THE STRUCTURE OF ALGEBRAIC COVARIANT DERIVATIVE CURVATURE TENSORS

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Abstract. We use the Nash embedding theorem to construct generators for the space of algebraic covariant derivative curvature tensors.

1. Introduction

Let M be an m dimensional Riemannian manifold. To a large extent, the geometry of M is the study of the Riemannian curvature $R \in \otimes^4 T^*M$ which is defined by the Levi-Civita connection ∇ and, to a lesser extent, the study of the covariant derivative ∇R. For example, M is a local symmetric space if and only if $\nabla R = 0$; note that local symmetric spaces are locally homogeneous.

It is convenient to work in the algebraic context. Let V be an m-dimensional real vector space. Let $\mathcal{A}(V) \subset \otimes^4 V^*$ and $\mathcal{A}_1(V) \subset \otimes^5 V^*$ be the spaces of all algebraic curvature tensors and all algebraic covariant derivative tensors, respectively, i.e. those tensors A and A_1 having the symmetries of R and of ∇R:

$$\begin{aligned}
&A(x,y,z,w) = A(z,w,x,y) = -A(y,x,z,w),\\
&A(x,y,z,w) + A(y,z,x,w) + A(z,x,y,w) = 0,\\
&A_1(x,y,z,w;v) = A_1(z,w,x,y;v) = -A_1(y,x,z,w;v),\\
&A_1(x,y,z,w;v) + A_1(y,z,x,w;v) + A_1(z,x,y,w;v) = 0,\\
&A_1(x,y,z,w;v) + A_1(x,y,w,v;z) + A_1(x,y,v,z;w) = 0\,.
\end{aligned}$$

Let $S^p(V) \subset \otimes^p V^*$ be the space of totally symmetric p forms. If $\Psi \in S^2(V)$ and if $\Psi_1 \in S^3(V)$, define $A_\Psi \in \mathcal{A}(V)$ and $A_{1,\Psi,\Psi_1} \in \mathcal{A}_1(V)$ by:

$$\begin{aligned}
A_\Psi(x,y,z,w): &= \Psi(x,w)\Psi(y,z) - \Psi(x,z)\Psi(y,w),\\
A_{1,\Psi,\Psi_1}(x,y,z,w;v): &= \Psi_1(x,w,v)\Psi(y,z) + \Psi(x,w)\Psi_1(y,z,v)\\
&- \Psi_1(x,z,v)\Psi(y,w) - \Psi(x,z)\Psi_1(y,w,v)\,.
\end{aligned}$$

If one thinks of Ψ_1 as the symmetrized covariant derivative of Ψ, then A_{1,Ψ,Ψ_1} can be regarded, at least formally speaking, as the covariant derivative of A_Ψ.

Fiedler [6, 7] used group representation theory to show:

Theorem 1.1 (Fiedler)**.**
(1) $\mathcal{A}(V) = \mathrm{Span}_{\Psi \in S^2(V)}\{A_\Psi\}$.
(2) $\mathcal{A}_1(V) = \mathrm{Span}_{\Psi \in S^2(V), \Psi_1 \in S^3(V)}\{A_{1,\Psi,\Psi_1}\}$.

Let $A \in \mathcal{A}(V)$ and $A_1 \in \mathcal{A}_1(V)$ be given. Choose $\nu(A)$ and $\nu_1(A_1)$ minimal so that there exist $\Psi_i \in S^2(V)$, $\tilde\Psi_j \in S^2(V)$, $\tilde\Psi_{1,j} \in S^3(V)$, and constants $\lambda_i, \lambda_{1,j}$ so:

$$A = \textstyle\sum_{1\le i\le\nu(A)} \lambda_i A_{\Psi_i} \quad \text{and} \quad A_1 = \sum_{1\le j\le\nu_1(A_1)} \lambda_{1,j} A_{1,\tilde\Psi_j,\tilde\Psi_{1,j}}\,.$$

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Set

$$\nu(m):=\sup_{A\in\mathcal{A}(V)}\nu(A)\quad\text{and}\quad\nu_1(m):=\sup_{A_1\in\mathcal{A}_1(V)}\nu_1(A_1)\,.$$

The main result of this paper is the following:

Theorem 1.2. *Let $m\ge 2$.*

(1) $\frac{1}{2}m\le\nu(m)$ *and* $\frac{1}{2}m\le\nu_1(m)$.
(2) $\nu(m)\le\frac{1}{2}m(m+1)$ *and* $\nu_1(m)\le\frac{1}{2}m(m+1)$.

We shall establish the lower bounds of Assertion (1) in Section 2. The upper bound given in Assertion (2) for $\nu(m)$ is due to Díaz-Ramos and García-Río [4] who used the Nash embedding theorem [17]; they also gave a separate argument to show $\nu(2)=1$ and $\nu(3)=2$. In Section 3, we shall generalize their approach to establish the following simultaneous 'diagonalization' result from which Theorem 1.2 (2) will follow as a Corollary:

Theorem 1.3. *Let V be an m dimensional vector space. Let $A\in\mathcal{A}(V)$ and let $A_1\in\mathcal{A}_1(V)$ be given. There exists $\Psi_i\in S^2(V)$ and $\Psi_{1,i}\in S^3(V)$ so that*

$$A=\textstyle\sum_{1\le i\le\frac{1}{2}m(m+1)}A_{\Psi_i}\quad\textit{and}\quad A_1=\sum_{1\le i\le\frac{1}{2}m(m+1)}A_{1,\Psi_i,\Psi_{1,i}}\,.$$

The study of the tensors A_Ψ arose in the original instance from the Osserman conjecture and related matters; we refer to [9, 11] for a more extensive discussion than is possible here, and content ourselves with only a very brief introduction to the subject.

1.1. **The Jacobi operator.** If M is a pseudo-Riemannian manifold of signature (p,q) and dimension $m=p+q$, let $S^+(M)$ (resp. $S^-(M)$) be the bundle of unit spacelike (resp. timelike) tangent vectors. The Jacobi operator $J(x)$ for $x\in TM$ is the self-adjoint endomorphism of TM characterized by the identity:

$$g(J(x)y,z)=R(y,x,x,z)\,.$$

One says that M is *spacelike Osserman* (resp. *timelike Osserman*) if the eigenvalues of $J(\cdot)$ are constant on $S^+(M)$ (resp. $S^-(M)$). It turns out these two notions are equivalent and such a manifold is simply said to be *Osserman*.

Restrict for the moment to the Riemannian setting ($p=0$). If M is a local rank 1 symmetric space or is flat, then the local isometries of M act transitively on the sphere bundle $S(M)=S^+(M)$ and hence the eigenvalues of $J(\cdot)$ are constant on $S(M)$ and M is Osserman. Osserman [22] wondered if the converse held; this question has been called the Osserman conjecture by subsequent authors. The conjecture has been answered in the affirmative if $m\ne 16$ by work of Chi [3] and Nikolayevsky [18, 19, 20].

In the Lorentzian setting ($p=1$), an Osserman manifold has constant sectional curvature [2, 8]. In the higher signature setting ($p>1$, $q>1$) it is more natural to work with the Jordan normal form rather than just the eigenvalue structure. One says that M is *spacelike Jordan Osserman* (resp. *timelike Jordan Osserman*) if the Jordan normal form of $J(\cdot)$ is constant on $S^+(M)$ (resp. $S^-(M)$); these two notions are not equivalent. The following example is instructive. Let $(\vec{x},\vec{y})$ for $\vec{x}=(x_1,...,x_p)$ and $\vec{y}=(y_1,...,y_p)$ be coordinates on $\mathbb{R}^{2p}$ where $p\ge 3$. Let $f=f(\vec{x})\in C^\infty(\mathbb{R}^p)$. Define a pseudo-Riemannian metric g_f of signature (p,p) on $\mathbb{R}^{2p}$ by setting

$$\text{(1.a)}\quad g_f(\partial_i^x,\partial_j^x)=\partial_i^x f\cdot\partial_j^x f,\ \ g_f(\partial_i^y,\partial_j^y)=0,\ \text{ and }\ g_f(\partial_i^x,\partial_j^y)=g_f(\partial_j^y,\partial_i^x)=\delta_{ij}\,.$$

Let Ψ be the Euclidean Hessian:

$$\Psi(\partial_i^x,\partial_j^x)=\partial_i^x\partial_j^x f,\quad\Psi(\partial_i^y,\partial_j^y)=0,\quad\text{and}\quad\Psi(\partial_i^x,\partial_j^y)=\Psi(\partial_j^y,\partial_i^x)=0\,.$$

One then has that $R = A_\Psi$. We suppose that the restriction of Ψ to $\operatorname{Span}\{\partial_i^x, \partial_j^x\}$ is positive definite henceforth. Then M is a complete pseudo-Riemannian manifold which is spacelike and timelike Jordan Osserman. Similarly set

$$\Psi_1(\partial_i^x, \partial_j^x, \partial_k^x) = \partial_i^x \partial_j^x \partial_k^x f$$

and extend Ψ_1 to vanish if any entry is ∂_ℓ^y. One has $\nabla R = A_{1,\Psi,\Psi_1}$; thus if f is not quadratic, M is not a local symmetric space. With a bit more work one can show that for generic such f, M is curvature homogeneous but not locally affine homogeneous. We refer to [5, 14] for further details.

1.2. **The skew-symmetric curvature operator.** Let $\{e_1, e_2\}$ be an orthonormal basis for an oriented spacelike (resp. timelike) 2 plane π. The skew-symmetric curvature operator $\mathcal{R}(\pi)$ is characterized by the identity

$$g(\mathcal{R}(\pi)y, z) = R(e_1, e_2, y, z)\,;$$

it is independent of the particular orthonormal basis chosen. One says that M is *spacelike Ivanov-Petrova* (resp. *timelike Ivanov-Petrova*) if the eigenvalues of $\mathcal{R}(\cdot)$ are constant on the Grassmannian of oriented spacelike (resp. timelike) 2-planes; these two notions are equivalent and such a manifold is simply said to be Ivanov-Petrova. The notions *spacelike Jordan Ivanov-Petrova* and *timelike Jordan Ivanov-Petrova* are defined similarly and are not equivalent.

The Riemannian Ivanov-Petrova manifolds have been classified [10, 13, 21]; they have also been classified in the Lorentzian setting [24] if $m \geq 10$. For all these manifolds, the curvature tensors have the form $R = A_\Psi$ where Ψ is an idempotent isometry and $\mathcal{R}(\pi)$ always has rank 2. Conversely, in the algebraic setting, if R is a spacelike Jordan Ivanov-Petrova algebraic curvature tensor on a vector space of signature (p, q) where $q \geq 5$ and where $\operatorname{Rank}\{\mathcal{R}(\cdot)\} = 2$, then there exist λ and Ψ so that $R = \lambda A_\Psi$. This once again motivates the study of these tensors. Unfortunately, the situation in the indefinite setting is again quite different. There exist spacelike Ivanov-Petrova manifolds of signature $(s, 2s)$ where $\mathcal{R}(\pi)$ has rank 4 and where the curvature tensor does not have the form $R = A_\Psi$. We refer to [15] for further details.

1.3. **The Szabó operator.** There is an analogous operator to the Jacobi operator which is defined by ∇R. The Szabó operator $J_1(x)$ is the self-adjoint endomorphism of TM characterized by $g(J_1(x)y, z) = \nabla R(y, x, x, z; x)$. One says that M is *spacelike Szabó* (resp. *timelike Szabó*) if the eigenvalues of $J_1(\cdot)$ are constant on $S^+(M)$ (resp. $S^-(M)$); these notions are equivalent and such a manifold is simply said to be Szabó. The notion *spacelike* (resp. *timelike*) *Jordan Szabó* is defined similarly.

In his study of 2 point symmetric spaces, Szabó [23] gave a very lovely topological argument showing that any Riemannian Szabó manifold is necessarily a local symmetric space – i.e. $\nabla R = 0$. This result was subsequently extended to the Lorentzian case [16]. In the higher signature setting, again the situation is unclear. The metric g_f described in Display (1.a) defines a Szabó pseudo-Riemannian manifolds of signature (p, p).

Even in the algebraic setting, there are no known non-zero elements $A_1 \in \mathcal{A}(V)$ which are spacelike Jordan Szabó. It has been shown [12] that if A_1 is a spacelike Jordan Szabó algebraic covariant derivative curvature tensor on a vector space of signature (p, q), where $q \equiv 1 \bmod 2$ and $p < q$ or where $q \equiv 2 \bmod 4$ and $p < q - 1$, then $A_1 = 0$. This algebraic result yields an elementary proof of the geometrical fact that any pointwise totally isotropic pseudo-Riemannian manifold with such a signature (p, q) is locally symmetric. The general question of finding non-trivial spacelike Jordan Szabó covariant algebraic curvature tensors, or conversely showing non exist, remains open.

The examples discussed above motivate consideration of the tensors A_{1,Ψ,Ψ_1} and more generally of tensors which are combinations of these. We hope that Theorems 1.2 and 1.3, although of interest in their own right, will play a central role in these investigations.

2. A lower bound for $\nu(m)$ and for $\nu_1(m)$

Let V be an m dimensional vector space, let $A \in \mathcal{A}(V)$, and let $A_1 \in \mathcal{A}_1(V)$. Give V a positive definite inner product $\langle\cdot,\cdot\rangle$. The associated curvature operators are then defined by the identities:

$$\begin{aligned}&\langle\mathcal{R}_A(\xi_1,\xi_2)z,w\rangle = A(\xi_1,\xi_2,z,w), \quad \text{and}\\ &\langle\mathcal{R}_{A_1}(\xi_1,\xi_2,\xi_3)z,w\rangle = A_1(\xi_1,\xi_2,z,w;\xi_3)\,.\end{aligned}$$

Theorem 1.2 (1) will follow from the following Lemma:

Lemma 2.1. *Let V be a vector space of dimension $m = 2\bar{m}$ or $m = 2\bar{m}+1$.*

(1) *If $\Psi \in S^2(V)$ and if $\Psi_1 \in S^3(V)$, then for any $\xi_1,\xi_2,\xi_3 \in V$ one has:*
$$\operatorname{Rank}\{\mathcal{R}_{A_\Psi}(\xi_1,\xi_2)\} \le 2 \quad \textit{and} \quad \operatorname{Rank}\{\mathcal{R}_{A_{1,\Psi,\Psi_1}}(\xi_1,\xi_2,\xi_3)\} \le 2\,.$$

(2) *If $A \in \mathcal{A}(V)$ and $A_1 \in \mathcal{A}_1(V)$, then for any $\xi_1,\xi_2,\xi_3 \in V$ one has:*
$$\operatorname{Rank}\{\mathcal{R}_A(\xi_1,\xi_2)\} \le 2\nu(A) \quad \textit{and} \quad \operatorname{Rank}\{\mathcal{R}_{A_1}(\xi_1,\xi_2,\xi_3)\} \le 2\nu_1(A_1)\,.$$

(3) *There exist $A \in \mathcal{A}(V)$, $A_1 \in \mathcal{A}_1(V)$, and $\xi_1,\xi_2,\xi_3 \in V$ so:*
$$\operatorname{Rank}\{\mathcal{R}_A(\xi_1,\xi_2)\} = 2\bar{m} \quad \textit{and} \quad \operatorname{Rank}\{\mathcal{R}_{A_1}(\xi_1,\xi_2,\xi_1)\} = 2\bar{m}\,.$$

Proof. If $\Psi \in S^2(V)$ and $\Psi_1 \in S^3(V)$, let ψ and $\psi_1(\cdot)$ be the associated self-adjoint endomorphisms characterized by the identities

$$\langle\psi x,y\rangle = \Psi(x,y) \quad \text{and} \quad \langle\psi_1(z)x,y\rangle = \Psi_1(x,y,z)\,.$$

Assertion (1) follows from the expression:

$$\begin{aligned}\mathcal{R}_{A_\Psi}(\xi_1,\xi_2)y &= \{\Psi(\xi_2,y)\psi\}\xi_1 - \{\Psi(\xi_1,y)\psi\}\xi_2, \quad \text{and}\\ \mathcal{R}_{A_{1,\Psi,\Psi_1}}(\xi_1,\xi_2,\xi_3)y &= \{\Psi(\xi_2,y)\psi_1(\xi_3) + \Psi_1(\xi_2,y,\xi_3)\psi\}\xi_1\\ &- \{\Psi(\xi_1,y)\psi_1(\xi_3) + \Psi_1(\xi_1,y,\xi_3)\psi\}\xi_2\,.\end{aligned}$$

Let $A_i := A_{\Psi_i}$, $A_{1,j} := A_{1,\tilde{\Psi}_j,\tilde{\Psi}_{1,j}}$, $\mathcal{R}_i := \mathcal{R}_{A_i}$, and $\mathcal{R}_{1,i} := \mathcal{R}_{A_{1,i}}$. Set

$$A = \textstyle\sum_{1\le i\le\nu(A)} A_i \quad \text{and} \quad A_1 = \sum_{1\le j\le\nu_1(A_1)} A_{1,j}\,.$$

Assertion (2) follows from Assertion (1) as

$$\begin{aligned}\operatorname{Rank}\{\mathcal{R}_A(\cdot)\} &= \operatorname{Rank}\{\textstyle\sum_{1\le i\le\nu(A)}\mathcal{R}_i(\cdot)\}\\ &\le \textstyle\sum_{1\le i\le\nu(A)}\operatorname{Rank}\{\mathcal{R}_i(\cdot)\} \le 2\nu(A),\\ \operatorname{Rank}\{\mathcal{R}_{A_1}(\cdot)\} &= \operatorname{Rank}\{\textstyle\sum_{1\le j\le\nu_1(A_1)}\mathcal{R}_{1,j}(\cdot)\}\\ &\le \textstyle\sum_{1\le j\le\nu_1(A_1)}\operatorname{Rank}\{\mathcal{R}_{1,j}(\cdot)\} \le 2\nu_1(A_1)\,.\end{aligned}$$

If $\dim(V) = 2\bar{m}$, let $\{e_1,...,e_{\bar{m}},f_1,...,f_{\bar{m}}\}$ be an orthonormal basis for V; if $\dim(V)$ is odd, the argument is similar and we simply extend A and A_1 to be trivial on the additional basis vector. Define the non-zero components of $\Psi_i \in S^2(V)$ and $\Psi_{1,i} \in S^3(V)$ by:

$$\begin{aligned}&\Psi_i(e_j,e_k) = \Psi_i(f_j,f_k) = \delta_{ij}\delta_{ik},\\ &\Psi_{1,i}(e_j,e_k,e_l) = \Psi_{1,i}(f_j,f_k,f_l) = \delta_{ij}\delta_{ik}\delta_{il};\end{aligned}$$

$\Psi_i(\cdot,\cdot)$ and $\Psi_{1,i}(\cdot,\cdot,\cdot)$ vanish if both an 'e' and an 'f' appear. Let

$$\begin{aligned}&A_i := A_{\Psi_i}, \quad \mathcal{R}_i := \mathcal{R}_{A_i}, \quad A_{1,i} := A_{1,\Psi_i,\Psi_{1,i}}, \quad \mathcal{R}_{1,i} := \mathcal{R}_{A_{1,i}},\\ &A := \textstyle\sum_{1\le i\le\bar{m}} A_i, \quad A_1 := \sum_{1\le i\le\bar{m}} A_{1,i},\\ &\xi_1 := e_1 + ... + e_{\bar{m}}, \quad \xi_2 := f_1 + ... + f_{\bar{m}}, \quad \xi_3 := \xi_1 + \xi_2\,.\end{aligned}$$

We may then complete the proof of Assertion (3) by computing:

$$\begin{aligned}
&\mathcal{R}_A(\xi_1,\xi_2)e_i=\mathcal{R}_i(e_i,f_i)e_i=-f_i,\\
&\mathcal{R}_A(\xi_1,\xi_2)f_i=\mathcal{R}_i(e_i,f_i)f_i=e_i,\\
&\mathcal{R}_{A_1}(\xi_1,\xi_2,\xi_3)e_i=\mathcal{R}_{1,i}(e_i,f_i,e_i+f_i)e_i=-2f_i\\
&\mathcal{R}_{A_1}(\xi_1,\xi_2,\xi_3)f_i=\mathcal{R}_{1,i}(e_i,f_i,e_i+f_i)f_i=2e_i\,. \quad\square
\end{aligned}$$

3. Geometric realizability

Henceforth, let $\langle\cdot,\cdot\rangle$ be a non-singular innerproduct on an m dimensional vector space V, let $A\in\mathcal{A}(V)$ and let $A_1\in\mathcal{A}(V)$.

Although the following is well-known, see for example Belger and Kowalski [1] where a more general result is established, we shall give the proof to keep the development as self-contained as possible and to establish notation needed subsequently.

Lemma 3.1.

(1) *If g is a pseudo-Riemannian metric on $\mathbb{R}^m$ with $\partial_i g_{jk}(0)=0$, then:*
  (a) $R_{ijkl}(0)=\frac12\{\partial_i\partial_k g_{jl}+\partial_j\partial_l g_{ik}-\partial_i\partial_l g_{jk}-\partial_j\partial_k g_{il}\}(0)$.
  (b) $R_{ijkl;n}(0)=\frac12\{\partial_i\partial_k\partial_n g_{jl}+\partial_j\partial_l\partial_n g_{ik}-\partial_i\partial_l\partial_n g_{jk}-\partial_j\partial_k\partial_n g_{il}\}(0)$.

(2) *There exists the germ of a pseudo-Riemannian metric g on $(\mathbb{R}^m,0)$ and an isomorphism Ξ from $T_0(\mathbb{R}^m)$ to V so that*
  (a) $\Xi^*\langle\cdot,\cdot\rangle=g|_{T_0(\mathbb{R}^m)}$.
  (b) $\Xi^*A=R_g|_{T_0(\mathbb{R}^m)}$.
  (c) $\Xi^*A_1=\nabla R_g|_{T_0(\mathbb{R}^m)}$.

Proof. Since the 1 jets of the metric vanish at the origin, we have

$$\begin{aligned}
&\Gamma_{ijk}:=g(\nabla_{\partial_i}\partial_j,\partial_k)=\tfrac12(\partial_i g_{jk}+\partial_j g_{ik}-\partial_k g_{ij})=O(|x|),\\
&R_{ijkl}(0)=\{\partial_i\Gamma_{jkl}-\partial_j\Gamma_{ikl}\}(0),\quad\text{and}\quad R_{ijkl;n}(0)=\{\partial_n R_{ijkl}\}(0)\,.
\end{aligned}$$

Assertion (1) now follows; see, for example, [11] [cf Lemma 1.11.1] for further details. To prove the second assertion, choose an orthonormal basis $\{e_1,...,e_m\}$ for V so that $\langle e_i,e_j\rangle=\pm\delta_{ij}$; we use this orthonormal basis to identify $V=\mathbb{R}^m$. Let A_{ijkl} and $A_{1,ijkl;n}$ denote the components of A and of A_1, respectively. Define

$$g_{ik}\quad=\quad\langle e_i,e_k\rangle-\tfrac13\textstyle\sum_{jl}A_{ijlk}x_jx_l-\tfrac16\sum_{jln}A_{1,ijlk;n}x_jx_lx_n\,.$$

Clearly $g_{ik}=g_{ki}$. As $g|T_0\mathbb{R}^m=\langle\cdot,\cdot\rangle$, g is non-degenerate on some neighborhood of 0. Since the 1 jets of the metric vanish at 0 we have by Assertion (1) that

$$\begin{aligned}
&R_{ijkl}(0)\\
=\;&\tfrac16\{-A_{jikl}-A_{jkil}-A_{ijlk}-A_{iljk}+A_{jilk}+A_{jlik}+A_{ijkl}+A_{ikjl}\}\\
=\;&\tfrac16\{4A_{ijkl}-2A_{iljk}-2A_{iklj}\}=A_{ijkl},\\
&R_{ijkl;n}(0)\\
=\;&\tfrac1{12}\{-A_{jikl;n}-A_{jkil;n}-A_{jnkl;i}-A_{jknl;i}-A_{jinl;k}-A_{jnil;k}\\
-\;&A_{ijlk;n}-A_{iljk;n}-A_{inlk;j}-A_{ilnk;j}-A_{ijnk;l}-A_{injk;l}\\
+\;&A_{jilk;n}+A_{jlik;n}+A_{jnlk;i}+A_{jlnk;i}+A_{jink;l}+A_{jnik;l}\\
+\;&A_{ijkl;n}+A_{ikjl;n}+A_{inkl;j}+A_{iknl;j}+A_{ijnl;k}+A_{injl;k}\}\\
=\;&\tfrac1{12}\{(4A_{ijkl;n}-2A_{jkil;n}+2A_{jlik;n})+(-2A_{jnkl;i}-2A_{inlk;j})\\
+\;&(-2A_{jinl;k}-2A_{ijnk;l})+(-A_{ilnk;j}-A_{jnil;k})\\
+\;&(-A_{injk;l}-A_{jknl;i})+(A_{jlnk;i}+A_{injl;k})+(A_{jnik;l}+A_{iknl;j})\}\\
=\;&\tfrac1{12}\{6A_{ijkl;n}+2A_{ijkl;n}+2A_{ijkl;n}+A_{ilkj;n}+A_{jkli;n}-A_{jlki;n}-A_{iklj;n}\}\\
=\;&\tfrac1{12}\{10A_{ijkl;n}+2A_{ilkj;n}+2A_{ikjl;n}\}=\tfrac1{12}\{10A_{ijkl;n}-2A_{ijlk;n}\}=A_{ijkl;n}\,.\quad\square
\end{aligned}$$

We suppose the inner product $\langle\cdot,\cdot\rangle$ is positive definite henceforth. We apply the Nash embedding theorem [17] to find an embedding $f:\mathbb{R}^m\to\mathbb{R}^{m+\kappa}$ realizing the metric g constructed in Lemma 3.1. By writing the submanifold as a graph over its tangent plane, we can choose coordinates (x,y) on $\mathbb{R}^{m+\kappa}$ where $x=(x_1,...,x_m)$ and $y=(y_1,...,y_\kappa)$ so that

$$f(x)=(x,f_1(x),...,f_\kappa(x))\quad\text{where}\quad df_\nu(0)=0\quad\text{for}\quad 1\le\nu\le\kappa\,.$$

Since $f_*(\partial_i^x)=(0,...,1,...,0,\partial_i^x f_1,....,\partial_i^x f_\kappa)$, we have

$$g_{ij}(x)=\delta_{ij}+\textstyle\sum_{1\le\sigma\le\kappa}\partial_i^x f_\sigma\cdot\partial_j^x f_\sigma\,.$$

Let $\Psi_{ij}^\sigma:=\partial_i^x\partial_j^x f_\sigma(0)$ and $\Psi_{ijk}^\sigma:=\partial_i^x\partial_j^x\partial_k^x f_\sigma(0)$. As $dg_{ij}(0)=0$, by Lemma 3.1:

$$\begin{aligned}
&R_{ijkl}(0)\\
=\;&\tfrac12\textstyle\sum_{1\le\sigma\le\kappa}\{(\Psi_{ij}^\sigma\Psi_{kl}^\sigma+\Psi_{il}^\sigma\Psi_{kj}^\sigma)+(\Psi_{ji}^\sigma\Psi_{lk}^\sigma+\Psi_{jk}^\sigma\Psi_{li}^\sigma)\\
-\;&(\Psi_{ij}^\sigma\Psi_{lk}^\sigma+\Psi_{ik}^\sigma\Psi_{lj}^\sigma)-(\Psi_{ji}^\sigma\Psi_{kl}^\sigma+\Psi_{jl}^\sigma\Psi_{ki}^\sigma)\}\\
=\;&\textstyle\sum_{1\le\sigma\le\kappa}\{\Psi_{il}^\sigma\Psi_{jk}^\sigma-\Psi_{ik}^\sigma\Psi_{jl}^\sigma\}=\sum_{1\le\sigma\le\kappa}A_{\Psi^\sigma},\\
&R_{ijkl;n}(0)\\
=\;&\tfrac12\textstyle\sum_{1\le\sigma\le\kappa}\{(\Psi_{jin}^\sigma\Psi_{lk}^\sigma+\Psi_{jkn}^\sigma\Psi_{li}^\sigma+\Psi_{ji}^\sigma\Psi_{lkn}^\sigma+\Psi_{jk}^\sigma\Psi_{lin}^\sigma+\Psi_{jik}^\sigma\Psi_{ln}^\sigma+\Psi_{jn}^\sigma\Psi_{lik}^\sigma)\\
+\;&(\Psi_{ijn}^\sigma\Psi_{kl}^\sigma+\Psi_{iln}^\sigma\Psi_{kj}^\sigma+\Psi_{ij}^\sigma\Psi_{kln}^\sigma+\Psi_{il}^\sigma\Psi_{kjn}^\sigma+\Psi_{ijl}^\sigma\Psi_{kn}^\sigma+\Psi_{in}^\sigma\Psi_{kjl}^\sigma)\\
-\;&(\Psi_{jin}^\sigma\Psi_{kl}^\sigma+\Psi_{jln}^\sigma\Psi_{ki}^\sigma+\Psi_{ji}^\sigma\Psi_{kln}^\sigma+\Psi_{jl}^\sigma\Psi_{kin}^\sigma+\Psi_{jil}^\sigma\Psi_{kn}^\sigma+\Psi_{jn}^\sigma\Psi_{kil}^\sigma)\\
-\;&(\Psi_{ijn}^\sigma\Psi_{lk}^\sigma+\Psi_{ikn}^\sigma\Psi_{lj}^\sigma+\Psi_{ij}^\sigma\Psi_{lkn}^\sigma+\Psi_{ik}^\sigma\Psi_{ljn}^\sigma+\Psi_{ijk}^\sigma\Psi_{ln}^\sigma+\Psi_{in}^\sigma\Psi_{ljk}^\sigma)\\
=\;&\textstyle\sum_{1\le\sigma\le\kappa}\{\Psi_{iln}^\sigma\Psi_{jk}^\sigma+\Psi_{jkn}^\sigma\Psi_{il}^\sigma-\Psi_{ikn}^\sigma\Psi_{jl}^\sigma-\Psi_{ik}^\sigma\Psi_{jln}^\sigma\}=\sum_{1\le\sigma\le\kappa}A_{1,\Psi^\sigma,\Psi^\sigma}\,.
\end{aligned}$$

Consequently, $\nu(A)\le\kappa$ and $\nu(A_1)\le\kappa$. Theorem 1.3 follows from the Nash embedding theorem as in the analytic category we may take $\kappa\le\frac12 m(m+1)$. □

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Dedication

11 de Marzo de 2004 Madrid: En memoria de todas las víctimas inocentes. Todos íbamos en ese tren. (In memory of all these innocent victims. We were all on that train.)

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