y_j(1-\theta)E_j \in \mathfrak{q}$ such that $c_\alpha(\xi) \neq c_\alpha(\eta)$ for some $\alpha \in \Psi_i$. This of course implies that $x_i, y_i \neq 0$. Taking the corresponding $\mathfrak{g}_\alpha$-component of $[\xi, \eta]$ as in the previous displayed formula we get

$$0 = y_i\alpha(\xi_0) - x_i\alpha(\eta_0) = y_i\langle\xi, H_\alpha\rangle - x_i\langle\eta, H_\alpha\rangle = x_i y_i \langle\alpha,\alpha\rangle(c_\alpha(\xi) - c_\alpha(\eta)),$$

which leads to a contradiction. Hence we can choose the c_α independently of $\xi \in \mathfrak{q}$, which allows us to define a function $c : \Psi \to \mathbb{R}$ by $\langle\xi, H_\alpha\rangle = c_\alpha\langle\alpha,\alpha\rangle x_i$, with the notation as above.

Finally, if $\xi = \xi_0 + \sum_{i=1}^k x_i(1-\theta)E_i \in \mathfrak{q}$ we can write $\xi_0 = \xi_0' + \sum_{\alpha\in\Psi} a_\alpha H_\alpha$ with $\xi_0' \in \mathfrak{a}_\Psi = \mathfrak{a} \ominus \left(\bigoplus_{\alpha\in\Psi} \mathbb{R}H_\alpha\right)$ and $a_\alpha \in \mathbb{R}$. Here, the a_α must satisfy $c_\alpha\langle\alpha,\alpha\rangle x_i = \langle\xi, H_\alpha\rangle = a_\alpha\langle\alpha,\alpha\rangle$, so $\mathfrak{q}$ is contained in one of the model spaces in the statement of the proposition. □

The following lemma is useful to understand how $\mathfrak{s}$ and $\mathfrak{s}_\mathfrak{p}^\perp$ are related.

Lemma 5.8. *If $\mathfrak{s} \subset \mathfrak{t}\oplus\mathfrak{a}\oplus\mathfrak{n}$ is a subalgebra and $\mathfrak{s}_\mathfrak{p}^\perp = \{\xi \in \mathfrak{p} : \xi \perp \mathfrak{s}\}$, then $\mathfrak{s}_n = \pi_{\mathfrak{a}\oplus\mathfrak{n}}(\mathfrak{s}) = \{X \in \mathfrak{a}\oplus\mathfrak{n} : X \perp \mathfrak{s}_\mathfrak{p}^\perp\}$.*

Proof. If $X \in \mathfrak{s}$, then for all $\xi \in \mathfrak{s}_\mathfrak{p}^\perp$ we have $\langle X, \xi\rangle = 0$ by definition. Since $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal, $\pi_{\mathfrak{a}\oplus\mathfrak{n}}(X) \perp \mathfrak{s}_\mathfrak{p}^\perp$.

Conversely, let $X \in \mathfrak{a} \oplus \mathfrak{n}$ such that $X \perp \mathfrak{s}_\mathfrak{p}^\perp$, and choose $Y \in (\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathfrak{s}_n$. We may write $Y = H + \sum_{\lambda \in \Sigma^+} Y_\lambda$ with $H \in \mathfrak{a}$ and $Y_\lambda \in \mathfrak{g}_\lambda$. Clearly, $Y - \sum_{\lambda \in \Sigma^+} \theta Y_\lambda = H + \sum_{\lambda \in \Sigma^+} (1 - \theta) Y_\lambda \in \mathfrak{p}$, and if $Z \in \mathfrak{s}$ we have $\langle Y - \sum_{\lambda \in \Sigma^+} \theta Y_\lambda, Z \rangle = \langle Y, Z \rangle - \sum_{\lambda \in \Sigma^+} \langle \theta Y_\lambda, Z \rangle = 0$, because Y and Z are perpendicular and so are $\mathfrak{g}_{-\lambda}$ and $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. This proves that $Y - \sum_{\lambda \in \Sigma^+} \theta Y_\lambda \in \mathfrak{s}_\mathfrak{p}^\perp$. By assumption we have $X \perp \mathfrak{s}_\mathfrak{p}^\perp$, and so $0 = \langle X, Y - \sum_{\lambda \in \Sigma^+} \theta Y_\lambda \rangle = \langle X, Y \rangle - \sum_{\lambda \in \Sigma^+} \langle X, \theta Y_\lambda \rangle = \langle X, Y \rangle$, again because $\mathfrak{g}_{-\lambda}$ and $\mathfrak{a} \oplus \mathfrak{n}$ are perpendicular. Since $Y \in (\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathfrak{s}$ is arbitrary, we conclude that $X \in (\mathfrak{a} \oplus \mathfrak{n}) \ominus ((\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathfrak{s}_n) = \mathfrak{s}_n$. ☐

Proposition 5.7 and Lemma 5.8 allow us to conclude the first step of the proof of our classification.

Theorem 5.9. *Let $\mathfrak{s} \subset \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be a subalgebra such that $\mathfrak{s}_\mathfrak{p}^\perp \subset \mathfrak{a} \oplus \mathfrak{p}^1$ is abelian. Then there are an orthogonal subset $\Phi \subset \Lambda$, numbers $a_\alpha \in \mathbb{R}$ and nonzero vectors $E_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Phi$, and a linear subspace $V \subset \mathfrak{a}_\Phi$ such that*

$$\mathfrak{s}_n = (V \oplus \mathfrak{a}^\Phi \oplus \mathfrak{n}) \ominus \left(\bigoplus_{\alpha \in \Phi} \mathbb{R}(a_\alpha H_\alpha + E_\alpha) \right).$$

Proof. Since $\mathfrak{s}_\mathfrak{p}^\perp \subset \mathfrak{a} \oplus \mathfrak{p}^1$ is abelian, by Proposition 5.7 we have that $\mathfrak{s}_\mathfrak{p}^\perp$ is a linear subspace of

$$\mathfrak{q} = \mathfrak{a}_\Phi \oplus \left(\bigoplus_{i=1}^k \mathbb{R} \left(\sum_{\alpha \in \Phi_i} a_\alpha H_\alpha + (1-\theta) E_i \right) \right).$$

Here, as usual, $\Phi = \{\alpha \in \Lambda : \pi_{\mathfrak{g}_\alpha}(\mathfrak{s}_\mathfrak{p}^\perp) \neq 0\} = \bigcup_{i=1}^k \Phi_i$, with Φ_i connected, and Φ_i disconnected to Φ_j whenever $i \neq j$, $E_i \in \bigoplus_{\alpha \in \Phi_i} \mathfrak{g}_\alpha$ with $\pi_{\mathfrak{g}_\alpha}(E_i) \neq 0$ for all $\alpha \in \Phi_i$, and $a : \Phi \to \mathbb{R}$ a real-valued function. Our first step is to prove that each Φ_i consists of exactly one root.

Fix $i \in \{1, \ldots, k\}$ and assume that Φ_i has more than one root. We may write $E_i = \sum_{\alpha \in \Phi_i} E_\alpha$ with $E_\alpha \in \mathfrak{g}_\alpha$. Also, take $\xi = \xi_0 + \sum_j y_j (\sum_{\alpha \in \Phi_j} a_\alpha H_\alpha + (1-\theta) E_j) \in \mathfrak{s}_\mathfrak{p}^\perp$ with $y_i \neq 0$. Since $\dim \left(\bigoplus_{\alpha \in \Phi_i} \mathbb{R} E_\alpha \right) > 1$, there exists a nonzero vector $X \in \bigoplus_{\alpha \in \Phi_i} \mathbb{R} E_\alpha$ such that X is orthogonal to E_i (or equivalently, to $(1-\theta)E_i$, or to ξ, or to $\mathfrak{s}_\mathfrak{p}^\perp$). By Lemma 5.8, there exists $S \in \mathfrak{k}$ such that $S + X \in \mathfrak{s}$. We write $X = \sum_{\alpha \in \Phi_i} x_\alpha E_\alpha$. Now, for each $\alpha \in \Phi_i$, and again for dimension reasons we can find a vector $Z_\alpha = H_\alpha + \sum_{\beta \in \Phi_i} z_{\alpha\beta} E_\beta$ such that Z_α is orthogonal to ξ (and hence to $\mathfrak{s}_\mathfrak{p}^\perp$). Lemma 5.8 ensures that there exists $T_\alpha \in \mathfrak{k}$ such that $T_\alpha + Z_\alpha \in \mathfrak{s}$ for each $\alpha \in \Phi_i$. By Lemma 5.5 we get $\langle [T_\alpha, X], \xi \rangle = \langle [S, Z_\alpha], \xi \rangle = 0$. Since $\mathfrak{s}$ is a subalgebra and $[E_\beta, E_\gamma] \in \mathfrak{n}^2$, we have

$$0 = \langle [T_\alpha + Z_\alpha, S + X], \xi \rangle = \langle [H_\alpha, X], \xi \rangle = \langle \sum_{\beta \in \Phi_i} x_\beta \langle \alpha, \beta \rangle E_\beta, \xi \rangle = y_i \sum_{\beta \in \Phi_i} \langle \alpha, \beta \rangle x_\beta \langle E_\beta, E_\beta \rangle$$

for each $\alpha \in \Phi_i$. Putting $A_{\beta\alpha} = \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$ and taking into account that $y_i \neq 0$, the previous equation is equivalent to $\sum_{\beta \in \Phi_i} A_{\beta\alpha} x_\beta \langle E_\beta, E_\beta \rangle = 0$. Of course, $(A_{\beta\alpha})$ is the Cartan matrix of the Dynkin subdiagram associated with Φ_i. Cartan matrices are known to be nonsingular (see for example [15, Proposition 2.52 (e)]) which means that $x_\beta \langle E_\beta, E_\beta \rangle = 0$ for all $\beta \in \Phi_i$.

This contradicts the fact that X is a nonzero vector and proves that Φ_i has exactly one root.

Thus from now one we can assume that $\mathfrak{s}_{\mathfrak{p}}^{\perp}$ is a linear subspace of

$$\mathfrak{q} = \mathfrak{a}_{\Phi} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathbb{R} \left(a_\alpha H_\alpha + (1-\theta) E_\alpha \right) \right),$$

with $E_\alpha \in \mathfrak{g}_\alpha$ and Φ an orthogonal subset of Λ. We have to prove that $\bigoplus_{\alpha\in\Phi} \mathbb{R}(a_\alpha H_\alpha + (1-\theta)E_\alpha) \subset \mathfrak{s}_{\mathfrak{p}}^{\perp}$. Consider a vector $\xi = \xi_0 + \sum_{\alpha\in\Phi} x_\alpha(a_\alpha H_\alpha + (1-\theta)E_\alpha) \in \mathfrak{q}$ orthogonal to $\mathfrak{s}_{\mathfrak{p}}^{\perp}$, with $\xi_0 \in \mathfrak{a}_\Phi$ and $x_\alpha \in \mathbb{R}$. Since ξ is orthogonal to $\mathfrak{s}_{\mathfrak{p}}^{\perp}$, by Lemma 5.8 we can find $S \in \mathfrak{t}$ such that the vector $X = S + \xi_0 + \sum_{\alpha\in\Phi} x_\alpha(a_\alpha H_\alpha + E_\alpha)$ is in $\mathfrak{s}$. On the other hand, it is clear, using again Lemma 5.8, that there exists $T_\alpha \in \mathfrak{t}$ such that $Z_\alpha = T_\alpha + \langle E_\alpha, E_\alpha\rangle H_\alpha - c_\alpha\langle\alpha,\alpha\rangle E_\alpha$ is a vector in $\mathfrak{s}$. Since $\mathfrak{s}$ is a subalgebra, $[Z_\alpha, X] \in \mathfrak{s}$ for each $\alpha \in \Phi$. A calculation using the facts that $\mathfrak{t} \oplus \mathfrak{a}$ is abelian, $\xi_0 \in \mathfrak{a}_\Phi$ and $\alpha + \beta \notin \Sigma$ if $\alpha, \beta \in \Phi$ and $\alpha \neq \beta$, gives

$$[Z_\alpha, X] = x_\alpha \langle \alpha,\alpha\rangle(\langle E_\alpha, E_\alpha\rangle + a_\alpha^2 \langle\alpha,\alpha\rangle)E_\alpha + [T_\alpha, \sum_{\beta\in\Phi} x_\beta E_\beta] + [S, a_\alpha\langle\alpha,\alpha\rangle E_\alpha].$$

Lemma 5.5 implies that the last two addends above are orthogonal to $\mathfrak{s}_{\mathfrak{p}}^{\perp}$. Since $[Z_\alpha, X]$ is orthogonal to $\mathfrak{s}_{\mathfrak{p}}^{\perp}$, the first addend must be orthogonal to $\mathfrak{s}_{\mathfrak{p}}^{\perp}$ as well. By definition of $\mathfrak{s}_{\mathfrak{p}}^{\perp}$, the only way this can happen is when $x_\alpha = 0$ for all $\alpha \in \Phi$. This implies $\xi = \xi_0 \in \mathfrak{a}_\Phi$ and proves $\bigoplus_{\alpha\in\Phi} \mathbb{R}(a_\alpha H_\alpha + (1-\theta)E_\alpha) \subset \mathfrak{s}_{\mathfrak{p}}^{\perp}$. Now the theorem follows after a straightforward application of Lemma 5.8. □

Motivated by Theorem 5.9 we introduce the following notation. Let $a : \Phi \to \mathbb{R}$ be a map and define $a_\alpha = a(\alpha)$ for all $\alpha \in \Phi$. Furthermore, for each $\alpha \in \Phi$ we choose a nonzero vector $E_\alpha \in \ell_\Phi \cap \mathfrak{g}_\alpha$. Consider

$$\mathfrak{s}_{\Phi,V,a} = (V \oplus \mathfrak{a}^\Phi \oplus \mathfrak{n}) \ominus \left(\bigoplus_{\alpha\in\Phi} \mathbb{R}(a_\alpha H_\alpha + E_\alpha) \right).$$

As above we will see in Proposition 5.10 that this does not depend on the particular choice of ℓ_Φ. We obviously have $\mathfrak{s}_{\Phi,V,0} = \mathfrak{s}_{\Phi,V}$ for the zero map $0 : \Phi \to \mathbb{R}$.

Proposition 5.10. *We have* $\mathrm{Ad}(g)\mathfrak{s}_{\Phi,V,a} = \mathfrak{s}_{\Phi,V}$ *with* $g = \mathrm{Exp}(-\sum_{\alpha\in\Phi} a_\alpha E_\alpha) \in N$ *if* $\Phi \neq \emptyset$ *and* $g = \mathrm{id}_G$ *if* $\Phi = \emptyset$. *In particular,* $\mathfrak{s}_{\Phi,V,a}$ *is a subalgebra of* $\mathfrak{a} \oplus \mathfrak{n}$. *Moreover, the corresponding connected subgroup* $S_{\Phi,V,a}$ *is conjugate to* $S_{\Phi,V}$ *and induces a hyperpolar homogeneous foliation. We also have*

$$(\mathfrak{s}_{\Phi,V,a})_{\mathfrak{p}}^{\perp} = (\mathfrak{a}_\Phi \ominus V) \oplus \left(\bigoplus_{\alpha\in\Phi} \mathbb{R}(a_\alpha H_\alpha + (1-\theta)E_\alpha) \right).$$

Proof. We define $\xi_\alpha = a_\alpha H_\alpha + E_\alpha$ for $\alpha \in \Phi$. Then the subalgebra $\mathfrak{s}_{\Phi,V,a}$ can equivalently be written as $\mathfrak{s}_{\Phi,V,a} = (V \oplus \mathfrak{a}^\Phi \oplus \mathfrak{n}) \ominus \left(\bigoplus_{\alpha\in\Phi} \mathbb{R}\xi_\alpha\right)$. Let $g_\alpha = \mathrm{Exp}(-a_\alpha E_\alpha)$ and $g = \prod_{\alpha\in\Phi} g_\alpha$. Since α and β are not connected in the Dynkin diagram, we have $[E_\alpha, E_\beta] = 0$, and so $g = \mathrm{Exp}(-\sum_{\alpha\in\Phi} a_\alpha E_\alpha)$. Our aim is to prove that $\mathrm{Ad}(g)\mathfrak{s}_{\Phi,V,a} = \mathfrak{s}_{\Phi,V}$.

We introduce the following notation:

$$\mathfrak{s}_\alpha = (V \oplus \mathfrak{a}^\Phi \oplus \mathfrak{n}) \ominus \mathbb{R}E_\alpha, \quad \hat{\mathfrak{s}}_\alpha = (V \oplus \mathfrak{a}^\Phi \oplus \mathfrak{n}) \ominus \mathbb{R}\xi_\alpha.$$

First we prove that $\operatorname{Ad}(g_\alpha)\hat{\mathfrak{s}}_\alpha = \mathfrak{s}_\alpha$ for each $\alpha \in \Phi$. Note that, since $-a_\alpha E_\alpha \in \mathfrak{a} \oplus \mathfrak{n}$, it follows that $\operatorname{Ad}(g_\alpha)(\mathfrak{a} \oplus \mathfrak{n}) = \mathfrak{a} \oplus \mathfrak{n}$. Now let $X \in \hat{\mathfrak{s}}_\alpha$. Since E_α is a unit vector and $X \in \mathfrak{a} \oplus \mathfrak{n}$, we have

$$\begin{aligned} \langle \operatorname{Ad}(g_\alpha)X, E_\alpha\rangle &= \langle X, \operatorname{Ad}(\operatorname{Exp}(a_\alpha\theta E_\alpha))E_\alpha\rangle = \langle X, e^{a_\alpha \operatorname{ad}(\theta E_\alpha)}E_\alpha\rangle \\ &= \langle X, E_\alpha + a_\alpha H_\alpha + \frac{a_\alpha^2}{2}|\alpha|^2\theta E_\alpha\rangle = \langle X, \xi_\alpha\rangle = 0. \end{aligned}$$

Also, if $H \in \mathfrak{a}_\Phi \ominus V$ then $\alpha(H) = 0$ for each $\alpha \in \Phi$, so

$$\begin{aligned} \langle \operatorname{Ad}(g_\alpha)X, H\rangle &= \langle X, \operatorname{Ad}(\operatorname{Exp}(a_\alpha\theta E_\alpha))H\rangle = \langle X, e^{a_\alpha \operatorname{ad}(\theta E_\alpha)}H\rangle \\ &= \langle X, H + a_\alpha\alpha(H)\theta E_\alpha\rangle = \langle X, H\rangle = 0. \end{aligned}$$

Altogether this proves that $\operatorname{Ad}(g_\alpha)\hat{\mathfrak{s}}_\alpha = \mathfrak{s}_\alpha$.

Now let $\alpha, \beta \in \Phi$ with $\alpha \neq \beta$. We prove that $\operatorname{Ad}(g_\alpha)\hat{\mathfrak{s}}_\beta = \hat{\mathfrak{s}}_\beta$. Since α and β are simple roots, $\beta - m\alpha$ is not a root for $m \geq 1$. Hence,

$$\operatorname{Ad}(\operatorname{Exp}(t\theta E_\alpha))E_\beta = e^{t \operatorname{ad}(\theta E_\alpha)}E_\beta = \sum_{m=0}^{\infty} \frac{t^m}{m!} \operatorname{ad}(\theta E_\alpha)^m E_\beta = E_\beta.$$

If $H \in \mathfrak{a}$ we also get

$$\operatorname{Ad}(\operatorname{Exp}(t\theta E_\alpha))H = e^{t \operatorname{ad}(\theta E_\alpha)}H = H + t\alpha(H)\theta E_\alpha.$$

Now, let $X \in \hat{\mathfrak{s}}_\beta$. Using the previous equations we obtain

$$\langle \operatorname{Ad}(g_\alpha)X, \xi_\beta\rangle = \langle X, \operatorname{Ad}(\operatorname{Exp}(a_\alpha\theta E_\alpha))\xi_\beta\rangle = \langle X, \xi_\beta\rangle + a_\alpha\langle \alpha, \beta\rangle\langle X, \theta E_\alpha\rangle = 0.$$

Also, if $H \in \mathfrak{a}_\Phi \ominus V$ we get

$$\langle \operatorname{Ad}(g_\alpha)X, H\rangle = \langle X, \operatorname{Ad}(\operatorname{Exp}(a_\alpha\theta E_\alpha))H\rangle = \langle X, H\rangle + a_\alpha\alpha(H)\langle X, \theta E_\alpha\rangle = 0.$$

Altogether this proves $\operatorname{Ad}(g_\alpha)\hat{\mathfrak{s}}_\beta = \hat{\mathfrak{s}}_\beta$. A similar argument shows also that $\operatorname{Ad}(g_\alpha)\mathfrak{s}_\beta = \mathfrak{s}_\beta$. However, since $\mathfrak{s}_{\Phi,V,a} = \bigcap_{\alpha\in\Phi} \hat{\mathfrak{s}}_\alpha$ and $\mathfrak{s}_{\Phi,V} = \bigcap_{\alpha\in\Phi} \mathfrak{s}_\alpha$, using the previous two equalities and a simple induction argument, we get $\operatorname{Ad}(g)\mathfrak{s}_{\Phi,V,a} = (\prod_{\alpha\in\Phi} \operatorname{Ad}(g_\alpha))\mathfrak{s}_{\Phi,V,a} = \mathfrak{s}_{\Phi,V}$. □

This is a good point to recall the contents of Theorem 5.9, which says that if $\mathfrak{s} \subset \mathfrak{t}\oplus\mathfrak{a}\oplus\mathfrak{n}$ is a subalgebra such that $\mathfrak{s}_\mathfrak{p}^\perp \subset \mathfrak{a}\oplus\mathfrak{p}^1$ is abelian, then there are an orthogonal subset $\Phi \subset \Lambda$, numbers $a_\alpha \in \mathbb{R}$ and nonzero vectors $E_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Phi$, and a linear subspace $V \subset \mathfrak{a}_\Phi$ such that $\mathfrak{s}_n = \mathfrak{s}_{\Phi,V,a}$.

Proposition 5.11. *Let $\mathfrak{s} \subset \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be a subalgebra such that*

$$\mathfrak{s}_n = \mathfrak{s}_{\Phi,V,a} = (V \oplus \mathfrak{a}^\Phi \oplus \mathfrak{n}) \ominus \left(\bigoplus_{\alpha\in\Phi} \mathbb{R}(a_\alpha H_\alpha + E_\alpha)\right)$$

with Φ a subset of orthogonal simple roots, $E_\alpha \in \mathfrak{g}_\alpha$ nonzero vectors, V a linear subspace of $\mathfrak{a}_\Phi$ and $a_\alpha \in \mathbb{R}$, and define $E = -\sum_{\alpha\in\Phi} a_\alpha E_\alpha$ and $g = \operatorname{Exp}(E)$. Then the following statements hold:

(i) $\mathrm{Ad}(g)\mathfrak{s}$ *is a subalgebra of* $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ *and* $(\mathrm{Ad}(g)\mathfrak{s})_n \subset \mathrm{Ad}(g)\mathfrak{s}_n = \mathfrak{s}_{\Phi,V}$.

(ii) *For each* $\alpha \in \Phi$ *the projection of* $(\mathfrak{t} \oplus \mathfrak{a} \oplus (\mathfrak{n} \ominus \mathfrak{g}_\alpha)) \cap \mathfrak{s}$ *onto* $\mathfrak{t}$ *centralizes* E_α.

(iii) $V \subset (\mathrm{Ad}(g)\mathfrak{s})_n$.

(iv) *Assume that* $\lambda \in \Sigma^+ \setminus \Phi$ *satisfies* $\lambda + \alpha \notin \Sigma$ *for each* $\alpha \in \Phi$. *Then* $\mathfrak{g}_\lambda \subset (\mathrm{Ad}(g)\mathfrak{s})_n$.

In addition, assume that the orbits of the connected subgroup S *of* G *whose Lie algebra is* $\mathfrak{s}$ *form a homogeneous foliation. Then the following further statements hold:*

(v) $(\mathrm{Ad}(g)\mathfrak{s})_n = \mathrm{Ad}(g)\mathfrak{s}_n = \mathfrak{s}_{\Phi,V}$.

(vi) *Denote by* $\mathfrak{s}_c$ *the projection of* $\mathfrak{s}$ *onto* $\mathfrak{t}$. *Then* $\mathfrak{s}_c$ *is an abelian subalgebra that centralizes each* E_α. *In particular,* $[\mathfrak{s}_c, \mathfrak{s}_{\mathfrak{p}}^\perp] = 0$.

(vii) *With the notation as in* (vi)*, let* S_c *be the connected subgroup of* G *whose Lie algebra is* $\mathfrak{s}_c$. *Then* S_c *acts trivially on* $\nu_o(S \cdot o)$.

Remark 5.12. Remark 4.4 shows that the hypothesis that the orbits of S form a homogeneous foliation is necessary in Proposition 5.11 (v), (vi) and (vii).

Proof. (i) First note that since $\mathfrak{n}$ is an ideal of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ we have $\mathrm{Ad}(g)(\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}) \subset \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $\mathrm{Ad}(g)\mathfrak{s} \subset \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Proposition 5.10 implies that $\mathrm{Ad}(g)\mathfrak{s}_n = \mathfrak{s}_{\Phi,V}$. We have to prove that $(\mathrm{Ad}(g)\mathfrak{s})_n \subset \mathfrak{s}_{\Phi,V}$. Let $T + H + X \in \mathfrak{s}$ with $T \in \mathfrak{t}$, $H \in \mathfrak{a}$ and $X \in \mathfrak{n}$. We already have $\mathrm{Ad}(g)(H + X) \in \mathrm{Ad}(g)\mathfrak{s}_n = \mathfrak{s}_{\Phi,V}$, so it suffices to prove that the projection of $\mathrm{Ad}(g)T$ onto $\mathfrak{a} \oplus \mathfrak{n}$ is in $\mathfrak{s}_{\Phi,V}$. Since $\mathfrak{n}$ is an ideal of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ this projection is $\mathrm{Ad}(g)T - T = \sum_{k=1}^{\infty} \frac{1}{k!} \mathrm{ad}(E)^k T \in \mathfrak{n}$. We have to prove that $\langle \mathrm{Ad}(g)T - T, E_\alpha \rangle = 0$ for all $\alpha \in \Phi$. Since $[E, T] \in \mathfrak{n}$ and $\mathfrak{n}$ is nilpotent it follows that $\mathrm{ad}(E)^k T \in \mathfrak{n} \ominus \mathfrak{n}^1$ for all $k \geq 2$, and for $k = 1$, we have $\langle [E, T], E_\alpha \rangle = 0$ for all $\alpha \in \Phi$ by Lemma 5.5. Hence, $(\mathrm{Ad}(g)\mathfrak{s})_n \subset \mathfrak{s}_{\Phi,V} = \mathrm{Ad}(g)\mathfrak{s}_n$.

(ii) Let $\alpha \in \Phi$ and T be in the image of the projection of $(\mathfrak{t} \oplus \mathfrak{a} \oplus (\mathfrak{n} \ominus \mathfrak{g}_\alpha)) \cap \mathfrak{s}$ onto $\mathfrak{t}$. Note that $[T, E_\alpha] \in \mathfrak{g}_\alpha \ominus \mathbb{R}E_\alpha$ since $\mathrm{ad}(T)$ preserves each root space and is skewsymmetric. Hence, we only have to show that $\langle \mathfrak{g}_\alpha \ominus \mathbb{R}E_\alpha, [T, E_\alpha] \rangle = 0$. Let $X \in \mathfrak{g}_\alpha \ominus \mathbb{R}E_\alpha$ be arbitrary. Since $X \in \mathfrak{s}_n$, there exists $S_X \in \mathfrak{t}$ such that $S_X + X \in \mathfrak{s}$. By definition of T, there exist $H \in \mathfrak{a}$ and $Y \in \mathfrak{n} \ominus \mathfrak{g}_\alpha$ such that $T + H + Y \in \mathfrak{s}$. As $\mathfrak{s}$ is a subalgebra we have $[S_X, Y] + [X, T] + [X, H] + [X, Y] = [S_X + X, T + H + Y] \in \mathfrak{s}$. Since $[S_X, Y] \in \mathfrak{n} \ominus \mathfrak{g}_\alpha$, $[X, H] = -\alpha(H)X \in \mathfrak{g}_\alpha \ominus \mathbb{R}E_\alpha$ and $[X, Y] \in \mathfrak{n} \ominus \mathfrak{n}^1$, the definition of $\mathfrak{s}_n$ yields

$$0 = \langle [S_X + X, T + H + Y], a_\alpha H_\alpha + (1 - \theta)E_\alpha \rangle = \langle [X, T], E_\alpha \rangle = -\langle X, [T, E_\alpha] \rangle,$$

which completes the proof of (ii).

(iii) Let $H \in V \subset \mathfrak{a}_\Phi$. Since $H \in \mathfrak{s}_n$ there is $T_H \in \mathfrak{t}$ such that $T_H + H \in \mathfrak{s}$. By (ii) we have $[T_H, E] = 0$ and by definition $\alpha(H) = 0$ for all $\alpha \in \Phi$, so $\mathrm{Ad}(g^{-1})(T_H + H) = e^{-\mathrm{ad}(E)}(T_H + H) = T_H + H \in \mathfrak{s}$. Hence $T_H + H \in \mathrm{Ad}(g)\mathfrak{s}$ and $H \in (\mathrm{Ad}(g)\mathfrak{s})_n$.

(iv) Assume $\lambda \in \Sigma^+ \setminus \Phi$ and $\lambda + \alpha \notin \Sigma$ for any $\alpha \in \Phi$. Take $X \in \mathfrak{g}_\lambda$ and $T_X \in \mathfrak{t}$ such that $T_X + X \in \mathfrak{s}$. By (ii) we have $[T_X, E] = 0$. Since $\lambda + \alpha \notin \Sigma$ for any $\alpha \in \Phi$, we also have $[X, E] = 0$. Hence, $\mathrm{Ad}(g^{-1})(T_X + X) = e^{-\mathrm{ad}(E)}(T_X + X) = T_X + X \in \mathfrak{s}$. This implies $T_X + X \in \mathrm{Ad}(g)\mathfrak{s}$ and $X \in (\mathrm{Ad}(g)\mathfrak{s})_n$, so (iv) follows.

(v) By (i) we already know $(\mathrm{Ad}(g)\mathfrak{s})_n \subset \mathrm{Ad}(g)\mathfrak{s}_n = \mathfrak{s}_{\Phi,V}$. We prove the equality by showing that $\dim(\mathrm{Ad}(g)\mathfrak{s})_n = \dim \mathrm{Ad}(g)\mathfrak{s}_n$.

By hypothesis and Proposition 2.1, all the orbits of S are principal and the same is true of $I_g(S)$. Hence the isotropy groups $S_o = S \cap K$ and $I_g(S)_o = I_g(S) \cap K$ are conjugate.

Their Lie algebras are $\mathfrak{s} \cap \mathfrak{t}$ and $(\mathrm{Ad}(g)\mathfrak{s}) \cap \mathfrak{t}$, respectively. By (ii) we have $[\mathfrak{s} \cap \mathfrak{t}, E] = 0$ so $\mathrm{Ad}(g) = e^{\mathrm{ad}(E)}$ acts as the identity on $\mathfrak{s} \cap \mathfrak{t}$. Hence $\mathfrak{s} \cap \mathfrak{t} \subset (\mathrm{Ad}(g)\mathfrak{s}) \cap \mathfrak{t}$ and thus equality follows by hypothesis. This implies $\dim(\mathrm{Ad}(g)\mathfrak{s})_n = \dim \mathrm{Ad}(g)\mathfrak{s} - \dim(\mathrm{Ad}(g)\mathfrak{s}) \cap \mathfrak{t} = \dim \mathfrak{s} - \dim \mathfrak{s} \cap \mathfrak{t} = \dim \mathfrak{s}_n = \dim \mathrm{Ad}(g)\mathfrak{s}_n$.

(vi) Obviously, $\mathfrak{s}_c$ is an abelian subalgebra because $\mathfrak{t}$ is abelian. For the second part, we assume first that $\mathfrak{s}_n = \mathfrak{s}_{\Phi,V}$. Fix $\alpha \in \Phi$ and let $X \in \mathfrak{g}_\alpha \ominus \mathbb{R}E_\alpha$ be arbitrary. Then there exists $S_X \in \mathfrak{t}$ such that $S_X + X \in \mathfrak{s}$. Since $H_\alpha \in \mathfrak{a}^\Phi \subset \mathfrak{s}_n$, there exists $T_{H_\alpha} \in \mathfrak{t}$ such that $T_{H_\alpha} + H_\alpha \in \mathfrak{s}$. As $\mathfrak{s}$ is a subalgebra we have $[T_{H_\alpha} + H_\alpha, S_X + X] = (\mathrm{ad}(T_{H_\alpha}) + \langle \alpha, \alpha \rangle 1_{\mathfrak{g}_\alpha})X \in \mathfrak{g}_\alpha \cap \mathfrak{s} \subset \mathfrak{g}_\alpha \ominus \mathbb{R}E_\alpha$, where $1_{\mathfrak{g}_\alpha}$ is the identity transformation of $\mathfrak{g}_\alpha$. Since $\langle \alpha, \alpha \rangle \neq 0$, $\mathrm{ad}(T_{H_\alpha}) + \langle \alpha, \alpha \rangle 1_{\mathfrak{g}_\alpha}$ is an isomorphism. Thus, the previous equality implies $\mathfrak{g}_\alpha \ominus \mathbb{R}E_\alpha \in \mathfrak{s}$. This implies $S_X \in \mathfrak{s} \cap \mathfrak{t}$ and thus $[S_X, E_\alpha] = 0$ by (ii). Also according to (ii), and since $\alpha \in \Phi$ is arbitrary, (vi) follows when $\mathfrak{s}_n = \mathfrak{s}_{\Phi,V}$.

Now we finish the proof of (vi). Let $\mathfrak{s} \subset \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be a subalgebra such that $\mathfrak{s}_n = \mathfrak{s}_{\Phi,V,a}$ and assume that all the orbits of the corresponding connected subgroup S of G whose Lie algebra is $\mathfrak{s}$ are principal. By (v) we get $(\mathrm{Ad}(g)\mathfrak{s})_n = \mathfrak{s}_{\Phi,V}$. Take an element $T + H + X \in \mathfrak{s}$ with $T \in \mathfrak{t}$, $H \in \mathfrak{a}$ and $X \in \mathfrak{n}$. Since $H + X \in \mathfrak{s}_n$, it follows that $\mathrm{Ad}(g)(H + X) \in \mathrm{Ad}(g)\mathfrak{s}_n = \mathfrak{s}_{\Phi,V}$ by Proposition 5.10. Hence, the projection of $\mathrm{Ad}(g)(T + H + X)$ onto $\mathfrak{t}$ is the same as the projection of $\mathrm{Ad}(g)T$ onto $\mathfrak{t}$, and as $g \in N$, that projection is T. Now, since $(\mathrm{Ad}(g)\mathfrak{s})_n = \mathfrak{s}_{\Phi,V}$, applying the argument in the previous paragraph to the subalgebra $\mathrm{Ad}(g)\mathfrak{s}$ we get $[T, E_\alpha] = [\pi_{\mathfrak{t}}(\mathrm{Ad}(g)(T + H + X)), E_\alpha] = 0$ for all $\alpha \in \Phi$. This already implies $[E, T] = 0$ and thus $\mathrm{Ad}(g)T = T$, so $\mathrm{Ad}(g)(T + H + X) = T + \mathrm{Ad}(g)(H + X)$ and $\mathrm{Ad}(g)(H + X) \in \mathfrak{a} \oplus \mathfrak{n}$. Since $[\mathfrak{t}, \mathfrak{a}] = 0$, we obtain $[T, (\mathfrak{a}_\Phi \ominus V) \oplus \big(\bigoplus_{\alpha \in \Phi} \mathbb{R}(a_\alpha H_\alpha + (1 - \theta)E_\alpha)\big)] = 0$ and the result follows.

(vii) Let $t \in S_c$ and $\xi \in \nu_o(S \cdot o)$. Since $\mathfrak{s}_{\mathfrak{p}}^\perp \subset \mathfrak{p}$ we may identify $\mathfrak{s}_{\mathfrak{p}}^\perp$ and $\nu_o(S \cdot o)$. By (vi), $\mathfrak{s}_c$ centralizes $\mathfrak{s}_{\mathfrak{p}}^\perp$, so with the above identification we get $t_* \xi = \mathrm{Ad}(t)\xi = \xi$. ☐

We will need the following result:

Lemma 5.13. *Let $\mathfrak{s}$ be a subalgebra of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $\mathfrak{s}_n$ its projection onto $\mathfrak{a} \oplus \mathfrak{n}$. Let λ and μ be two positive roots (not necessarily different). If $\mathfrak{g}_\lambda + \mathfrak{g}_\mu \subset \mathfrak{s}_n$, then $\mathfrak{g}_{\lambda+\mu} \subset \mathfrak{s}_n$.*

Proof. We may assume that $\lambda + \mu$ is a root; otherwise there is nothing to prove. Let $X \in \mathfrak{g}_\lambda$ and $Y \in \mathfrak{g}_\mu$. By definition there exist $S, T \in \mathfrak{t}$ such that $S + X, T + Y \in \mathfrak{s}$. Then, $[S + X, T + Y] = [S, Y] - [T, X] + [X, Y] \in \mathfrak{s}$. Recall that $[\mathfrak{k}_0, \mathfrak{g}_\nu] \subset \mathfrak{g}_\nu$ for any $\nu \in \Sigma$. The vector $[S, Y] - [T, X] + [X, Y]$ is in $\mathfrak{n}$ and hence in $\mathfrak{s}_n$. On the other hand, $[S, Y] - [T, X] \in \mathfrak{g}_\mu + \mathfrak{g}_\lambda \subset \mathfrak{s}_n$, so $[X, Y] \in \mathfrak{s}_n$. Since, X and Y are arbitrary, $\mathfrak{g}_{\lambda+\mu} = [\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{s}_n$. ☐

We now drop the assumption $\mathfrak{s}_{\mathfrak{p}}^\perp \subset \mathfrak{a} \oplus \mathfrak{p}^1$.

Let $\mathfrak{s}$ be a subalgebra of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ such that $\mathfrak{s}_{\mathfrak{p}}^\perp$ is abelian. From now on we assume that the orbits of the connected closed subgroup S of G whose Lie algebra is $\mathfrak{s}$ form a homogeneous foliation on M. As usual, we denote by $\mathfrak{s}_n = \pi_{\mathfrak{a} \oplus \mathfrak{n}}(\mathfrak{s})$ the projection of $\mathfrak{s}$ onto the noncompact part of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

We define $\tilde{\mathfrak{s}} = \mathfrak{s} + (\mathfrak{n}^2 \oplus \cdots \oplus \mathfrak{n}^m) = \mathfrak{s} + (\mathfrak{n} \ominus \mathfrak{n}^1)$ where $m = m_\emptyset$ is the level of the highest root of Σ. Since $\mathfrak{n} \ominus \mathfrak{n}^1$ is an ideal of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ it follows that $\tilde{\mathfrak{s}}$ is a subalgebra of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Also, $\mathfrak{s} \subset \tilde{\mathfrak{s}}$ and thus $\tilde{\mathfrak{s}}_{\mathfrak{p}}^\perp \subset \mathfrak{s}_{\mathfrak{p}}^\perp$, which means that $\tilde{\mathfrak{s}}_{\mathfrak{p}}^\perp$ is also an abelian subspace of $\mathfrak{p}$.

It is obvious by definition that $\tilde{\mathfrak{s}}_{\mathfrak{p}}^{\perp} \subset \mathfrak{a} \oplus \mathfrak{p}^1$. Hence Theorem 5.9 implies that $\tilde{\mathfrak{s}}_n := \pi_{\mathfrak{a}\oplus\mathfrak{n}}(\tilde{\mathfrak{s}}) = \mathfrak{s}_{\Phi,V,a}$ with $\Phi \subset \Lambda$ a subset of orthogonal simple roots, $a_\alpha \in \mathbb{R}$, $0 \neq E_\alpha \in \mathfrak{g}_\alpha$ and $V \subset \mathfrak{a}_\Phi$ as usual. By Proposition 5.10, there exists $g \in N$ such that $\mathrm{Ad}(g)\tilde{\mathfrak{s}}_n = \mathfrak{s}_{\Phi,V}$. This element can be taken to be $g = \mathrm{Exp}(E)$ with $E = -\sum_{\alpha\in\Phi} a_\alpha E_\alpha$.

We define $\hat{\mathfrak{s}} = \mathrm{Ad}(g)\mathfrak{s}$. The subgroup of G whose Lie algebra is $\hat{\mathfrak{s}}$ is $\hat{S} = I_g(S)$. Obviously, $\hat{S}$ induces a hyperpolar homogeneous foliation on M. By Proposition 5.11 (i) we get $\hat{\mathfrak{s}}_n := \pi_{\mathfrak{a}\oplus\mathfrak{n}}(\hat{\mathfrak{s}}) \subset (\mathrm{Ad}(g)\tilde{\mathfrak{s}})_n \subset \mathrm{Ad}(g)\tilde{\mathfrak{s}}_n = \mathfrak{s}_{\Phi,V}$. Then it follows that $(\mathfrak{a}_\Phi \ominus V) \oplus \left(\bigoplus_{\alpha\in\Phi} \mathbb{R}(1-\theta)E_\alpha\right) \subset \hat{\mathfrak{s}}_{\mathfrak{p}}^{\perp}$. Since $\hat{\mathfrak{s}}_{\mathfrak{p}}^{\perp}$ is abelian, we have that $\hat{\mathfrak{s}}_{\mathfrak{p}}^{\perp}$ must be contained in the centralizer $Z_{\mathfrak{p}}((\mathfrak{a}_\Phi \ominus V) \oplus \left(\bigoplus_{\alpha\in\Phi} \mathbb{R}(1-\theta)E_\alpha\right))$ of $(\mathfrak{a}_\Phi \ominus V) \oplus \left(\bigoplus_{\alpha\in\Phi} \mathbb{R}(1-\theta)E_\alpha\right)$ in $\mathfrak{p}$. Our first aim is essentially to calculate this centralizer. Eventually, this will allow us to determine $\hat{\mathfrak{s}}_n$ and later $\mathfrak{s}_n$.

We start with $\bigoplus_{\alpha\in\Phi} \mathbb{R}(1-\theta)E_\alpha$ where the situation is a bit more involved. We deal with this in a series of lemmas.

Lemma 5.14. *Let $\alpha \in \Phi$ and let $\xi \in \mathfrak{p}$ be written as $\xi = \xi_0 + \sum_{\lambda\in\Sigma^+}(1-\theta)\xi_\lambda$ with $\xi_0 \in \mathfrak{a}$ and $\xi_\lambda \in \mathfrak{g}_\lambda$ for each $\lambda \in \Sigma^+$. Then ξ is in the centralizer $Z_{\mathfrak{p}}(\mathbb{R}(1-\theta)E_\alpha)$ of $\mathbb{R}(1-\theta)E_\alpha$ in $\mathfrak{p}$ if and only if $\xi_0 \in \mathfrak{a}_{\{\alpha\}}$, $\xi_\alpha \in \mathbb{R}E_\alpha$, $\xi_{2\alpha} = 0$ and $[\xi_{\lambda-\alpha}, E_\alpha] = [\xi_{\lambda+\alpha}, \theta E_\alpha]$ for all $\lambda \in \Sigma^+ \setminus \{\alpha, 2\alpha\}$.*

Proof. If the vector ξ commutes with $(1-\theta)E_\alpha$ a simple calculation yields

$$0 = [\xi, (1-\theta)E_\alpha] = (1+\theta)\left\{\alpha(\xi_0)E_\alpha + \sum_{\lambda\in\Sigma^+}[\xi_\lambda, E_\alpha] - \sum_{\lambda\in\Sigma^+\setminus\Lambda}[\theta\xi_\lambda, E_\alpha] - [\theta\xi_\alpha, E_\alpha]\right\}.$$

The above vector is zero if and only if each of its components in $\mathfrak{k}_\lambda$, $\lambda \in \Sigma^+ \cup \{0\}$, is zero.

The $\mathfrak{k}_0$-component is zero if and only if $[\theta\xi_\alpha, E_\alpha] \in \mathfrak{a}$, and the $\mathfrak{k}_{2\alpha}$-component is zero if and only if $[\xi_\alpha, E_\alpha] = 0$. Denote by $\mathfrak{v}$ the vector subspace of $\mathfrak{g}_\alpha$ spanned by E_α and ξ_α. The above two conditions imply $[\mathfrak{v}, \theta\mathfrak{v}] \subset \mathfrak{a}$ and $[\mathfrak{v}, \mathfrak{v}] = 0$. Since α is a simple root, Lemma 5.3 implies that $\mathfrak{v}$ is 1-dimensional and hence $\xi_\alpha \in \mathbb{R}E_\alpha$.

The $\mathfrak{k}_\alpha$-component vanishes if and only if $\alpha(\xi_0)E_\alpha - [\xi_{2\alpha}, \theta E_\alpha] = 0$. Taking inner product with E_α yields

$$0 = \langle \alpha(\xi_0)E_\alpha - [\xi_{2\alpha}, \theta E_\alpha], E_\alpha\rangle = \alpha(\xi_0)\langle E_\alpha, E_\alpha\rangle - \langle \xi_{2\alpha}, [E_\alpha, E_\alpha]\rangle = \alpha(\xi_0)\langle E_\alpha, E_\alpha\rangle.$$

Hence, $\xi_0 \in \mathfrak{a}_{\{\alpha\}} = \mathfrak{a} \ominus \mathbb{R}H_\alpha$. Taking into account the above equation, this also implies $[\xi_{2\alpha}, \theta E_\alpha] = 0$. Using the Jacobi identity we get

$$0 = [[\xi_{2\alpha}, \theta E_\alpha], E_\alpha] = -[[\theta E_\alpha, E_\alpha], \xi_{2\alpha}] = -\langle E_\alpha, E_\alpha\rangle[H_\alpha, \xi_{2\alpha}] = -2\langle\alpha, \alpha\rangle\langle E_\alpha, E_\alpha\rangle\xi_{2\alpha},$$

which implies $\xi_{2\alpha} = 0$.

Finally, if $\lambda \in \Sigma^+ \setminus \{\alpha, 2\alpha\}$, the $\mathfrak{k}_\lambda$-component is $(1+\theta)([\xi_{\lambda-\alpha}, E_\alpha] - [\xi_{\lambda+\alpha}, \theta E_\alpha])$. This vanishes if and only if $[\xi_{\lambda-\alpha}, E_\alpha] - [\xi_{\lambda+\alpha}, \theta E_\alpha] = 0$ because $\mathfrak{g}_\lambda$ and $\mathfrak{g}_{-\lambda}$ are linearly independent. Since the "only if " part is elementary, the result follows. ☐

Lemma 5.15. *Let $\alpha \in \Phi$ and $\lambda \in \Sigma^+ \setminus \{\alpha, 2\alpha\}$, and assume that the α-string of λ has length greater than one. Then $\bigoplus_{m\in\mathbb{Z}} \mathfrak{g}_{\lambda+m\alpha} \subset \hat{\mathfrak{s}}_n$.*

Proof. Since $(1-\theta)E_\alpha \in \hat{\mathfrak{s}}_{\mathfrak{p}}^\perp$ and $\hat{\mathfrak{s}}_{\mathfrak{p}}^\perp$ is abelian, we have $\hat{\mathfrak{s}}_{\mathfrak{p}}^\perp \subset Z_{\mathfrak{p}}(\mathbb{R}(1-\theta)E_\alpha)$. Let $\xi \in \hat{\mathfrak{s}}_{\mathfrak{p}}^\perp$ and write as usual $\xi = \xi_0 + \sum_{\lambda\in\Sigma^+}(1-\theta)\xi_\lambda$ with $\xi_0 \in \mathfrak{a}$ and $\xi_\lambda \in \mathfrak{g}_\lambda$ for each $\lambda \in \Sigma^+$. Lemma 5.14 already implies that $\xi_0 \in \mathfrak{a}_{\{\alpha\}}$, $\xi_\alpha \in \mathbb{R}E_\alpha$ and $\xi_{2\alpha} = 0$. We have to prove that $\xi_{\lambda+m\alpha} = 0$ for all $m \in \mathbb{Z}$. We prove this assertion depending on the whether the length of the α-string of λ is 2, 3 or 4. Note that $\lambda \notin \Phi$.

Assume that the length of the α-string of λ is 2. In this case we may assume $\lambda-\alpha, \lambda+2\alpha \notin \Sigma^+$ and $\lambda, \lambda+\alpha \in \Sigma^+$ (switch to $\lambda-\alpha$ if necessary). Then α and λ span a root system of type A_2. Since $\lambda+2\alpha \notin \Sigma^+$, Lemma 5.14 implies $[\xi_\lambda, E_\alpha] = [\xi_{\lambda+2\alpha}, \theta E_\alpha] = 0$. Since $\lambda-\alpha \notin \Sigma$ and $\lambda+\alpha \in \Sigma$ we get from Lemma 5.4 that $\xi_\lambda = 0$. Similarly, by Lemma 5.14 we have $[\xi_{\lambda+\alpha}, \theta E_\alpha] = [\xi_{\lambda-\alpha}, E_\alpha] = 0$. Since $\lambda+\alpha-(-\alpha) = \lambda+2\alpha \notin \Sigma$ and $\lambda+\alpha+(-\alpha) = \lambda \in \Sigma$, Lemma 5.4 yields $\xi_{\lambda+\alpha} = 0$ which finishes the proof in this case.

Assume that the α-string of λ has length 3. In this case we may assume $\lambda-\alpha, \lambda+3\alpha \notin \Sigma^+$ and $\lambda, \lambda+\alpha, \lambda+2\alpha \in \Sigma^+$. Then, λ and α span a root system of type B_2 or BC_2. First we claim that $\xi_{\lambda+\alpha} \in [\mathfrak{g}_\alpha \ominus \mathbb{R}E_\alpha, \mathfrak{g}_\lambda]$. Since the root system spanned by λ and α is of type B_2 or BC_2, $\lambda+\alpha$ and α are orthogonal and have the same length. This implies that there exists an element of the Weyl group that maps α to $\lambda+\alpha$. Hence $\mathfrak{g}_{\lambda+\alpha}$ and $\mathfrak{g}_\alpha$ have the same dimension. By Lemma 5.4, for any nonzero $Z_\lambda \in \mathfrak{g}_\lambda$, $\operatorname{ad}(Z_\lambda) : \mathfrak{g}_\alpha \to \mathfrak{g}_{\lambda+\alpha}$ is injective, hence bijective. We write $\xi_{\lambda+\alpha} = [Z_\lambda, cE_\alpha + Z_\alpha]$ with $c \in \mathbb{R}$ and $Z_\alpha \in \mathfrak{g}_\alpha \ominus \mathbb{R}E_\alpha$. Since $\langle Z_\alpha, E_\alpha\rangle = 0$, Lemma 5.2 yields $(1-\theta)[\theta E_\alpha, Z_\alpha] = 0$ and thus $[\theta E_\alpha, Z_\alpha] \in \mathfrak{k}_0$. Hence $\operatorname{ad}([\theta E_\alpha, Z_\alpha])$ is skewsymmetric. Lemma 5.14 implies $[\xi_{\lambda+\alpha}, \theta E_\alpha] = [\xi_{\lambda-\alpha}, E_\alpha] = 0$. Then, using the Jacobi identity, we get

$$\begin{aligned} 0 &= \langle[\xi_{\lambda+\alpha}, \theta E_\alpha], Z_\lambda\rangle = \langle[[Z_\lambda, cE_\alpha + Z_\alpha], \theta E_\alpha], Z_\lambda\rangle \\ &= -c\langle[[E_\alpha, \theta E_\alpha], Z_\lambda], Z_\lambda\rangle - \langle[[Z_\alpha, \theta E_\alpha], Z_\lambda], Z_\lambda\rangle \\ &= c\langle\lambda, \alpha\rangle\langle E_\alpha, E_\alpha\rangle\langle Z_\lambda, Z_\lambda\rangle - \langle\operatorname{ad}([Z_\alpha, \theta E_\alpha])Z_\lambda, Z_\lambda\rangle = c\langle\lambda, \alpha\rangle\langle E_\alpha, E_\alpha\rangle\langle Z_\lambda, Z_\lambda\rangle. \end{aligned}$$

Hence $c = 0$ and our assertion follows.

Now we claim that $\mathfrak{g}_\lambda \oplus \mathfrak{g}_{\lambda+2\alpha} \subset \hat{\mathfrak{s}}_n$. By Lemma 5.14 we have

$$Z_{\mathfrak{p}_\lambda\oplus\mathfrak{p}_{\lambda+2\alpha}}(\mathbb{R}(1-\theta)E_\alpha) = \{(1-\theta)(\eta_\lambda + \eta_{\lambda+2\alpha}) \in \mathfrak{p}_\lambda \oplus \mathfrak{p}_{\lambda+2\alpha} : [\eta_\lambda, E_\alpha] = [\eta_{\lambda+2\alpha}, \theta E_\alpha]\},$$

whose dimension coincides with $\dim \mathfrak{p}_\lambda$. Note that $\eta_{\lambda+2\alpha}$ is uniquely determined by η_λ since

$$[[\eta_\lambda, E_\alpha], E_\alpha] = [[\eta_{\lambda+2\alpha}, \theta E_\alpha], E_\alpha] = -[[\theta E_\alpha, E_\alpha], \eta_{\lambda+2\alpha}] = -\langle E_\alpha, E_\alpha\rangle\langle\alpha, \lambda+2\alpha\rangle\eta_{\lambda+2\alpha}.$$

Since $H_\alpha \in \hat{\mathfrak{s}}_n$, by Lemma 5.14 there exists $S \in \mathfrak{t}$ such that $S + H_\alpha \in \hat{\mathfrak{s}}$. We prove that $Z_{\mathfrak{p}_\lambda\oplus\mathfrak{p}_{\lambda+2\alpha}}(\mathbb{R}(1-\theta)E_\alpha)$ is invariant under $\operatorname{ad}(S)$. Let $X \in \mathfrak{g}_\alpha \ominus \mathbb{R}E_\alpha$. Then there exists $T \in \mathfrak{t}$ such that $T + X \in \hat{\mathfrak{s}}$ by Lemmas 5.8 and 5.14. Hence, $[S + H_\alpha, T + X] = (\operatorname{ad}(S) + \langle\alpha, \alpha\rangle 1_{\mathfrak{g}_\alpha})X \in \hat{\mathfrak{s}} \cap \mathfrak{g}_\alpha \subset \hat{\mathfrak{s}}_n$. Using again Lemma 5.14 we get $0 = \langle(\operatorname{ad}(S) + \langle\alpha, \alpha\rangle 1_{\mathfrak{g}_\alpha})X, (1-\theta)E_\alpha\rangle = -\langle X, \operatorname{ad}(S)E_\alpha\rangle$. Since $X \in \mathfrak{g}_\alpha \ominus \mathbb{R}E_\alpha$ and $\operatorname{ad}(S)E_\alpha \in \mathfrak{g}_\alpha \ominus \mathbb{R}E_\alpha$ (because $\operatorname{ad}(S)$ is skewsymmetric), the above equation implies $[S, E_\alpha] = 0$. Note that $[S, \theta E_\alpha] = \theta[S, E_\alpha] = 0$. Now assume that $(1-\theta)(\eta_\lambda + \eta_{\lambda+2\alpha}) \in Z_{\mathfrak{p}_\lambda\oplus\mathfrak{p}_{\lambda+2\alpha}}(\mathbb{R}(1-\theta)E_\alpha)$. We have to show that $(1-\theta)([S, \eta_\lambda] + [S, \eta_{\lambda+2\alpha}]) \in Z_{\mathfrak{p}_\lambda\oplus\mathfrak{p}_{\lambda+2\alpha}}(\mathbb{R}(1-\theta)E_\alpha)$. Indeed, using the Jacobi identity and $[S, E_\alpha] = 0$ we get

$$[[S, \eta_\lambda], E_\alpha] = -[[\eta_\lambda, E_\alpha], S] = -[[\eta_{\lambda+2\alpha}, \theta E_\alpha], S] = [[S, \eta_{\lambda+2\alpha}], E_\alpha].$$

This proves that $Z_{\mathfrak{p}_\lambda \oplus \mathfrak{p}_{\lambda+2\alpha}}(\mathbb{R}(1-\theta)E_\alpha)$ is invariant under $\mathrm{ad}(S)$.

Let $Z_\lambda \in \mathfrak{g}_\lambda$. Then there exists $Z_{\lambda+2\alpha} \in \mathfrak{g}_{\lambda+2\alpha}$ such that $Z_\lambda + Z_{\lambda+2\alpha}$ is perpendicular to $Z_{\mathfrak{p}_\lambda \oplus \mathfrak{p}_{\lambda+2\alpha}}(\mathbb{R}(1-\theta)E_\alpha)$. Thus Lemma 5.8 yields that $Z_\lambda + Z_{\lambda+2\alpha} \in \hat{\mathfrak{s}}_n$, and so there exists $T \in \mathfrak{t}$ such that $T + Z_\lambda + Z_{\lambda+2\alpha} \in \hat{\mathfrak{s}}$. Hence $[S + H_\alpha, T + Z_\lambda + Z_{\lambda+2\alpha}] = [S, Z_\alpha + Z_{\lambda+2\alpha}] + \langle \lambda, \alpha \rangle Z_\lambda + \langle \lambda + 2\alpha, \alpha \rangle Z_{\lambda+2\alpha} \in \hat{\mathfrak{s}} \cap (\mathfrak{g}_\lambda \oplus \mathfrak{g}_{\lambda+2\alpha})$. As $Z_\lambda + Z_{\lambda+2\alpha}$ is perpendicular to $Z_{\mathfrak{p}_\lambda \oplus \mathfrak{p}_{\lambda+2\alpha}}(\mathbb{R}(1-\theta)E_\alpha)$ and $Z_{\mathfrak{p}_\lambda \oplus \mathfrak{p}_{\lambda+2\alpha}}(\mathbb{R}(1-\theta)E_\alpha)$ is $\mathrm{ad}(S)$-invariant, it follows that $[S, Z_\lambda + Z_{\lambda+2\alpha}]$ is also perpendicular to $Z_{\mathfrak{p}_\lambda \oplus \mathfrak{p}_{\lambda+2\alpha}}(\mathbb{R}(1-\theta)E_\alpha)$, and so $[S, Z_\lambda + Z_{\lambda+2\alpha}] \in \hat{\mathfrak{s}}_n$ by Lemma 5.5. Hence, $\langle \lambda, \alpha \rangle Z_\lambda + \langle \lambda + 2\alpha, \alpha \rangle Z_{\lambda+2\alpha} \in \hat{\mathfrak{s}}_n$. Since λ and α span a root system of type B_2 or BC_2 we know that $\langle \lambda, \alpha \rangle < 0$ and $\langle \lambda + 2\alpha, \alpha \rangle > 0$. Therefore we have $Z_\lambda \in \hat{\mathfrak{s}}_n$, which implies $\mathfrak{g}_\lambda \subset \hat{\mathfrak{s}}_n$. Similarly, one can show $\mathfrak{g}_{\lambda+2\alpha} \subset \hat{\mathfrak{s}}_n$, which proves our claim. Hence, $\xi_\lambda, \xi_{\lambda+2\alpha} = 0$.

We have that $(\mathfrak{g}_\alpha \ominus \mathbb{R}E_\alpha) \oplus \mathfrak{g}_\lambda \subset \hat{\mathfrak{s}}_n$. Let $X \in \mathfrak{g}_\alpha \ominus \mathbb{R}E_\alpha$ and $Y \in \mathfrak{g}_\lambda$. There exist $S, T \in \mathfrak{t}$ such that $S + X, T + Y \in \hat{\mathfrak{s}}$. Hence, $[S + X, T + Y] = [S, Y] - [T, X] + [X, Y] \in \hat{\mathfrak{s}} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_\lambda \oplus \mathfrak{g}_{\alpha+\lambda}) \subset \hat{\mathfrak{s}}_n$. Then, by Lemma 5.8, the right-hand side of the previous equation is orthogonal to $(1-\theta)E_\alpha$ so $[T, X] \in \mathfrak{g}_\alpha \ominus \mathbb{R}E_\alpha \subset \hat{\mathfrak{s}}_n$. Since $[S, Y] \in \mathfrak{g}_\lambda \subset \hat{\mathfrak{s}}_n$ we conclude $[X, Y] \in \hat{\mathfrak{s}}_n$. As X and Y are arbitrary, $\xi_{\lambda+\alpha} \in [\mathfrak{g}_\alpha \ominus \mathbb{R}E_\alpha, \mathfrak{g}_\lambda] \subset \hat{\mathfrak{s}}_n$. This implies $\xi_{\lambda+\alpha} = 0$ and finishes the proof for α-strings of λ of length 3.

Finally, assume that the length of the α-string of λ is 4. In this case we may assume $\lambda - \alpha, \lambda + 4\alpha \notin \Sigma^+$ and $\lambda, \lambda + \alpha, \lambda + 2\alpha, \lambda + 3\alpha \in \Sigma^+$. Then α and λ span a root system of type G_2. A consequence of this fact is that all four restricted root spaces have the same dimension. Now, by Lemma 5.4, the linear map $\mathrm{ad}(E_\alpha) : \mathfrak{g}_\lambda \to \mathfrak{g}_{\lambda+\alpha}$ is injective, and hence bijective. Thus, we can write $\xi_{\lambda+\alpha} = [E_\alpha, X_\lambda]$ with $X_\lambda \in \mathfrak{g}_\lambda$. We get from Lemma 5.14 that

$$0 = [\xi_{\lambda-\alpha}, E_\alpha] = [\xi_{\lambda+\alpha}, \theta E_\alpha] = [[E_\alpha, X_\lambda], \theta E_\alpha] = -[[\theta E_\alpha, E_\alpha], X_\lambda] = -\langle \lambda, \alpha \rangle \langle E_\alpha, E_\alpha \rangle X_\lambda.$$

Since $\langle \lambda, \alpha \rangle \neq 0$, this implies $X_\lambda = 0$, and hence $\xi_{\lambda+\alpha} = 0$. Since $\lambda + 3\alpha - (-\alpha) \notin \Sigma$ and $\lambda + 3\alpha + (-\alpha) \in \Sigma$, Lemma 5.4 and $[\xi_{\lambda+3\alpha}, \theta E_\alpha] = [\xi_{\lambda+\alpha}, E_\alpha] = 0$ imply $\xi_{\lambda+3\alpha} = 0$.

Another application of Lemma 5.4 implies that $\mathrm{ad}(\theta E_\alpha) : \mathfrak{g}_{\lambda+3\alpha} \to \mathfrak{g}_{\lambda+2\alpha}$ is injective and hence bijective. A similar argument as before writing $\xi_{\lambda+2\alpha} = [\theta E_\alpha, X_{\lambda+3\alpha}]$ with $X_{\lambda+3\alpha} \in \mathfrak{g}_{\lambda+3\alpha}$ yields, using Lemma 5.14,

$$\begin{aligned} 0 &= [\xi_{\lambda+4\alpha}, \theta E_\alpha] = [\xi_{\lambda+2\alpha}, E_\alpha] = [[\theta E_\alpha, X_{\lambda+3\alpha}], E_\alpha] \\ &= -[[E_\alpha, \theta E_\alpha], X_{\lambda+3\alpha}] = \langle \lambda + 3\alpha, \alpha \rangle \langle E_\alpha, E_\alpha \rangle X_{\lambda+3\alpha}. \end{aligned}$$

Hence, $X_{\lambda+3\alpha} = 0$ and $\xi_{\lambda+2\alpha} = 0$ because $\langle \lambda + 3\alpha, \alpha \rangle \neq 0$. Since $\lambda - \alpha \notin \Sigma$ and $\lambda + \alpha \in \Sigma$, Lemma 5.4 and $[\xi_\lambda, E_\alpha] = [\xi_{\lambda+2\alpha}, \theta E_\alpha] = 0$ imply $\xi_\lambda = 0$. ☐

We define $\Psi = \{\gamma \in \Lambda : \langle \gamma, \alpha \rangle = 0 \text{ for all } \alpha \in \Phi\}$. The root subsystem of Σ generated by Ψ is denoted by Σ_Ψ. We also denote by 2Φ the set of roots of the form 2α with $\alpha \in \Phi$. Of course, the number of elements of 2Φ is at most the number of irreducible components of Σ. The root subsystem generated by $\Psi \cup \Phi$ is $\Sigma_\Psi \cup \Phi \cup 2\Phi$.

Lemma 5.16. *We have*

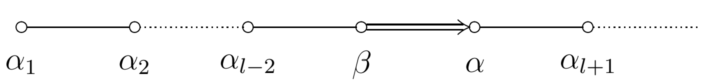
$$\hat{\mathfrak{s}}_{\mathfrak{p}}^{\perp} \subset \mathfrak{a}_{\Phi} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathbb{R}(1-\theta)E_{\alpha} \right) \oplus (\mathfrak{p}_{\Psi} \ominus \mathfrak{a}) \,.$$

Proof. If $\Phi = \emptyset$ then $\Psi = \Lambda$ and $\Sigma_{\Psi} = \Sigma$, and so the assertion is that $\hat{\mathfrak{s}}_{\mathfrak{p}}^{\perp} \subset \mathfrak{p}$ and there is nothing to prove in that case. Hence, we may assume that $\Phi \neq \emptyset$. It can happen that $\Phi \cup \Psi = \Lambda$. By definition of Ψ this implies that Σ is reducible, and in fact it is the direct sum of two root systems, one generated by Ψ and the other one generated by Φ. Moreover, each element of Φ is in an irreducible component of rank one of Σ. In that case, $\Sigma_{\Psi} = \Sigma \setminus (\Phi \cup 2\Phi)$ and the result follows readily from Lemma 5.14. Hence, we may also assume that $\Phi \cup \Psi \neq \Lambda$.

Let $Z = H^{\Phi \cup \Psi}$ be the characteristic element in $\mathfrak{a}$ of the gradation $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^{k}_{\Phi \cup \Psi}$ of $\mathfrak{g}$ corresponding to the parabolic subalgebra $\mathfrak{q}_{\Phi \cup \Psi}$. We claim that if $\lambda \in \Sigma^{+}$ and $\lambda(Z) = 1$ then there exists $\alpha \in \Phi$ such that $\lambda + \alpha \in \Sigma^{+}$ or $\lambda - \alpha \in \Sigma^{+}$.

In order to prove this, let $\lambda \in \Sigma^{+}$ be such that $\lambda(Z) = 1$. If $\langle \lambda, \alpha \rangle \neq 0$ for some $\alpha \in \Phi$, then $A_{\lambda\alpha} \neq 0$, where $A_{\lambda\alpha}$ is the corresponding Cartan integer. This clearly implies our claim. Thus we may assume $\langle \lambda, \alpha \rangle = 0$ for all $\alpha \in \Phi$. Write $\lambda = \sum_{\gamma \in \Lambda} n_{\gamma}\gamma$ with $n_{\gamma} \geq 0$. Then by hypothesis, $1 = \lambda(Z) = \sum_{\gamma \in \Lambda \setminus (\Phi \cup \Psi)} n_{\gamma}$. Since $n_{\gamma} \geq 0$ we can then write $\lambda = \sum_{\alpha \in \Phi} n_{\alpha}\alpha + \beta + \mu$, where $\beta \in \Lambda \setminus (\Phi \cup \Psi)$ and $\mu \in \operatorname{span} \Psi$. Now, since Φ consists of orthogonal roots, for each $\alpha \in \Phi$ we have $0 = \langle \lambda, \alpha \rangle = n_{\alpha}\langle \alpha, \alpha \rangle + \langle \beta, \alpha \rangle = (n_{\alpha} + A_{\beta\alpha}/2)\langle \alpha, \alpha \rangle$ so the Cartan integer satisfies $A_{\beta\alpha} = -2n_{\alpha}$. It cannot happen that $n_{\alpha} = 0$ for all $\alpha \in \Phi$ because in that case the previous equality implies $\langle \beta, \alpha \rangle = 0$ for all $\alpha \in \Phi$ and hence $\beta \in \Psi$, contradiction. Therefore we can find $\alpha \in \Phi$ such that $n_{\alpha} > 0$. We will see that $\lambda \pm \alpha \in \Sigma^{+}$, from where the claim will follow.

By the properties of Cartan integers, the equation $A_{\beta\alpha} = -2n_{\alpha}$ can only hold when $n_{\alpha} = 1$ and $\{\alpha, \beta\}$ spans a root system of type B_2 (or BC_2). It is also obvious that λ must be a root of the root subsystem of Σ determined by the irreducible component where both α and β lie. Hence, the connected component of α and β in the Dynkin diagram of the original root system Σ has a double arrow pointing to α. Therefore, this connected component is one of B_r, C_r, BC_r or F_4. Relabel the corresponding reduced Dynkin diagram as indicated in the following figure:



According to this labeling $\mu \in \operatorname{span}\{\alpha_1, \dots, \alpha_{l-2}, \alpha_{l+2}, \dots\}$ (whenever the corresponding simple roots exist). However, since α_i is orthogonal to α and β for $i \geq l+2$, it follows that λ can be a root only if $\mu \in \operatorname{span}\{\alpha_1, \dots, \alpha_{l-2}\}$, so the problem reduces to studying a root of the form $\lambda = \sum_{i=1}^{l} n_i\alpha_i$ in a root system of type B_l or BC_l with the labeling as above and with $n_{l-1} = 1$, $n_l \geq 1$. By the description of all roots for these root systems, λ must be of the form $\lambda = \alpha_i + \cdots + \alpha_l$ or $\lambda = \alpha_i + \cdots + \alpha_{l-1} + 2\alpha_l$. Only the first of these two possibilities can be orthogonal to α, and in that case it follows that $\lambda \pm \alpha \in \Sigma^{+}$.

Therefore, if $\lambda \in \Sigma^+$ and $\lambda(Z) = 1$ there exists $\alpha \in \Phi$ such that $\lambda + \alpha \in \Sigma^+$ or $\lambda - \alpha \in \Sigma^+$. Since $\lambda \neq \alpha, 2\alpha$, we can now apply Lemma 5.15 to obtain that $\mathfrak{g}^1_{\Phi\cup\Psi} \subset \hat{\mathfrak{s}}_n$. Since the gradation is of type α_0, it follows from Lemma 5.13 that $\bigoplus_{k\geq 1} \mathfrak{g}^k_{\Phi\cup\Psi} \subset \hat{\mathfrak{s}}_n$. This implies that $\hat{\mathfrak{s}}_{\mathfrak{p}}^{\perp}$ is contained in the projection of $\mathfrak{g}^0$ onto $\mathfrak{p}$. Obviously $\lambda(Z) = 0$ if and only if $\lambda \in \Sigma_\Psi \cup \Phi \cup 2\Phi$. Combining this with Lemma 5.14 we get the result. □

Now we turn our attention to the $\mathfrak{a}$-part of $\hat{\mathfrak{s}}_{\mathfrak{p}}^{\perp}$ and define $\bar{\Sigma} = \{\lambda \in \Sigma : \lambda(\mathfrak{a}_\Phi \ominus V) = 0\} = \{\lambda \in \Sigma : H_\lambda \in V \oplus \mathfrak{a}^\Phi\}$. It is obvious that $\bar{\Sigma}$ is a (possibly empty) root subsystem of Σ. We denote by $\bar{\Sigma}^+$ a set of positive roots with respect to an ordering consistent with that of Σ. We define $\Pi = \Lambda \cap \bar{\Sigma}$ and denote by Σ_Π the corresponding root subsystem of Σ generated by Π. Consider in Σ_Π an ordering compatible with that of Σ so that $\Sigma_\Pi^+ = \Sigma_\Pi \cap \Sigma^+$.

Lemma 5.17. *We have*

$$\hat{\mathfrak{s}}_{\mathfrak{p}}^{\perp} \subset \mathfrak{a}_\Phi \oplus \left(\bigoplus_{\alpha\in\Phi} \mathbb{R}(1-\theta)E_\alpha\right) \oplus (\mathfrak{p}_\Pi \ominus \mathfrak{a}).$$

Proof. Write $\xi \in \mathfrak{p}$ as $\xi = \xi_0 + \sum_{\lambda\in\Sigma^+}(1-\theta)\xi_\lambda$ with $\xi_0 \in \mathfrak{a}$ and $\xi_\lambda \in \mathfrak{g}_\lambda$ for each $\lambda \in \Sigma^+$. Given $H \in \mathfrak{a}_\Phi \ominus V$ we get $[H, \xi] = (1+\theta)\sum_{\lambda\in\Sigma^+} \lambda(H)\xi_\lambda$, which implies that the centralizer of $\mathfrak{a}_\Phi \ominus V$ in $\mathfrak{p}$ is $Z_{\mathfrak{p}}(\mathfrak{a}_\Phi \ominus V) = \mathfrak{a} \oplus \left(\bigoplus_{\lambda\in\bar{\Sigma}^+} \mathfrak{p}_\lambda\right)$.

For any $\alpha \in \Phi$ it is obvious that $\alpha(\mathfrak{a}_\Phi \ominus V) = 0$, and so $\Phi \subset \bar{\Sigma}$. Using Lemma 5.16 and $Z_{\mathfrak{p}}(\mathfrak{a}_\Phi \ominus V) = \mathfrak{a} \oplus \left(\bigoplus_{\lambda\in\bar{\Sigma}^+} \mathfrak{p}_\lambda\right)$ we easily get

$$\hat{\mathfrak{s}}_{\mathfrak{p}}^{\perp} \subset \mathfrak{a}_\Phi \oplus \left(\bigoplus_{\alpha\in\Phi} \mathbb{R}(1-\theta)E_\alpha\right) \oplus \left(\bigoplus_{\lambda\in\Sigma_\Psi^+\cap\bar{\Sigma}} \mathfrak{p}_\lambda\right).$$

This implies that for any $\lambda \in \Sigma^+ \setminus (\Phi \cup (\Sigma_\Psi \cap \bar{\Sigma}))$ we have $\mathfrak{g}_\lambda \in \hat{\mathfrak{s}}_n$. We will use this fact several times during this proof.

Let $Z = H^{\Lambda\setminus(\Psi\setminus\Pi)}$ be the characteristic element in $\mathfrak{a}$ of the gradation $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}^k_{\Lambda\setminus(\Psi\setminus\Pi)}$ of $\mathfrak{g}$ corresponding to the parabolic subalgebra $\mathfrak{q}_{\Lambda\setminus(\Psi\setminus\Pi)}$. Let $\lambda \in \Sigma_\Psi^+$ be written as $\lambda = \sum_{\gamma\in\Psi} n_\gamma\gamma$ and assume $\lambda(Z) = 1$. Then $1 = \lambda(Z) = \sum_{\gamma\in\Psi\setminus\Pi} n_\gamma$, so we can write $\lambda = \alpha + \mu$, where $\alpha \in \Psi \setminus \Pi$ and $\mu \in \operatorname{span}\Pi$. By definition of μ it is obvious that $\mu(\mathfrak{a}_\Phi \ominus V) = 0$, and by definition of α it is clear that $\lambda(\mathfrak{a}_\Phi \ominus V) = \alpha(\mathfrak{a}_\Phi \ominus V) \neq 0$, that is, $\lambda \notin \bar{\Sigma}$. Thus we have $\mathfrak{g}_\lambda \subset \hat{\mathfrak{s}}_n$. On the other hand, assume $\lambda \in \Sigma^+ \setminus \Sigma_\Psi^+$ satisfies $\lambda(Z) = 1$. Then we conclude $\mathfrak{g}_\lambda \subset \hat{\mathfrak{s}}_n$ unless $\lambda \in \Phi$. The latter case is not possible since $\Phi \subset \Lambda \setminus \Psi \subset \Lambda \setminus (\Psi \setminus \Pi)$, which would imply $\lambda(Z) = 0$. Hence, the conclusion is that for any $\lambda \in \Sigma^+$ satisfying $\lambda(Z) = 1$ we have $\mathfrak{g}_\lambda \subset \hat{\mathfrak{s}}_n$. This implies that $\mathfrak{g}^1_{\Lambda\setminus(\Psi\setminus\Pi)} \subset \hat{\mathfrak{s}}_n$, and hence, by Lemma 4.13 we have $\bigoplus_{k\geq 1} \mathfrak{g}^k_{\Lambda\setminus(\Psi\setminus\Pi)} \subset \hat{\mathfrak{s}}_n$. Combining this with the above inclusion for $\hat{\mathfrak{s}}_{\mathfrak{p}}^{\perp}$ implies the result. □

We are now ready to determine $\mathfrak{s}_n$.

Lemma 5.18. *We have* $\mathfrak{s}_n = \mathfrak{s}_{\Phi,V,a}$.

Proof. Fix $\alpha \in \Pi$. Since $\alpha \in \bar{\Sigma}$ we have $\alpha(\mathfrak{a}_\Phi \ominus V) = 0$ and hence $H_\alpha \in V \oplus \mathfrak{a}^\Phi$. On the other hand, $\alpha \in \Sigma_\Psi^+$ so $H_\alpha \in \mathfrak{a}_\Phi$. Since $\mathfrak{a}_\Phi \cap \mathfrak{a}^\Phi = \{0\}$, Proposition 5.11 (iii) applied to $\tilde{\mathfrak{s}}$

implies $H_\alpha \in V \subset (\mathrm{Ad}(g)\tilde{\mathfrak{s}})_n$. Thus there exists $S \in \mathfrak{t}$ such that $S + H_\alpha \in \mathrm{Ad}(g)\tilde{\mathfrak{s}}$. Let $X \in \mathfrak{g}_\alpha$. By definition of Π and Proposition 5.11 (iv) we get $\mathfrak{g}_\alpha \subset (\mathrm{Ad}(g)\tilde{\mathfrak{s}})_n$. Then there exists $T \in \mathfrak{t}$ such that $T + X \in \mathrm{Ad}(g)\tilde{\mathfrak{s}}$. Since $\mathrm{Ad}(g)\tilde{\mathfrak{s}}$ is a Lie algebra and $[\mathfrak{t}, \mathfrak{g}_\alpha] \subset \mathfrak{g}_\alpha$ we have $[S + H_\alpha, T + X] = [S, X] + [H_\alpha, X] = (\mathrm{ad}(S) + \langle \alpha, \alpha \rangle 1_{\mathfrak{g}_\alpha})X \in (\mathrm{Ad}(g)\tilde{\mathfrak{s}}) \cap \mathfrak{g}_\alpha$, where $1_{\mathfrak{g}_\alpha}$ is the identity of $\mathfrak{g}_\alpha$. Since the linear map $\mathrm{ad}(S)$ is skewsymmetric, it follows that $\mathrm{ad}(S) + \langle \alpha, \alpha \rangle 1_{\mathfrak{g}_\alpha} : \mathfrak{g}_\alpha \to \mathfrak{g}_\alpha$ is an isomorphism. However, $X \in \mathfrak{g}_\alpha$ is arbitrary, and so $\mathfrak{g}_\alpha \subset \mathrm{Ad}(g)\tilde{\mathfrak{s}}$. Since $\alpha \in \Pi$ is also arbitrary it follows that for all $\alpha \in \Pi$ we have $\mathfrak{g}_\alpha \subset \mathrm{Ad}(g)\tilde{\mathfrak{s}}$.

Let us denote by $\bar{\mathfrak{n}}^s = \mathfrak{n}^s \cap \mathfrak{n}_\Pi$ the direct sum of root spaces associated with roots of Σ_Π^+ of level s (note that the level of a root in Σ_Π^+ coincides with the level of this root as a root of Σ). The previous argument shows that $\bar{\mathfrak{n}}^1 \subset \mathrm{Ad}(g)\tilde{\mathfrak{s}}$. Choose a basis $\{E_1, \dots, E_k\}$ of $\bar{\mathfrak{n}}^1$. Since $g \in N$ it follows that $\mathrm{Ad}(g)(\mathfrak{n} \ominus \mathfrak{n}^1) \subset \mathfrak{n} \ominus \mathfrak{n}^1$, and as $\mathrm{Ad}(g)$ is an automorphism equality holds. Hence, by definition of $\tilde{\mathfrak{s}}$ and $\hat{\mathfrak{s}}$ it is obvious that $\mathrm{Ad}(g)\tilde{\mathfrak{s}} = \hat{\mathfrak{s}} + (\mathfrak{n} \ominus \mathfrak{n}^1)$. Therefore, for each i, there exists $X_i \in \mathfrak{n} \ominus \mathfrak{n}^1$ such that $E_i + X_i \in \hat{\mathfrak{s}}$.

We introduce the following notation. Define $[Y_1, Y_2, \dots, Y_l] = [Y_1, [Y_2, \dots, Y_l]]$ inductively, being $[Y_1, Y_2]$ the usual Lie bracket. Denote by k the level of the highest root of Σ_Π^+. Let s be the smallest integer for which $\bar{\mathfrak{n}}^s \oplus \cdots \oplus \bar{\mathfrak{n}}^k \subset \hat{\mathfrak{s}}_n$. Our aim is to prove $s = 1$.

First we prove $s \leq k$, that is, $\bar{\mathfrak{n}}^k \subset \hat{\mathfrak{s}}_n$. Since $[\mathfrak{n}^a, \mathfrak{n}^b] \subset \mathfrak{n}^{a+b}$ and $\mathfrak{n} \ominus (\mathfrak{n}^1 \oplus \bar{\mathfrak{n}}^2 \oplus \cdots \oplus \bar{\mathfrak{n}}^k) \subset \hat{\mathfrak{s}}_n$ by Lemma 5.17, we have $[E_{i_1} + X_{i_1}, \dots, E_{i_k} + X_{i_k}] \equiv [E_{i_1}, \dots, E_{i_m}] \mod \hat{\mathfrak{s}}_n$. Here we have used the fact that the level of a root in Σ_Π^+ is the same as the level of that root as a root of Σ^+. The brackets of k vectors in the right-hand side of the previous formula span $\bar{\mathfrak{n}}^k$ whereas the brackets on the left-hand side belong to $\hat{\mathfrak{s}} \cap \mathfrak{n}$ because $\hat{\mathfrak{s}} \cap \mathfrak{n}$ is a subalgebra. Since $\hat{\mathfrak{s}} \cap \mathfrak{n} \subset \hat{\mathfrak{s}}_n$, this implies $\bar{\mathfrak{n}}^k \subset \hat{\mathfrak{s}}_n$.

Now assume $s > 1$. Hence $\bar{\mathfrak{n}}^s \oplus \cdots \oplus \bar{\mathfrak{n}}^k \subset \hat{\mathfrak{s}}_n$ but $\bar{\mathfrak{n}}^{s-1} \not\subset \hat{\mathfrak{s}}_n$. We use again $[\mathfrak{n}^a, \mathfrak{n}^b] \subset \mathfrak{n}^{a+b}$ and $\mathfrak{n} \ominus (\mathfrak{n}^1 \oplus \bar{\mathfrak{n}}^2 \oplus \cdots \oplus \bar{\mathfrak{n}}^{s-1}) \subset \hat{\mathfrak{s}}_n$, which follows from Lemma 5.17 and the definition of s. Thus we get $[E_{i_1} + X_{i_1}, \dots, E_{i_{s-1}} + X_{i_{s-1}}] \equiv [E_{i_1}, \dots, E_{i_{s-1}}] \mod \hat{\mathfrak{s}}_n$. Again, the brackets in the right-hand side of the congruency span $\bar{\mathfrak{n}}^{s-1}$ whereas the brackets in the left-hand side belong to $\hat{\mathfrak{s}} \cap \mathfrak{n} \subset \hat{\mathfrak{s}}_n$. Then we get $\bar{\mathfrak{n}}^{s-1} \subset \hat{\mathfrak{s}}_n$, which is a contradiction. Therefore $s = 1$ and thus $\bar{\mathfrak{n}}^1 \oplus \cdots \oplus \bar{\mathfrak{n}}^k \subset \hat{\mathfrak{s}}_n$. Altogether this implies

$$(\mathfrak{a}_\Phi \ominus V) \oplus \left(\bigoplus_{\alpha \in \Phi} \mathbb{R}(1 - \theta)E_\alpha \right) \subset \hat{\mathfrak{s}}_{\mathfrak{p}}^\perp \subset \mathfrak{a}_\Phi \oplus \left(\bigoplus_{\alpha \in \Phi} \mathbb{R}(1 - \theta)E_\alpha \right).$$

Since $V \subset (\mathrm{Ad}(g)\tilde{\mathfrak{s}})_n$ by Proposition 5.11 (iii) and $\mathfrak{n} \ominus \mathfrak{n}^1 \subset \hat{\mathfrak{s}}_n$ by the above equation, it follows from $\mathrm{Ad}(g)\tilde{\mathfrak{s}} = \hat{\mathfrak{s}} + (\mathfrak{n} \ominus \mathfrak{n}^1)$ that $\hat{\mathfrak{s}}_n = \mathfrak{s}_{\Phi,V}$.

Let $T + H + X \in \hat{\mathfrak{s}}$ with $T \in \mathfrak{t}$, $H \in \mathfrak{a}$ and $X \in \mathfrak{n}$. By hypothesis, the connected subgroup $\hat{S}$ of G with Lie algebra $\hat{\mathfrak{s}}$ induces a hyperpolar foliation. Let $E = -\sum_{\alpha \in \Phi} a_\alpha E_\alpha$ and $g = \mathrm{Exp}(E)$. It follows from Proposition 5.11 (vi) that $[\hat{\mathfrak{s}}_c, E] = 0$, and hence $\mathrm{Ad}(g^{-1})(T + H + X) = T + \mathrm{Ad}(g^{-1})(H + X)$. Proposition 5.10 shows that $\mathrm{Ad}(g^{-1})\mathfrak{s}_{\Phi,V} = \mathfrak{s}_{\Phi,V,a}$. Since $\mathfrak{s} = \mathrm{Ad}(g^{-1})\hat{\mathfrak{s}}$, the result follows. □

To conclude our proof we need to prove the following result:

Proposition 5.19. *Let $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be a maximally noncompact Borel subalgebra of $\mathfrak{g}$. Let $\mathfrak{s}$ be a subalgebra of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ such that $\mathfrak{s}_n = \pi_{\mathfrak{a}\oplus\mathfrak{n}}(\mathfrak{s}) = \mathfrak{s}_{\Phi,V,a}$ with some orthogonal subset Φ of Λ. Assume that the orbits of the connected subgroup S of G whose Lie algebra is $\mathfrak{s}$ form a homogeneous foliation on M. Then the actions of S and $S_{\Phi,V}$ are orbit equivalent.*

Proof. First, assume $\mathfrak{s}$ is a subalgebra of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ such that $\mathfrak{s}_n = \mathfrak{s}_{\Phi,V}$. Certainly, $\mathfrak{s}_n$ is a subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$. Denote by S and S_n the corresponding connected subgroups of G. Also, denote by $\bar{N}$ the connected subgroup of G whose subalgebra is $\mathfrak{n} \ominus (\bigoplus_{\alpha\in\Phi} \mathbb{R}E_\alpha)$. We prove that S and S_n have the same orbits.

Assume that $T + H \in \mathfrak{s}$ with $H \in \mathfrak{a}$ and $t \in \mathfrak{t}$. Let $X \in \mathfrak{n} \ominus \left(\bigoplus_{\alpha\in\Phi} \mathbb{R}E_\alpha\right)$. By definition, there exists $R \in \mathfrak{s}$ such that $R + X \in \mathfrak{s}$. As $\mathfrak{t} \oplus \mathfrak{a}$ is abelian, $[T + H, X] = [T + H, R + X] \in \mathfrak{s} \cap \mathfrak{n}$. Hence, if $tan \in S$, there exists $n' \in \bar{N}$ such that $tan = n'ta$. Since $\mathfrak{t} \oplus \mathfrak{a}$ is abelian we have $ta = at$, and since $\mathfrak{a}$ normalizes $\mathfrak{n} \ominus \left(\bigoplus_{\alpha\in\Phi} \mathbb{R}E_\alpha\right)$, there exists $n'' \in \bar{N}$ such that $n'a = an''$. Altogether this implies $tan = n'ta = n'at = an''t$. Thus, $tan(o) = an''t(o) = an''(o)$ and hence $S \cdot o \subset S_n \cdot o$. Since both orbits $S \cdot o$ and $S_n \cdot o$ have the same dimension and are connected and complete we conclude $S \cdot o = S_n \cdot o$. Now, let $p = \exp_o(\xi)$ with $\xi \in \nu_o(S \cdot o)$. Using the fact that S acts isometrically on M and that $t_*\xi = \xi$ by Proposition 5.11 (vii) we get

$$tan(p) = tan(\exp_o(\xi)) = \exp_{tan(o)}((tan)_*\xi) = \exp_{an''(o)}((an'')_*\xi) = an''(exp_o(\xi)) = an''(p).$$

Hence, $S \cdot p \subset S_n \cdot p$, and thus equality holds. Since the action of S is hyperpolar, all the orbits can be obtained in this way, and so S and S_n have the same orbits as claimed above.

Now we deal with the general case, that is, $\mathfrak{s}_n = \mathfrak{s}_{\Phi,V,a}$. Let S be the connected subgroup of G with Lie algebra $\mathfrak{s}$. By Proposition 5.11 (v), there exists $g \in N$ such that $(\mathrm{Ad}(g)\mathfrak{s})_n = \mathrm{Ad}(g)\mathfrak{s}_n = \mathfrak{s}_{\Phi,V}$. The subgroup $I_g(S_n)$ whose Lie algebra is $\mathrm{Ad}(g)\mathfrak{s}_n = (\mathrm{Ad}(g)\mathfrak{s})_n$ has the same orbits as $I_g(S)$ by the previous argument. Then $I_g(S_n)$ and S have the same orbits and hence the theorem follows. ☐

Now we finish the proof of Theorem 4.11 (ii). Let H be a closed subgroup of the isometry group of M inducing a hyperpolar homogeneous foliation on M. By Proposition 5.1, the action of H is orbit equivalent to the action of a closed solvable subgroup S whose Lie algebra $\mathfrak{s}$ is contained in a maximally noncompact Borel subalgebra. Then, there exists a Cartan decomposition of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and a root space decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\lambda\in\Sigma} \mathfrak{g}_\lambda\right)$ with respect to a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ such that the projection of $\mathfrak{s}$ onto $\mathfrak{a} \oplus \mathfrak{n}$ is given by $\mathfrak{s}_n = \mathfrak{s}_{\Phi,V,a}$ by Lemma 5.18. Proposition 5.19 then implies that the actions of the connected subgroups of G with Lie algebras $\mathfrak{s}$ and $\mathfrak{s}_{\Phi,V,a}$ are orbit equivalent. Hence, the action of H is orbit equivalent to the action of $S_{\Phi,V}$ on M, which concludes the proof of Theorem 4.11.

6. Geometry of the leaves of hyperpolar homogeneous foliations

In this section we study the extrinsic geometry of the leaves of hyperpolar homogeneous foliations on noncompact symmetric spaces.

Proposition 6.1. *The orbit $S_{\Phi,V} \cdot p$ is isometrically congruent to $S_{\Phi,V,a} \cdot o$ for some $a : \Phi \to \mathbb{R}$.*

Proof. Let $\mathcal{D}$ be the left-invariant distribution on M determined by $(\mathfrak{a} \ominus V) \oplus \ell_\Phi$. Obviously, $(\mathfrak{a} \ominus V) \oplus \ell_\Phi$ is a subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$, and since AN is simply connected, the leaf of $\mathcal{D}$ through o is $\mathcal{D}_o = \mathrm{Exp}((\mathfrak{a} \ominus V) \oplus \ell_\Phi) \cdot o$. We prove that $\mathcal{D}$ is autoparallel. Using the formula for the Levi-Civita connection it is easy to obtain $\nabla_H H' = \nabla_H E_\alpha = 0$, $\nabla_{E_\alpha} H = -\alpha(H) E_\alpha$ and $\nabla_{E_\alpha} E_\beta = \langle E_\alpha, E_\beta \rangle H_\alpha$ for any $H, H' \in \mathfrak{a} \ominus V$ and $\alpha, \beta \in \Phi$.

In particular, the leaf $\mathcal{D}_o$ is totally geodesic in M and since $\nu_o(S_{\Phi,V} \cdot o) = (\mathfrak{a}_\Phi \ominus V) \oplus \ell_\Phi$, it contains the section of $S_{\Phi,V}$ through o. Since a section of $S_{\Phi,V}$ intersects all the orbits, we may assume that p lies in that section and so, the point p can be written as $p = g(o)$ with $g = \mathrm{Exp}(X)$ and $X \in (\mathfrak{a} \ominus V) \oplus \ell_\Phi$. Hence, $S_{\Phi,V} \cdot p = g(g^{-1} S_{\Phi,V} g) \cdot o = g I_g(S_{\Phi,V}) \cdot o$. Since g is an isometry of M, the orbit $S_{\Phi,V} \cdot p$ is isometrically congruent to $I_g(S_{\Phi,V}) \cdot o$. We will now prove that $I_g(S_{\Phi,V}) = S_{\Phi,V,a}$ for some $a : \Phi \to \mathbb{R}$, and for that we will show that $\mathrm{Ad}(g)\mathfrak{s}_{\Phi,V} = \mathfrak{s}_{\Phi,V,a}$.

Let $H \in \mathfrak{a}$. We show that $\mathrm{Ad}(\mathrm{Exp}\, H)\mathfrak{s}_{\Phi,V} = \mathfrak{s}_{\Phi,V}$. First notice that $\mathrm{Ad}(\mathrm{Exp}\, H)\mathfrak{s}_{\Phi,V} \subset \mathfrak{a} \oplus \mathfrak{n}$, so it suffices to prove that $\mathrm{Ad}(\mathrm{Exp}\, H)\mathfrak{s}_{\Phi,V}$ is orthogonal to $(\mathfrak{a}_\Phi \ominus V) \oplus \ell_\Phi$. Let $X \in \mathfrak{s}_{\Phi,V}$, $H' \in \mathfrak{a}_\Phi \ominus V$ and $\alpha \in \Phi$. Then our assertion follows from $\langle \mathrm{Ad}(\mathrm{Exp}\, H)X, H' \rangle = \langle X, \mathrm{Ad}(\mathrm{Exp}(-\theta H))H' \rangle = \langle X, e^{\mathrm{ad}(H)} H' \rangle = \langle X, H' \rangle = 0$, and

$$\langle \mathrm{Ad}(\mathrm{Exp}\, H)X, E_\alpha \rangle \;=\; \langle X, e^{\mathrm{ad}(H)} E_\alpha \rangle \;=\; \sum_{k=0}^{\infty} \frac{\alpha(H)^k}{k!} \langle X, E_\alpha \rangle \;=\; e^{\alpha(H)} \langle X, E_\alpha \rangle \;=\; 0.$$

Write $X = H + \sum_{\alpha \in \Phi}(x_\alpha H_\alpha + y_\alpha E_\alpha)$ with $H \in \mathfrak{a}_\Phi \ominus V$ and $x_\alpha, y_\alpha \in \mathbb{R}$. Since $\mathbb{R}H_\alpha \oplus \mathbb{R}E_\alpha$ is a subalgebra, there exist constants $a_\alpha, b_\alpha \in \mathbb{R}$ such that $\mathrm{Exp}(a_\alpha E_\alpha) \cdot \mathrm{Exp}(b_\alpha H_\alpha) = \mathrm{Exp}(x_\alpha H_\alpha + y_\alpha E_\alpha)$. This equation and $[\mathfrak{a}_\Phi, \mathfrak{g}_\Phi] = [\mathfrak{g}_{\{\alpha\}}, \mathfrak{g}_{\{\beta\}}] = \{0\}$ for any $\alpha, \beta \in \Phi$, $\alpha \neq \beta$, imply

$$\begin{aligned} g &= \left(\prod_{\alpha \in \Phi} \mathrm{Exp}(x_\alpha H_\alpha + y_\alpha E_\alpha) \right) \mathrm{Exp}\, H = \left(\prod_{\alpha \in \Phi} \mathrm{Exp}(a_\alpha E_\alpha)\, \mathrm{Exp}(b_\alpha H_\alpha) \right) \mathrm{Exp}\, H \\ &= \mathrm{Exp} \left(\sum_{\alpha \in \Phi} a_\alpha E_\alpha \right) \mathrm{Exp} \left(\sum_{\alpha \in \Phi} b_\alpha H_\alpha \right) \mathrm{Exp}\, H. \end{aligned}$$

Hence, the equality $\mathrm{Ad}(\mathrm{Exp}\, H')\mathfrak{s}_{\Phi,V} = \mathfrak{s}_{\Phi,V}$ for any $H' \in \mathfrak{a}$ and Proposition 5.10 imply

$$\mathrm{Ad}(g)\mathfrak{s}_{\Phi,V} = \mathrm{Ad} \left(\mathrm{Exp} \left(\sum_{\alpha \in \Phi} a_\alpha E_\alpha \right) \right) \mathrm{Ad} \left(\mathrm{Exp} \left(\sum_{\alpha \in \Phi} b_\alpha H_\alpha \right) \right) \mathrm{Ad}(\mathrm{Exp}\, H)\mathfrak{s}_{\Phi,V} = \mathfrak{s}_{\Phi,V,a},$$

where $a : \Phi \to \mathbb{R}$, $\alpha \mapsto a_\alpha$. This concludes the proof. □

In view of Proposition 6.1, in order to calculate the geometry of the orbits of $S_{\Phi,V}$, it suffices to study the geometry of the orbit through the origin o of $S_{\Phi,V,a}$, where Φ is an orthogonal subset of Λ, V is a linear subspace of $\mathfrak{a}_\Phi$ and $a : \Phi \to \mathbb{R}$ is a function. Hence, we consider

$$\mathfrak{s}_{\Phi,V,a} = (V \oplus \mathfrak{a}^\Phi \oplus \mathfrak{n}) \ominus \left(\bigoplus_{\alpha\in\Phi} \mathbb{R}(a_\alpha H_\alpha + E_\alpha)\right),$$

for certain $a_\alpha \in \mathbb{R}$ and nonzero $E_\alpha \in \mathfrak{g}_\alpha$. Note that,

$$\mathfrak{s}_{\Phi,V,a} = V \oplus \left(\bigoplus_{\alpha\in\Phi} \mathbb{R}X_\alpha\right) \oplus \left(\bigoplus_{\alpha\in\Phi}((\mathfrak{g}_\alpha \ominus \mathbb{R}E_\alpha) \oplus \mathfrak{g}_{2\alpha})\right) \oplus \left(\bigoplus_{\lambda\in\Sigma^+\setminus(\Phi\cup 2\Phi)} \mathfrak{g}_\lambda\right),$$

where $X_\alpha = \frac{1}{|\alpha|^2}H_\alpha - a_\alpha E_\alpha$. We denote by 2Φ the set of roots of the form 2α with $\alpha \in \Phi$.

Let $X \in \mathfrak{s}_{\Phi,V,a}$ and $\xi \in (\mathfrak{a}\oplus\mathfrak{n})\ominus\mathfrak{s}_{\Phi,V,a}$. Using the formula for the Levi-Civita connection with respect to left-invariant vector fields we easily obtain

$$A_\xi X = \frac{1}{2}[(1-\theta)\xi, X]_{\mathfrak{s}_{\Phi,V,a}},$$

where the subscript denotes orthogonal projection onto $\mathfrak{s}_{\Phi,V,a}$.

Assume first that $\xi \in \mathfrak{a}_\Phi \ominus V$. If $X \in V$, then $A_\xi X = [\xi, X] = 0$. Since $\xi \in \mathfrak{a}_\Phi$ we also have $A_\xi X_\alpha = [\xi, X_\alpha]_{\mathfrak{s}_{\Phi,V,a}} = -a_\alpha \alpha(\xi)(E_\alpha)_{\mathfrak{s}_{\Phi,V,a}} = 0$. Analogously, if $X \in \mathfrak{g}_\alpha \ominus \mathbb{R}E_\alpha$ then $A_\xi X = \alpha(\xi)X = 0$, and similarly, if $X \in \mathfrak{g}_{2\alpha}$ then also $A_\xi X = 0$. Finally, if $X \in \mathfrak{g}_\lambda$ with $\lambda \in \Sigma^+ \setminus (\Phi \cup 2\Phi)$, then $A_\xi X = [\xi, X]_{\mathfrak{s}_{\Phi,V,a}} = \lambda(\xi)X$. In particular, $\operatorname{tr}A_\xi = \sum_{\lambda\in\Sigma^+\setminus(\Phi\cup2\Phi)}(\dim\mathfrak{g}_\lambda)\lambda(\xi) = 2\delta(\xi)$, where as usual $\delta = \frac{1}{2}\sum_{\lambda\in\Sigma^+}(\dim\mathfrak{g}_\lambda)\lambda$ (see [15, p. 329]).

Now let $\xi = a_\alpha H_\alpha + E_\alpha$ for some $\alpha \in \Phi$. If $X \in V$, then it follows easily that

$$A_{\xi_\alpha} X = \frac{1}{2}[(1-\theta)\xi, X]_{\mathfrak{s}_{\Phi,V,a}} = -\frac{1}{2}\alpha(X)((1+\theta)\xi)_{\mathfrak{s}_{\Phi,V,a}} = 0.$$

For $\beta \in \Phi$ we calculate

$$A_\xi X_\beta = \left(-a_\alpha a_\beta\langle\alpha,\beta\rangle E_\beta - \frac{1}{2|\alpha_\beta|^2}\langle\alpha,\beta\rangle(1+\theta)E_\alpha + \frac{1}{2}a_\beta[\theta E_\alpha, E_\beta]\right)_{\mathfrak{s}_{\Phi,V,a}}.$$

If $\alpha \neq \beta$, we have as usual that α and β are orthogonal and hence $A_\xi X_\beta = 0$. If $\alpha = \beta$, we can write the above expression in terms of ξ and X_α to get

$$A_\xi X_\alpha = \left(a_\alpha|\alpha|^2 X_\alpha - \frac{1}{2}\xi - \frac{1}{2}\theta E_\alpha\right)_{\mathfrak{s}_{\Phi,V,a}} = a_\alpha|\alpha|^2 X_\alpha.$$

Assume now that $X \in \mathfrak{g}_\beta \ominus \mathbb{R}E_\beta$ with $\beta \in \Phi$ and $\alpha \neq \beta$. Since α and β are orthogonal we get $[(1-\theta)\xi, X] = 2a_\alpha\langle\alpha,\beta\rangle X + [E_\alpha, X] - [\theta E_\alpha, X] = 0$, and thus $A_\xi X = 0$. One can prove in a similar way that $A_\xi X = 0$ if $X \in \mathfrak{g}_{2\beta}$ with $\beta \in \Phi$ and $\alpha \neq \beta$.

Now we turn our attention to the subspace $(\mathfrak{g}_\alpha \ominus \mathbb{R}\xi) \oplus \mathfrak{g}_{2\alpha}$. Let $X \in \mathfrak{g}_\alpha \ominus \mathbb{R}\xi$. Clearly, $[\theta E_\alpha, X] \in \mathfrak{g}_0$ and $\langle[\theta E_\alpha, X], H\rangle = -\alpha(H)\langle X, E_\alpha\rangle = 0$ for all $H \in \mathfrak{a}$. Then we get

$$A_\xi X = \left[a_\alpha H_\alpha + \frac{1}{2}E_\alpha - \frac{1}{2}\theta E_\alpha, X\right]_{\mathfrak{s}_{\Phi,V,a}} = a_\alpha|\alpha|^2 X + \frac{1}{2}[E_\alpha, X].$$

On the other hand, if $Y \in \mathfrak{g}_{2\alpha}$ we have $\langle [\theta E_\alpha, Y], \xi\rangle = a_\alpha \langle \alpha, \alpha\rangle \langle Y, E_\alpha\rangle = 0$, and so

$$A_\xi Y = \left[a_\alpha H_\alpha + \frac{1}{2}E_\alpha - \frac{1}{2}\theta E_\alpha, Y\right]_{\mathfrak{s}_{\Phi,V,a}} = 2a_\alpha|\alpha|^2 Y - \frac{1}{2}[\theta E_\alpha, Y].$$

It is then clear that A_ξ leaves the subspace $(\mathfrak{g}_\alpha \ominus \mathbb{R}\xi) \oplus \mathfrak{g}_{2\alpha}$ invariant. Moreover, since for all $Y \in \mathfrak{g}_{2\alpha}$ we have $[E_\alpha, [\theta E_\alpha, Y]] = -[Y, [E_\alpha, \theta E_\alpha]] = -2|\alpha|^2 Y$, the linear map $\operatorname{ad}(E_\alpha)|\operatorname{ad}(\theta E_\alpha)(\mathfrak{g}_{2\alpha}) : \operatorname{ad}(\theta E_\alpha)(\mathfrak{g}_{2\alpha}) \to \mathfrak{g}_{2\alpha}$ is an isomorphism. From here we obtain the decomposition $\mathfrak{g}_\alpha = \operatorname{Ker}(\operatorname{ad}(E_\alpha)|\mathfrak{g}_\alpha) \oplus \operatorname{ad}(\theta E_\alpha)(\mathfrak{g}_{2\alpha})$. Hence, if $X \in \operatorname{Ker}(\operatorname{ad}(E_\alpha)|\mathfrak{g}_\alpha)$ we get from the previous expression that $A_\xi X = a_\alpha|\alpha|^2 X$, so A_ξ restricted to $\operatorname{Ker}(\operatorname{ad}(E_\alpha)|\mathfrak{g}_\alpha)$ is $a_\alpha|\alpha|^2 1_{\mathfrak{g}_\alpha}$ and the multiplicity is $\dim \mathfrak{g}_\alpha - \dim \mathfrak{g}_{2\alpha} - 1$. On the other hand, for nonzero $Y \in \mathfrak{g}_{2\alpha}$ define $X = [\theta E_\alpha, Y] \in \mathfrak{g}_\alpha$. Then the previous formulas read $A_\xi X = a_\alpha|\alpha|^2 X - |\alpha|^2 Y$ and $A_\xi Y = -\frac{1}{2}X + 2a_\alpha|\alpha|^2 Y$. The eigenvalues of the matrix

$$\begin{pmatrix} a_\alpha|\alpha|^2 & -\frac{1}{2} \\ -|\alpha|^2 & 2a_\alpha|\alpha|^2 \end{pmatrix}$$

are $\frac{|\alpha|}{2}\left(3a_\alpha|\alpha| \pm \sqrt{2 + a_\alpha^2|\alpha|^2}\right)$.

Before continuing we need the following (recall that $\xi = \xi = a_\alpha H_\alpha + E_\alpha$)

Lemma 6.2. $\operatorname{ad}((1-\theta)\xi)\left(\mathfrak{n} \ominus \left(\bigoplus_{\gamma\in\Phi}(\mathfrak{g}_\gamma \oplus \mathfrak{g}_{2\gamma})\right)\right) \subset \mathfrak{n} \ominus \left(\bigoplus_{\gamma\in\Phi}(\mathfrak{g}_\gamma \oplus \mathfrak{g}_{2\gamma})\right)$.

Proof. Let $Y \in \mathfrak{n} \ominus \left(\bigoplus_{\gamma\in\Phi}(\mathfrak{g}_\gamma \oplus \mathfrak{g}_{2\gamma})\right)$. By the properties of root systems, it is clear that $\operatorname{ad}((1-\theta)\xi)Y \subset \mathfrak{n}$. For $\beta \in \Phi$ and $Z \in \mathfrak{g}_\beta$ we calculate

$$\langle \operatorname{ad}((1-\theta)\xi)Y, Z\rangle = \langle \operatorname{ad}((1-\theta)\xi)Z, Y\rangle = 2a_\alpha\langle \alpha, \beta\rangle\langle Z, Y\rangle + \langle [E_\alpha, Z], Y\rangle - \langle [\theta E_\alpha, Z], Y\rangle.$$

By assumption we have $\langle Z, Y\rangle = 0$. Now, if $\alpha \neq \beta$, $[E_\alpha, Z] \in \mathfrak{g}_{\alpha+\beta} = 0$ and $[\theta E_\alpha, Z] \in \mathfrak{g}_{\beta-\alpha} = 0$. If $\alpha = \beta$, $[E_\alpha, Z] \in \mathfrak{g}_{2\alpha}$ and so $\langle [E_\alpha, Z], Y\rangle = 0$, and $[\theta E_\alpha, Z] \in \mathfrak{g}_0$, and so $\langle [\theta E_\alpha, Z], Y\rangle = 0$. In any case, $\langle \operatorname{ad}((1-\theta)\xi)Y, Z\rangle = 0$.

If $Z \in \mathfrak{g}_{2\beta}$ a similar calculation shows $\langle \operatorname{ad}((1-\theta)\xi)Y, Z\rangle = \langle [E_\alpha, Z], Y\rangle - \langle [\theta E_\alpha, Z], Y\rangle$. If $\alpha \neq \beta$, $[E_\alpha, Z] \in \mathfrak{g}_{\alpha+2\beta} = 0$ and $[\theta E_\alpha, Z] \in \mathfrak{g}_{2\beta-\alpha} = 0$. If $\alpha = \beta$, $[E_\alpha, Z] \in \mathfrak{g}_{3\alpha} = 0$ and $[\theta E_\alpha, Z] \in \mathfrak{g}_\alpha$ so $\langle [\theta E_\alpha, Z], Y\rangle = 0$. In any case, $\langle \operatorname{ad}((1-\theta)\xi)Y, Z\rangle = 0$ and the result is proved. $\square$

Now define $\phi = \operatorname{Exp}(\frac{\pi}{\sqrt{2}|\alpha|}(1+\theta)E_\alpha))$ for $\alpha \in \Phi$. It is well known that $\phi \in N_K(\mathfrak{a})$. Moreover, we have

Lemma 6.3. $\operatorname{Ad}(\phi)\left(\mathfrak{n} \ominus \left(\bigoplus_{\gamma\in\Phi}(\mathfrak{g}_\gamma \oplus \mathfrak{g}_{2\gamma})\right)\right) \subset \mathfrak{n} \ominus \left(\bigoplus_{\gamma\in\Phi}(\mathfrak{g}_\gamma \oplus \mathfrak{g}_{2\gamma})\right)$.

Proof. The argument given in the proof of the previous lemma can be applied here to show that $\operatorname{ad}((1+\theta)\xi)\left(\mathfrak{n} \ominus \left(\bigoplus_{\gamma\in\Phi}(\mathfrak{g}_\gamma \oplus \mathfrak{g}_{2\gamma})\right)\right) \subset \mathfrak{n} \ominus \left(\bigoplus_{\gamma\in\Phi}(\mathfrak{g}_\gamma \oplus \mathfrak{g}_{2\gamma})\right)$. Since $\operatorname{Ad}(\phi) = \sum_{m=1}^{\infty} \frac{1}{m!}(\frac{\pi}{\sqrt{2}|\alpha|})^m \operatorname{ad}((1+\theta)\xi)^m$, the result follows. $\square$

Finally, let $X \in \mathfrak{g}_\lambda$ with $\lambda \in \Sigma^+ \setminus \Phi$. Then, using the previous two lemmas, $\mathrm{Ad}(\phi)H_\alpha = -H_\alpha$ and $\mathrm{Ad}(\phi)((1-\theta)E_\alpha) = -(1-\theta)E_\alpha$ we get

$$\begin{aligned} A_\xi \,\mathrm{Ad}(\phi)X &= \left[\frac{1-\theta}{2}\xi, \mathrm{Ad}(\phi)X\right] = \frac{1}{2}\mathrm{Ad}(\phi)[\mathrm{Ad}(\phi^{-1})(1-\theta)\xi, X] \\ &= \frac{1}{2}\mathrm{Ad}(\phi)\left[\mathrm{Ad}(\phi^{-1})\left(2a_\alpha H_\alpha + (1-\theta)E_\alpha\right), X\right] \\ &= \frac{1}{2}\mathrm{Ad}(\phi)\left[-2a_\alpha H_\alpha - (1-\theta)E_{i_j}, X\right] \\ &= \mathrm{Ad}(\phi)\left[-\frac{1-\theta}{2}\xi_{i_j}, X\right] = -\mathrm{Ad}(\phi)A_\xi X. \end{aligned}$$

In particular this implies $\mathrm{tr}A_\xi = a_\alpha|\alpha|^2(\dim\mathfrak{g}_\alpha + 2\dim\mathfrak{g}_{2\alpha})$.

We summarize all these calculations in the following

Proposition 6.4. *Let $S_{\Phi,V,a}$ be the connected subgroup of G whose Lie algebra is $\mathfrak{s}_{\Phi,V,a}$. Let us write $X_\alpha = \frac{1}{|\alpha|^2}H_\alpha - a_\alpha E_\alpha$ and denote by A_ξ the shape operator of $S_{\Phi,V,a}\cdot o$ with respect to a normal vector $\xi \in (\mathfrak{a}\oplus\mathfrak{n})\ominus\mathfrak{s}_{\Phi,V,a}$. We have:*

1. *If $\xi \in \mathfrak{a}_\Phi \ominus V$, then the restriction of A_ξ to $V \oplus \left(\bigoplus_{\gamma\in\Phi}\mathbb{R}X_\gamma\right)\oplus\left(\bigoplus_{\gamma\in\Phi}(\mathfrak{g}_\gamma\ominus\mathbb{R}E_\gamma)\right)\oplus\left(\bigoplus_{\gamma\in\Phi}\mathfrak{g}_{2\gamma}\right)$ is zero and the restriction of A_ξ to $\mathfrak{g}_\lambda$ for $\lambda\in\Sigma^+\setminus(\Phi\cup 2\Phi)$ is $\lambda(\xi)1_{\mathfrak{g}_\lambda}$.*
2. *If $\xi = a_\alpha H_\alpha + E_\alpha$ we have:*
   (a) *The restriction of A_ξ to $V \oplus \left(\bigoplus_{\gamma\in\Phi\setminus\{\alpha\}}\mathbb{R}X_\gamma\right)\oplus\left(\bigoplus_{\gamma\in\Phi\setminus\{\alpha\}}(\mathfrak{g}_\gamma\ominus\mathbb{R}E_\gamma)\right)\oplus\left(\bigoplus_{\gamma\in\Phi\setminus\{\alpha\}}\mathfrak{g}_{2\gamma}\right)$ is zero.*
   (b) *$A_\xi X_\alpha = a_\alpha|\alpha|^2 X_\alpha$.*
   (c) *We can decompose $\mathfrak{g}_\alpha$ as $\mathfrak{g}_\alpha = \mathrm{Ker}(\mathrm{ad}(E_\alpha)|\mathfrak{g}_\alpha)\oplus\mathrm{ad}(\theta E_\alpha)(\mathfrak{g}_{2\alpha})$. The restriction of A_ξ to $\mathrm{Ker}(\mathrm{ad}(E_\alpha)|\mathfrak{g}_\alpha)\ominus\mathbb{R}E_\alpha$ is $a_\alpha|\alpha|^2 1_{\mathrm{Ker}(\mathrm{ad}(E_\alpha)|\mathfrak{g}_\alpha)\ominus\mathbb{R}E_\alpha}$ and the dimension of $\mathrm{Ker}(\mathrm{ad}(E_\alpha)|\mathfrak{g}_\alpha)\ominus\mathbb{R}E_\alpha$ is $\dim\mathfrak{g}_\alpha - \dim\mathfrak{g}_{2\alpha} - 1$. The subspace $\mathrm{ad}(\theta E_\alpha)(\mathfrak{g}_{2\alpha})\oplus\mathfrak{g}_{2\alpha}$ is invariant under A_ξ and A_ξ acts with eigenvalues*
   $$\frac{|\alpha|}{2}\left(3a_\alpha|\alpha| \pm \sqrt{2 + a_\alpha^2|\alpha|^2}\right)$$
   whose multiplicities are $\dim\mathfrak{g}_{2\alpha}$.
   (d) *If $\phi = \mathrm{Exp}\left(\frac{\pi}{\sqrt{2}|\alpha|}(1+\theta)E_\alpha\right) \in N_K(\mathfrak{a})$ then $\mathfrak{n}\ominus\left(\bigoplus_{\gamma\in\Phi}(\mathfrak{g}_\gamma\oplus\mathfrak{g}_{2\gamma})\right)$ is invariant by A_ξ and $\mathrm{Ad}(\phi)$, and $A_\xi\,\mathrm{Ad}(\phi) = -\mathrm{Ad}(\phi)A_\xi$. In particular, if $A_\xi X = cX$, then $A_\xi\,\mathrm{Ad}(\phi)X = -c\,\mathrm{Ad}(\phi)X$.*

The mean curvature vector $\mathcal{H}$, defined with respect to an orthonormal basis $\{e_i\}$ as $\mathcal{H} = \sum_i II(e_i, e_i)$, is in our case

$$\mathcal{H} = 2\pi_{\mathfrak{a}_\Phi\ominus V}(H_\delta) - \sum_{\alpha\in\Phi} a_\alpha|\alpha|^2(\dim\mathfrak{g}_\alpha + 2\dim\mathfrak{g}_{2\alpha})(a_\alpha H_\alpha + E_\alpha),$$

where as usual $\pi_{\mathfrak{a}_\Phi\ominus V}$ denotes orthogonal projection onto $\mathfrak{a}_\Phi\ominus V$.

We recall that the horocycle foliation is induced by the group N, the action of the nilpotent part of some Iwasawa decomposition. In this case we have:

Corollary 6.5. *The orbits of the horocycle foliation are isometrically congruent to each other and their shape operator with respect to a vector $\xi \in \mathfrak{a}$ is given by $A_\xi = \operatorname{ad}(\xi)_{|\mathfrak{n}} = \bigoplus_{\lambda \in \Sigma^+} \lambda(\xi) 1_{\mathfrak{g}_\lambda}$.*

Remark 6.6. Let M be a symmetric space of rank one. Assume that $\Lambda = \{\alpha\}$. There are up to congruency two possible hyperpolar homogeneous foliations, namely, the horosphere foliation, which is the same as the horocycle foliation in this case, and the solvable foliation, where $\Phi = \{\alpha\}$. In both cases the foliation is by homogeneous hypersurfaces.

All the leaves of the horosphere foliation are congruent. The principal curvatures of horospheres are $|\alpha|$ and $2|\alpha|$ with multiplicities $\dim \mathfrak{g}_\alpha$ and $\dim \mathfrak{g}_{2\alpha}$ respectively.

Now we consider the solvable foliation, whose leaves are the orbits of the group $S_{\{\alpha\},\{0\}}$. If γ is a geodesic parametrized by unit speed such $\gamma(0) = o$ and $\dot\gamma(0)$ is orthogonal to $S_{\{\alpha\},\{0\}} \cdot o$, then the path of γ is a section of this hyperpolar foliation. The principal curvatures of the orbit $S_{\{\alpha\},\{0\}} \cdot \gamma(r)$ are

$$-|\alpha| \tanh(|\alpha| r), \quad -\frac{3|\alpha|}{2} \tanh(|\alpha| r) \pm \frac{|\alpha|}{2}\sqrt{2 - \tanh(|\alpha| r)},$$

with multiplicities $\dim \mathfrak{g}_\alpha$, $\dim \mathfrak{g}_{2\alpha}$ and $\dim \mathfrak{g}_{2\alpha}$, respectively.

For cohomogeneity one homogeneous foliations see [4].

References

[1] J. Berndt: Homogeneous hypersurfaces in hyperbolic spaces, *Math. Z.* **229** (1998), 589–600.
[2] J. Berndt, S. Console, C. Olmos: *Submanifolds and holonomy*, Chapman & Hall/CRC, Boca Raton, 2003.
[3] J. Berndt, J.C. Díaz-Ramos: Real hypersurfaces with constant principal curvatures in complex hyperbolic spaces, *J. London Math. Soc., II. Ser.* **74** (2006), 778–798.
[4] J. Berndt, H. Tamaru: Homogeneous codimension one foliations on noncompact symmetric spaces, *J. Differential Geom.* **63** (2003), no. 1, 1–40.
[5] J. Berndt, H. Tamaru: Cohomogeneity one actions on noncompact symmetric spaces with a totally geodesic singular orbit, *Tohoku Math. J.* (2) **56** (2004), no. 2, 163–177.
[6] A. Borel, L. Ji: *Compactifications of symmetric and locally symmetric spaces*, Birkhäuser, Boston, 2006.
[7] J. Dadok: Polar coordinates induced by actions of compact Lie groups, *Trans. Amer. Math. Soc.* **288** (1985), 125–137.
[8] J.C. Díaz-Ramos: Proper isometric actions, preprint arXiv:math.DG/0811.0547.
[9] P.B. Eberlein: *Geometry of nonpositively curved manifolds*, University of Chicago Press, Chicago, London, 1996.
[10] D. Gromoll, G. Walschap: *Metric foliations and curvature*, Birkhäuser, Boston, Basel, 2008.
[11] E. Heintze, X. Liu, C. Olmos: Isoparametric submanifolds and a Chevalley-type restriction theorem, in: *Integrable systems, geometry, and topology* (Ed. C.L. Terng). Providence, RI: American Mathematical Society (AMS). Somerville, MA: International Press. AMS/IP Studies in Advanced Mathematics 36, 151–234 (2006).
[12] E. Heintze, R. Palais, C.L. Terng, G. Thorbergsson: Hyperpolar actions on symmetric spaces. *Geometry, topology, & physics*, 214–245, Conf. Proc. Lecture Notes Geom. Topology, IV, Int. Press, Cambridge, MA, 1995.

[13] S. Helgason: *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, **80**, Academic Press, Inc., New York-London, 1978.
[14] S. Kaneyuki, H. Asano: Graded Lie algebras and generalized Jordan Lie triple systems, *Nagoya Math. J.* **112** (1988), 81–115.
[15] A.W. Knapp: *Lie groups beyond an introduction*, Birkhäusser, Boston MA, 1996.
[16] T. Kobayashi: Multiplicity-free representations and visible actions on complex manifolds, *Publ. Res. Inst. Math. Sci.* **41** (2005), 497–549.
[17] N. Koike: Examples of a complex hyperpolar action without singular orbit, arXiv:0807.1609v1 [math.DG]
[18] A, Kollross: A classification of hyperpolar and cohomogeneity one actions, *Trans. Amer. Math. Soc.* **354** (2002), 571–612.
[19] A. Kollross: Polar actions on symmetric spaces, *J. Differential Geom.* **77** (2007), no. 3, 425–482.
[20] G.D. Mostow: On maximal subgroups of real Lie groups, *Ann. of Math.* (2) **74** (1961), 503–517.
[21] A.L. Onishchik (Ed.): *Lie groups and Lie algebras. I. Foundations of Lie theory. Lie transformation groups*, A translation of Current problems in mathematics. Fundamental directions, Vol. 20 (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1988. Translation by A. Kozlowski. Translation edited by A.L. Onishchik. Encyclopaedia of Mathematical Sciences, 20, Springer-Verlag, Berlin, 1993.
[22] A.L. Onishchik, È. B. Vinberg (Eds.): *Lie groups and Lie algebras III. Structure of Lie groups and Lie algebras*, A translation of Current problems in mathematics, Fundamental directions, Vol. 41 (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1990. Translation by V. Minachin, Translation edited by A.L. Onishchik and È.B. Vinberg, Encyclopaedia of Mathematical Sciences, 41, Springer-Verlag, Berlin, 1994.
[23] F. Podestà, G. Thorbergsson: Polar actions on rank-one symmetric spaces, *J. Differential Geom.* **53** (1999), 131–175.

J Berndt: Department of Mathematics, University College Cork, Ireland.
Email: `j.berndt@ucc.ie`

JC Díaz Ramos: Department of Mathematics, University College Cork, Ireland.
Email: `jc.diazramos@ucc.ie`

H Tamaru: Department of Mathematics, Hiroshima University, Japan.
Email: `tamaru@math.sci.hiroshima-u.ac.jp`