Homogeneous CPC submanifolds in symmetric spaces of non-compact type

Víctor Sanmartín López

Department of Mathematics, Universidade de Santiago de Compostela

Joint work with Jürgen Berndt

Symmetry and shape

Celebrating the 60th birthday of Jürgen Berndt

Contents

Contents

Introduction

- $\bar{M}$ Riemannian manifold
- M submanifold of $\bar{M}$
- G Lie group

Homogeneous submanifold

$M \subset \bar{M}$ homogeneous $:\Leftrightarrow \forall p, q \in M$ there exists $g \in \mathrm{Isom}(\bar{M})$ such that:

- $g(M) = M$
- $g(p) = q$

Introduction

- $\bar{M}$ Riemannian manifold
- M submanifold of $\bar{M}$
- G Lie group

Homogeneous submanifold

$M \subset \bar{M}$ homogeneous $:\Leftrightarrow \forall p, q \in M$ there exists $g \in \mathrm{Isom}(\bar{M})$ such that:

- $g(M) = M$
- $g(p) = q$

Cohomogeneity one action

$G \times \bar{M} \to \bar{M}$ isometric action whose maximal orbits are hypersurfaces

Introduction

- $\bar{M}$ Riemannian manifold
- M submanifold of $\bar{M}$
- G Lie group

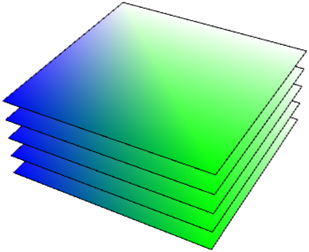
Homogeneous submanifold

$M \subset \bar{M}$ homogeneous $:\Leftrightarrow \forall p, q \in M$ there exists $g \in \mathrm{Isom}(\bar{M})$ such that:

- $g(M) = M$
- $g(p) = q$

Cohomogeneity one action

$G \times \bar{M} \to \bar{M}$ isometric action whose maximal orbits are hypersurfaces



Introduction

- $\bar{M}$ Riemannian manifold
- M submanifold of $\bar{M}$
- G Lie group

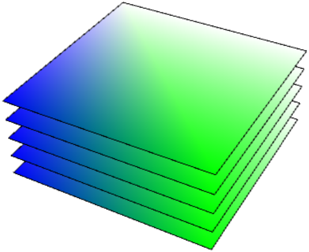
Homogeneous submanifold

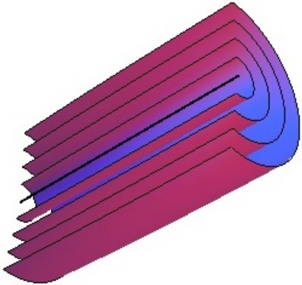
$M \subset \bar{M}$ homogeneous $:\Leftrightarrow \forall p, q \in M$ there exists $g \in \mathrm{Isom}(\bar{M})$ such that:

- $g(M) = M$
- $g(p) = q$

Cohomogeneity one action

$G \times \bar{M} \to \bar{M}$ isometric action whose maximal orbits are hypersurfaces





Introduction

Fact

If M is the singular orbit of a cohomogeneity one action $\Rightarrow$ the principal curvatures of M are independent on the normal

Introduction

Fact

If M is the singular orbit of a cohomogeneity one action $\Rightarrow$ the principal curvatures of M are independent on the normal $\Leftrightarrow$: CPC

Introduction

Fact

If M is the singular orbit of a cohomogeneity one action $\Rightarrow$ the principal curvatures of M are independent on the normal $\Leftrightarrow$: CPC

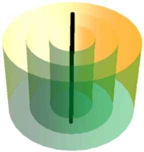
Totally geodesic $\subset$ CPC $\subset$ Austere $\subset$ Minimal

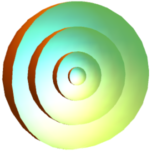
Introduction

Fact

If M is the singular orbit of a cohomogeneity one action $\Rightarrow$ the principal curvatures of M are independent on the normal $\Leftrightarrow$: CPC

Totally geodesic $\subset$ CPC $\subset$ Austere $\subset$ Minimal



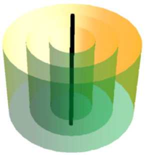


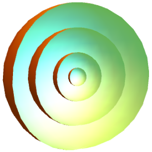
Introduction

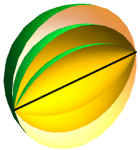
Fact

If M is the singular orbit of a cohomogeneity one action $\Rightarrow$ the principal curvatures of M are independent on the normal $\Leftrightarrow$: CPC

Totally geodesic $\subset$ CPC $\subset$ Austere $\subset$ Minimal





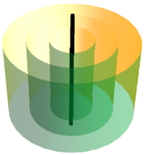


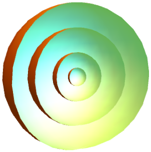
Introduction

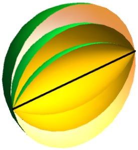
Fact

If M is the singular orbit of a cohomogeneity one action $\Rightarrow$ the principal curvatures of M are independent on the normal $\Leftrightarrow$: CPC

Totally geodesic $\subset$ CPC $\subset$ Austere $\subset$ Minimal







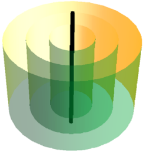
$\mathbb{R}^n$, $\mathbb{R}H^n$: Minimal + Homogeneous $\Rightarrow$ Totally geodesic [Di Scala]

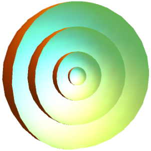
Introduction

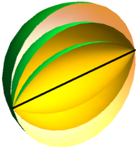
Fact

If M is the singular orbit of a cohomogeneity one action $\Rightarrow$ the principal curvatures of M are independent on the normal $\Leftrightarrow$: CPC

Totally geodesic $\subset$ CPC $\subset$ Austere $\subset$ Minimal







$\mathbb{R}^n$, $\mathbb{R}H^n$: Minimal + Homogeneous $\Rightarrow$ Totally geodesic [Di Scala]

In $\mathbb{S}^n$

Inhomogenous examples of non-totally geodesic CPC submanifolds

Symmetric spaces of non-compact type

$\bar{M} \cong G/K$ symmetric space of non-compact type, where
$G = \mathrm{Isom}^0(\bar{M}) \quad K = \{g \in G \ : \ g(o) = o\}$

Symmetric spaces of non-compact type

$\bar{M} \cong G/K$ symmetric space of non-compact type, where
$G = \mathrm{Isom}^0(\bar{M}) \quad K = \{g \in G \ : \ g(o) = o\}$

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition $\rightsquigarrow \theta$ Cartan involution
- B Killing form, non-degenerate
- Inner product in $\mathfrak{g}$: $\langle X, Y \rangle_{B_\theta} = -B(\theta X, Y)$, $X, Y \in \mathfrak{g}$

Symmetric spaces of non-compact type

$\bar{M} \cong G/K$ symmetric space of non-compact type, where
$G = \mathrm{Isom}^0(\bar{M}) \quad K = \{g \in G \ : \ g(o) = o\}$

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition $\rightsquigarrow \theta$ Cartan involution
- B Killing form, non-degenerate
- Inner product in $\mathfrak{g}$: $\langle X, Y \rangle_{B_\theta} = -B(\theta X, Y)$, $X, Y \in \mathfrak{g}$
- $\{\mathrm{ad}(H) \ : \ H \in \mathfrak{a}\}$ self-adjoint commutative endomorphisms
- $\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha\right)$ root space decomposition
- $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [A, X] = \lambda(A)X, \text{ for all } A \in \mathfrak{a}\}$, $\lambda \in \mathfrak{a}^*$
- $\Sigma = \{\lambda \in \mathfrak{a}^* \ : \ \mathfrak{g}_\lambda \neq 0\}$ set of roots $\rightsquigarrow \Sigma = \Sigma^+ \cup \Sigma^-$

Symmetric spaces of non-compact type

$\bar{M} \cong G/K$ symmetric space of non-compact type, where
$G = \mathrm{Isom}^0(\bar{M}) \quad K = \{g \in G \ : \ g(o) = o\}$

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition $\rightsquigarrow \theta$ Cartan involution
- B Killing form, non-degenerate
- Inner product in $\mathfrak{g}$: $\langle X, Y \rangle_{B_\theta} = -B(\theta X, Y)$, $X, Y \in \mathfrak{g}$
- $\{\mathrm{ad}(H) \ : \ H \in \mathfrak{a}\}$ self-adjoint commutative endomorphisms
- $\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha\right)$ root space decomposition
- $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [A, X] = \lambda(A)X, \text{ for all } A \in \mathfrak{a}\}$, $\lambda \in \mathfrak{a}^*$
- $\Sigma = \{\lambda \in \mathfrak{a}^* \ : \ \mathfrak{g}_\lambda \neq 0\}$ set of roots $\rightsquigarrow \Sigma = \Sigma^+ \cup \Sigma^-$

Iwasawa decomposition

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$
- $\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ nilpotent

$\bar{M}$ as a Lie group

- $\mathfrak{a} \oplus \mathfrak{n}$ Lie subalgebra $\rightsquigarrow AN \curvearrowright \bar{M}$ free, transitive
- $\bar{M} \cong AN$ solvable Lie group with left invariant metric

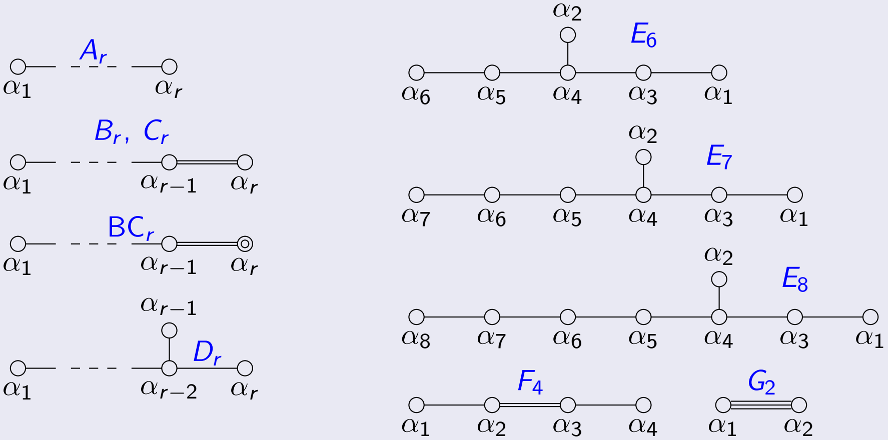
Contents

CPC submanifolds

$\Pi \subset \Sigma$ set of simple roots

A positive root is simple :$\Leftrightarrow$ cannot be written as sum of positive roots

Dynkin diagram



CPC submanifolds

$\Pi \subset \Sigma$ set of simple roots

A positive root is simple :$\Leftrightarrow$ cannot be written as sum of positive roots

Setup of the problem

- $\mathfrak{a} \oplus \mathfrak{n}$
- $\Pi' = \{\alpha \in \Pi : 2\alpha \notin \Sigma\}$
- $V \subset \bigoplus_{\alpha \in \Pi'} \mathfrak{g}_\alpha$

$$\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{n} \ominus V) \rightsquigarrow S \cdot o$$

In which cases is $S \cdot o$ a CPC submanifold?

CPC submanifolds

$\Pi \subset \Sigma$ set of simple roots

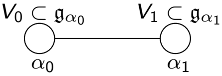
A positive root is simple $:\Leftrightarrow$ cannot be written as sum of positive roots

Setup of the problem

- $\mathfrak{a} \oplus \mathfrak{n}$
- $\Pi' = \{\alpha \in \Pi : 2\alpha \notin \Sigma\}$
- $V \subset \bigoplus_{\alpha \in \Pi'} \mathfrak{g}_\alpha$

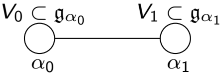
$$\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{n} \ominus V) \rightsquigarrow S \cdot o$$

In which cases is $S \cdot o$ a CPC submanifold?



$$V = V_0 \oplus V_1 \rightsquigarrow S \cdot o \begin{cases} T_o(S \cdot o) = \mathfrak{s} \\ \nu_o(S \cdot o) = V_0 \oplus V_1 \end{cases}$$

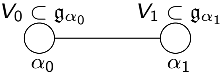
- $\mathcal{S}|_{\mathfrak{a}} = 0 \quad (A \cdot o$ totally geodesic$)$



$$V = V_0 \oplus V_1 \rightsquigarrow S \cdot o \begin{cases} T_o(S \cdot o) = \mathfrak{s} \\ \nu_o(S \cdot o) = V_0 \oplus V_1 \end{cases}$$

- $\langle \bar{\nabla}_X \xi, Y \rangle_{AN} = \dfrac{1}{4} \langle [X, \xi] + [\theta X, \xi] - [X, \theta \xi], Y \rangle_{B_\theta}$
- $\mathcal{S}_\xi$ shape operator w.r.t. $\xi = \xi_0 + \xi_1 \in V_0 \oplus V_1 \subset \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$

CPC submanifolds



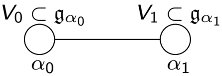
$$V = V_0 \oplus V_1 \rightsquigarrow S \cdot o \begin{cases} T_o(S \cdot o) = \mathfrak{s} \\ \nu_o(S \cdot o) = V_0 \oplus V_1 \end{cases}$$

- $\langle \bar{\nabla}_X \xi, Y \rangle_{AN} = \frac{1}{4} \langle [X, \xi] + [\theta X, \xi] - [X, \theta \xi], Y \rangle_{B_\theta}$
- $\mathcal{S}_\xi$ shape operator w.r.t. $\xi = \xi_0 + \xi_1 \in V_0 \oplus V_1 \subset \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$

- If $X \in \mathfrak{g}_\lambda^\top$:

$$\mathcal{S}_\xi X = -\frac{1}{2}\Big(\overbrace{[X, \xi_0 + \xi_1]}^{\mathfrak{g}_{\lambda+\alpha_0} \oplus \mathfrak{g}_{\lambda+\alpha_1}} + \overbrace{[\theta X, \xi_0 + \xi_1]}^{\mathfrak{g}_{-\lambda+\alpha_0} \oplus \mathfrak{g}_{-\lambda+\alpha_1}} - \overbrace{[X, \theta\xi_0 + \theta\xi_1]}^{\mathfrak{g}_{\lambda-\alpha_0} \oplus \mathfrak{g}_{\lambda-\alpha_1}}\Big)^\top$$

CPC submanifolds



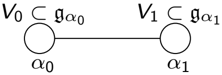
$$V = V_0 \oplus V_1 \rightsquigarrow S \cdot o \begin{cases} T_o(S \cdot o) = \mathfrak{s} \\ \nu_o(S \cdot o) = V_0 \oplus V_1 \end{cases}$$

- $\langle \bar{\nabla}_X \xi, Y \rangle_{AN} = \dfrac{1}{4} \langle [X, \xi] + [\theta X, \xi] - [X, \theta \xi], Y \rangle_{B_\theta}$
- $\mathcal{S}_\xi$ shape operator w.r.t. $\xi = \xi_0 + \xi_1 \in V_0 \oplus V_1 \subset \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$

- If $X \in \mathfrak{g}_\lambda^\top$:

$$\mathcal{S}_\xi X = -\frac{1}{2}\Big(\overbrace{[X, \xi_0 + \xi_1]}^{\mathfrak{g}_{\lambda+\alpha_0} \oplus \mathfrak{g}_{\lambda+\alpha_1}} + \overbrace{\cancel{[\theta X, \xi_0 + \xi_1]}}^{\mathfrak{g}_{-\lambda+\alpha_0} \oplus \mathfrak{g}_{-\lambda+\alpha_1}} - \overbrace{[X, \theta\xi_0 + \theta\xi_1]}^{\mathfrak{g}_{\lambda-\alpha_0} \oplus \mathfrak{g}_{\lambda-\alpha_1}}\Big)^\top$$

CPC submanifolds



$$V = V_0 \oplus V_1 \rightsquigarrow S \cdot o \begin{cases} T_o(S \cdot o) = \mathfrak{s} \\ \nu_o(S \cdot o) = V_0 \oplus V_1 \end{cases}$$

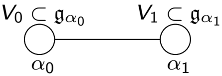
- $\langle \bar{\nabla}_X \xi, Y \rangle_{AN} = \dfrac{1}{4} \langle [X, \xi] + [\theta X, \xi] - [X, \theta \xi], Y \rangle_{B_\theta}$
- $\mathcal{S}_\xi$ shape operator w.r.t. $\xi = \xi_0 + \xi_1 \in V_0 \oplus V_1 \subset \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$

- If $X \in \mathfrak{g}_\lambda^\top$:

$$\mathcal{S}_\xi X = -\frac{1}{2}\Big(\overbrace{[X, \xi_0 + \xi_1]}^{\mathfrak{g}_{\lambda+\alpha_0} \oplus \mathfrak{g}_{\lambda+\alpha_1}} + \overbrace{\cancel{[\theta X, \xi_0 + \xi_1]}}^{\mathfrak{g}_{-\lambda+\alpha_0} \oplus \mathfrak{g}_{-\lambda+\alpha_1}} - \overbrace{[X, \theta\xi_0 + \theta\xi_1]}^{\mathfrak{g}_{\lambda-\alpha_0} \oplus \mathfrak{g}_{\lambda-\alpha_1}}\Big)^\top$$

- $\gamma_1 \sim \gamma_2 :\Leftrightarrow$ belong to the same (α_0, α_1)-string
- (α_0, α_1)-string of γ : $\{\gamma + n\alpha_0 + m\alpha_1 \in \Sigma : n, m \in \mathbb{Z}\}$

CPC submanifolds



$$V = V_0 \oplus V_1 \rightsquigarrow S \cdot o \begin{cases} T_o(S \cdot o) = \mathfrak{s} \\ \nu_o(S \cdot o) = V_0 \oplus V_1 \end{cases}$$

- $\langle \bar{\nabla}_X \xi, Y \rangle_{AN} = \dfrac{1}{4} \langle [X, \xi] + [\theta X, \xi] - [X, \theta \xi], Y \rangle_{B_\theta}$
- $\mathcal{S}_\xi$ shape operator w.r.t. $\xi = \xi_0 + \xi_1 \in V_0 \oplus V_1 \subset \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$

- If $X \in \mathfrak{g}_\lambda^\top$:

$$\mathcal{S}_\xi X = -\frac{1}{2}\Big(\overbrace{[X, \xi_0 + \xi_1]}^{\mathfrak{g}_{\lambda+\alpha_0} \oplus \mathfrak{g}_{\lambda+\alpha_1}} + \overbrace{\cancel{[\theta X, \xi_0 + \xi_1]}}^{\mathfrak{g}_{-\lambda+\alpha_0} \oplus \mathfrak{g}_{-\lambda+\alpha_1}} - \overbrace{[X, \theta\xi_0 + \theta\xi_1]}^{\mathfrak{g}_{\lambda-\alpha_0} \oplus \mathfrak{g}_{\lambda-\alpha_1}}\Big)^\top$$

- $\gamma_1 \sim \gamma_2 :\Leftrightarrow$ belong to the same (α_0, α_1)-string
- (α_0, α_1)-string of γ : $\{\gamma + n\alpha_0 + m\alpha_1 \in \Sigma : n, m \in \mathbb{Z}\}$

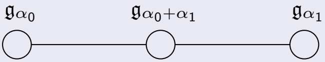
$$T_o(S \cdot o) \ominus \mathfrak{a} = \bigoplus_{\lambda \in (\Sigma^+/\sim)} \Big(\bigoplus_{\gamma \in [\lambda]} \mathfrak{g}_\gamma^\top \Big)$$

$\mathcal{S}_\xi$-invariant decomposition

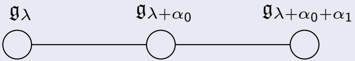
$$\mathcal{S}_\xi = \left[\begin{array}{c|c|c} \ddots & & \\ \hline & \bigoplus_{\gamma \in [\lambda]} \mathfrak{g}_\gamma^\top & \\ \hline & & \ddots \end{array} \right]$$

CPC submanifolds

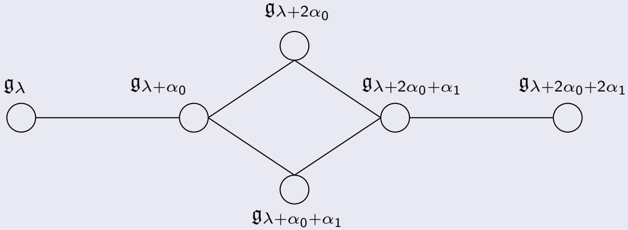
1. $[\alpha_0] = \{\alpha_0, \alpha_1, \alpha_0 + \alpha_1\}$



2. $|\alpha_0| \geq |\lambda|$:

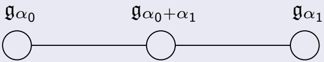


3. $|\alpha_0| < |\lambda|$:

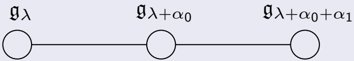


CPC submanifolds

1. $[\alpha_0] = \{\alpha_0, \alpha_1, \alpha_0 + \alpha_1\}$

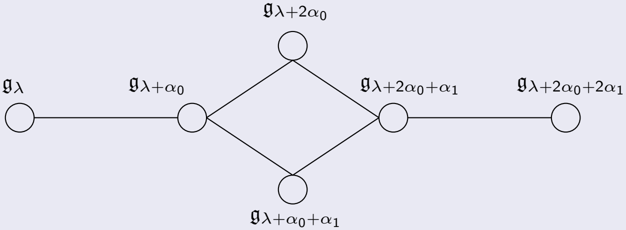


2. $|\alpha_0| \geq |\lambda|$:



✓**CPC**

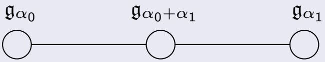
3. $|\alpha_0| < |\lambda|$:



✓**CPC**

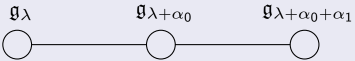
CPC submanifolds

1. $[\alpha_0] = \{\alpha_0, \alpha_1, \alpha_0 + \alpha_1\}$



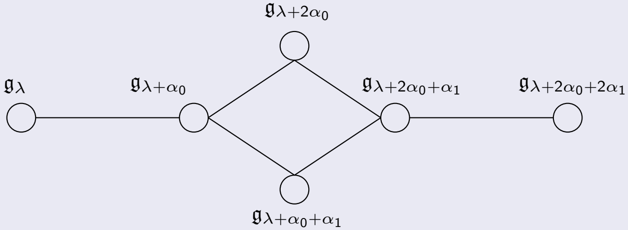
- $SL_3(\mathbb{R})/SO_3$
- $SL_3(\mathbb{C})/SU_3$
- SU_6^*/Sp_3
- E_6^{-26}/F_4

2. $|\alpha_0| \geq |\lambda|$:



✓**CPC**

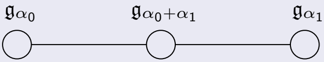
3. $|\alpha_0| < |\lambda|$:



✓**CPC**

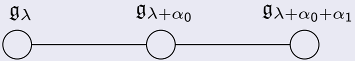
CPC submanifolds

1. $[\alpha_0] = \{\alpha_0, \alpha_1, \alpha_0 + \alpha_1\}$



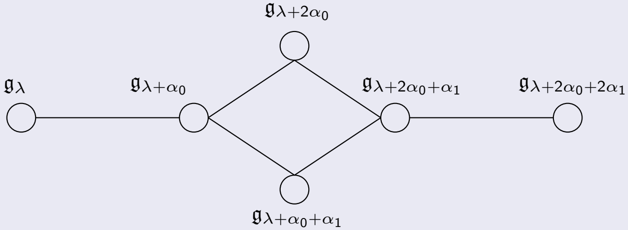
- $SL_3(\mathbb{R})/SO_3$
- $SL_3(\mathbb{C})/SU_3$
- SU_6^*/Sp_3
- E_6^{-26}/F_4

2. $|\alpha_0| \geq |\lambda|$:



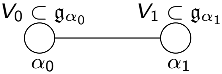
✓**CPC**

3. $|\alpha_0| < |\lambda|$:

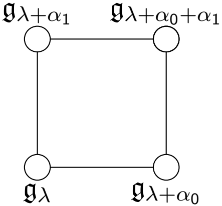


✓**CPC**

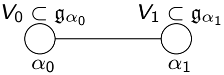
CPC submanifolds



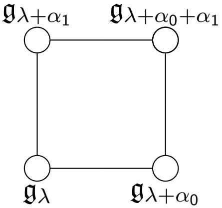
- More than two roots? $\Rightarrow$ At least two orthogonal roots



CPC submanifolds



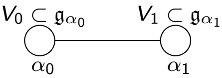
- More than two roots? $\Rightarrow$ At least two orthogonal roots



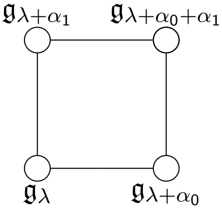
- Why not roots of different length?

$$\xi \in V_\alpha \subset \mathfrak{g}_\alpha \Rightarrow \mathcal{S}_\xi X = \pm \frac{|\alpha|}{2} X$$

CPC submanifolds



- More than two roots? $\Rightarrow$ At least two orthogonal roots



- Why not roots of different length?

$$\xi \in V_\alpha \subset \mathfrak{g}_\alpha \Rightarrow \mathcal{S}_\xi X = \pm \frac{|\alpha|}{2} X$$

- Why not $V \subset \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$ diagonally?

$$\text{If } \mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{n} \ominus V) \text{ is a subalgebra} \Rightarrow V = V_0 \oplus V_1$$

CPC submanifolds

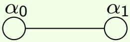
Π' set of reduced simple roots ($\alpha \in \Pi, 2\alpha \notin \Sigma$)

Main Theorem

Let $\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{n} \ominus V)$, with V subspace of $\bigoplus_{\alpha \in \Pi'} \mathfrak{g}_\alpha$. Then, $S \cdot o$ is CPC if and only if:

(I) There exists a simple root $\lambda \in \Pi'$ such that $V \subset \mathfrak{g}_\lambda$

(II) $V = V_0 \oplus V_1$, with $V_k \subset \mathfrak{g}_{\alpha_k}$ for $k \in \{0, 1\}$, where



are simple roots with the same length and one of the following statements holds:

- (a) $V_0 \oplus V_1 = \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$
- (b) V_0, V_1 isomorphic to $\mathbb{R}$
- (c) V_0, V_1 isomorphic to $\mathbb{C}$ w.r.t. $\mathrm{ad}(T)$ for some $T \in \mathfrak{k}_0$
- (d) V_0, V_1 isomorphic to $\mathbb{H}$ w.r.t. $\mathrm{ad}(\mathfrak{l})$ for some $\mathfrak{l} \subset \mathfrak{k}_0$

J. Berndt, V. Sanmartín-López, Submanifolds with constant principal curvatures in Riemannian symmetric spaces, arXiv:1805.10088

Homogeneous CPC submanifolds in symmetric spaces of non-compact type

Víctor Sanmartín López

Department of Mathematics, Universidade de Santiago de Compostela

Joint work with Jürgen Berndt

Symmetry and shape

Celebrating the 60th birthday of Jürgen Berndt