A rigidity theorem for self-shrinkers of MCF.

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Definition 1

A complete isometric immersion $X : \Sigma^n \to \mathbb{R}^{n+m}$ is a λ-soliton of the MCF with respect to $\vec{0} \in \mathbb{R}^{n+m}$, ($\lambda \in \mathbb{R}$), if and only if

$$\vec{H} = -\lambda X^{\perp}$$

where $X^{\perp}$ stands for the normal component of X and $\vec{H}$ is the mean curvature vector of the immersion X.

Definition 2

A λ-soliton for the MCF with respect to $\vec{0} \in \mathbb{R}^{n+m}$ is called a self-shrinker if and only if $\lambda \geq 0$. It is called a self-expander if and only if $\lambda < 0$.

Remark 3

Given a complete immersion $X : \Sigma^n \to \mathbb{R}^{n+m}$ *satisfying*

$$\vec{H} = -\lambda X^{\perp}$$

the family of homothetic immersions

$$X_t = \sqrt{1 - 2\lambda t}X$$

satisfies the equation of the MCF

$$\begin{cases} (\frac{\partial}{\partial t}X(p,t))^{\perp} & = & \vec{H}(p,t) \;\; \forall p \in \Sigma, \; \forall t \in [0,T) \\ X(p,0) & = & X_0(p), \;\; \forall p \in \Sigma \end{cases}$$

so X *becomes the* 0*-slice of the family* $\{X_t\}_{t=0}^{\infty}$ *of solutions of equation above.*

Example 4

- A compact λ-self-shrinker $X : \Sigma^n \to \mathbb{R}^{n+m}$ is $\Sigma = S^{n+m-1}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$
- Complete non-compact self-shrinkers:
  - $\Gamma \times \mathbb{R}^{n-1} \subseteq \mathbb{R}^{n+m}$, where Γ is an Abresch-Langer curve
  - $S^k(\sqrt{\frac{k}{\lambda}}) \times \mathbb{R}^{n-k} \subseteq \mathbb{R}^{n+m}$, generalized cylinders
  - $\Sigma = \mathbb{R}^n \subseteq \mathbb{R}^{n+m}$ is an Euclidean subespace, (case $\lambda = 0$).

H. D. Cao and H. Li proved the following classification result for properly immersed self-shrinkers

Theorem. H. D. Cao and H. Li, Calc. Var. 46 (2013)

Let $X:\Sigma^n \to \mathbb{R}^{n+m}$ be a complete and proper λ-self-shrinker, with bounded norm of the second fundamental form by

$$\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 \leq \lambda,$$

Then Σ is one of the following:

1. Σ is a round sphere $S^n(\sqrt{\frac{n}{\lambda}})$, (and hence $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 = \lambda$).
2. Σ is a cylinder $S^k(\sqrt{\frac{k}{\lambda}}) \times \mathbb{R}^{n-k}$, (and hence $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 = \lambda$).
3. Σ is an hyperplane, (and hence $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 = 0$).

Definition 5

Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be an isometric immersion. We say that the sphere $S^{n+m-1}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$ **separates** $X(\Sigma)$ if and only if $X(\Sigma) \cap B^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0}) \neq \emptyset$ and $X(\Sigma) \cap \left(\mathbb{R}^{n+m} \setminus \bar{B}^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})\right) \neq \emptyset$.

Namely, there exists $p, q \in \Sigma$ such that
$r_{\vec{0}}(p) = dist_{\mathbb{R}^{n+m}}(\vec{0}, X(p)) = \|X(p)\| < \sqrt{\frac{n}{\lambda}}$ and
$r_{\vec{0}}(q) = \|X(q)\| > \sqrt{\frac{n}{\lambda}}$.

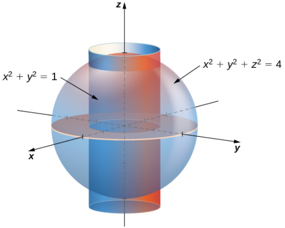
Definition 6

Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be an isometric immersion. We say that the sphere $S^{n+m-1}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$ **does not separate** $X(\Sigma)$ if and only if $X(\Sigma) \cap B^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0}) = \emptyset$ or $X(\Sigma) \cap \left(\mathbb{R}^{n+m} \setminus \bar{B}^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})\right) = \emptyset$.

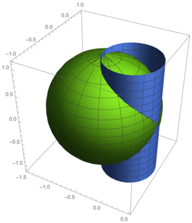
Namely, $\forall p \in \Sigma$, we have $r_{\vec{0}}(p) = \|X(p)\| \leq \sqrt{\frac{n}{\lambda}}$ or $r_{\vec{0}}(p) = \|X(q)\| \geq \sqrt{\frac{n}{\lambda}}$

Part I. Introduction: When the sphere separates a soliton.

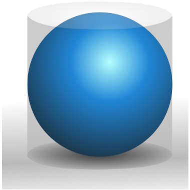
A cylinder *separated* by one sphere



A cylinder *separated* by one sphere



A cylinder *non separated* by one sphere



Theorem 1. V. Gimeno and V. P., JGA, 2019

Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete and proper λ-self-shrinker. Let us suppose that the sphere $S^{n+m-1}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$ does not separate $X(\Sigma)$. Then Σ^n is compact and $X : \Sigma \to S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$ is a minimal immersion.

Corollary 1. M. P. Cavalcante-J.M. Espinar, Bull. London Math. Soc. 48 (2016), V. Gimeno and V. P., JGA, 2019

Let $X : \Sigma^n \to \mathbb{R}^{n+1}$ be a complete, connected and proper λ-self-shrinker.

Let us suppose that the sphere $S^{n+m-1}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$ does not separate $X(\Sigma)$.

Then, Σ^n is isometric to $S^n\left(\sqrt{\frac{n}{\lambda}}\right)$

Sketch of proof

- No separation by the sphere implies, (Theorem 1), that $X : \Sigma \to S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$ is a minimal immersion.
- The local isometry $X : \Sigma^n \to S^n\left(\sqrt{\frac{n}{\lambda}}\right)$ among connected/simply connected spaces becomes a Riemannian covering and hence, an isometry.

Theorem 2. V. Gimeno and V. P., JGA, 2019

Let $X : \Sigma^n \to \mathbb{R}^{n+m}$, $(m \geq 2)$, be a complete and proper λ-self-shrinker, such that:

i) The sphere $S^{n+m-1}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$ does not separate $X(\Sigma)$.

ii) The second fundamental form of Σ is bounded by

$$\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 < \frac{5}{3}\lambda$$

Then, Σ^n is isometric to $S^n\left(\sqrt{\frac{n}{\lambda}}\right)$.

- We would like to emphasize the analogy of our notion of "separation by spheres" with the notion of "separation by planes" used in the Halfspace theorem for self-shrinkers.

Halfspace theorem for self-shrinkers, see M. P. Cavalcante-J.M. Espinar, Bull. London Math. Soc. 48 (2016) and S. Pigola-M. Rimoldi, Ann. Global Analysis 45 (2014)

Let P^n be an hyperplane in $\mathbb{R}^{n+1}$ passing through the origin. The only properly immersed self-shrinker Σ^n contained in one of the closed half-space determined by P is $\Sigma = P$.

- In this sense, Corollary 1 above could be stated as:

Theorem, (Corollary 1)

The only properly immersed and connected self-shrinker Σ^n contained in one of the closed domains determined by the sphere $S^n_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$, is $\Sigma^n = S^n_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$

Proposition 7 (K. Smoczyk, Int. Math. Res. Not. 48 (2005))

Let $X : \Sigma^n \to \mathbb{S}^{n+m-1}(R)$ be a complete spherical immersion. Then, the following affirmations are equivalent:

1. $X : \Sigma^n \to \mathbb{S}^{n+m-1}(R)$ *is a minimal immersion into* $\mathbb{S}^{n+m-1}(R)$.
2. X *is a* λ*- self-shrinker with* $\lambda = \frac{n}{R^2}$, *i.e.,* $R = \sqrt{\frac{n}{\lambda}}$

Proof of Proposition 7

- To see 1) $\Rightarrow$ 2), use the equation
$$\vec{H}_{\Sigma\subseteq\mathbb{R}^{n+m}} = \vec{H}_{\Sigma\subseteq S^{n+m-1}(R)} - \frac{n}{R^2}X = -\frac{n}{R^2}X = -\frac{n}{R^2}X^{\perp}$$
- To see 2)$\Rightarrow$ 1), use that X is a λ-self-shrinker and the extrinsic distance function $r_{\vec{0}}(p) := dist_{\mathbb{R}^{n+m}}(\vec{0}, X(p))$ defined on Σ.
- Given $F(p) := r^2(p) = \|X\|^2 = R^2$ on Σ, apply

Lemma 8

Given $F : \Sigma \to \mathbb{R}$, $F \in C^2(\Sigma)$, for all $x \in \Sigma$ such that $r(x) > 0$, we have

$$\begin{aligned}\Delta^{\Sigma} F(r(x)) &= \left(\frac{F''(r(x))}{r^2(x)} - \frac{F'(r(x))}{r^3(x)}\right)\|X^T\|^2 \\ &+ \frac{F'(r(x))}{r(x)}\left(n + \langle X, \vec{H}_{\Sigma\subseteq\mathbb{R}^{n+m}}\rangle\right)\end{aligned}$$

Part III. Proof of our results. Theorem 1.

We are going to prove

Theorem 1

Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete properly immersed λ-self-shrinker.
Let us suppose that the sphere $S^{n+m-1}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$ does not separate $X(\Sigma)$.
Then Σ^n is compact and $X : \Sigma \to S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$ is a minimal immersion.

Proof of Theorem 1

- As $S^{n+m-1}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$ does not separate $X(\Sigma)$ then $X(\Sigma) \subseteq \bar{B}^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$ or $X(\Sigma) \subseteq \mathbb{R}^{n+m} \setminus B^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$.
- Suppose first that $X(\Sigma) \subseteq \bar{B}^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$.
- Then $\sqrt{\frac{n}{\lambda}} \geq r(p)\ \forall p \in \Sigma$.
- Hence $\|X^{\perp}\|^2 \leq \|X\|^2 \leq \frac{n}{\lambda}$.
- Then, compute, (using Lemma 8 above):

$$\triangle^{\Sigma} r^2(x) = 2(n - \lambda \|X^{\perp}\|^2) \geq 0$$

Proof of Theorem 1

- As X is proper and $\Sigma = X^{-1}(\bar{B}^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0}))$, then Σ is compact and hence, by Hopf's Lemma:
- $r^2(x) = R^2 \ \forall x \in \Sigma$, so we have the spherical immersion $X : \Sigma \to S^{n+m-1}(R)$, for some $R \leq \sqrt{\frac{n}{\lambda}}$.
- As Σ is a λ-soliton for the MCF, then $R = \sqrt{\frac{n}{\lambda}}$, and $X : \Sigma \to S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$ is minimal by Proposition 7.

Proof of Theorem 1

- Suppose now that $X(\Sigma) \subseteq \mathbb{R}^{n+m} \setminus B^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$.
- Then $\sqrt{\frac{n}{\lambda}} \leq r(p)\ \forall p \in \Sigma$.
- Assume that $X(\Sigma) \not\subseteq S^{n+m-1}(R)$ for any radius $R > 0$ and that $\inf_\Sigma r > \sqrt{\frac{n}{\lambda}}$. We will reach a contradiction.
- First, as $\inf_\Sigma r > \sqrt{\frac{n}{\lambda}}$, we have that, for any $p \in \Sigma$

$$1 - \frac{\lambda}{n} r^2(p) < 0$$

Proof of Theorem 1

- Hence, given the extrinsic ball $D_R = X^{-1}(B_R^{n+m}(\vec{0})) = \{p \in \Sigma : \|X(p)\| < R\} \subseteq \Sigma$ and integrating

$$\int_{D_R} \left(1 - \frac{\lambda}{n} r^2\right) e^{\frac{\lambda}{2}(R^2 - r^2)} d\sigma < 0 \tag{1}$$

- Now, we need the following

Lemma 9

Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete properly immersed λ-self-shrinker in $\mathbb{R}^{n+m}$. Let us suppose that $X(\Sigma) \not\subseteq S^{n+m-1}(R)$ for any radius $R > 0$. Given the extrinsic ball $D_R = X^{-1}(B_R^{n+m}(\vec{0}))$, if $\operatorname{Vol}(D_R) > 0$, we have, for all $R > 0$:

$$0 \leq 1 - \frac{\int_{D_R} \|\vec{H}_\Sigma\|^2 d\sigma}{n\lambda \operatorname{Vol}(D_R)} = \frac{\int_{D_R} \left(1 - \frac{\lambda}{n} r^2\right) e^{\frac{\lambda}{2}(R^2 - r^2)} d\sigma}{\operatorname{Vol}(D_R)} \tag{2}$$

Proof of Theorem 1

- Now, applying inequality (1) and Lemma 9, we have

$$0 \leq 1 - \frac{\int_{D_R} \|\vec{H}_\Sigma\|^2 d\sigma}{n\lambda \operatorname{Vol}(D_R)} = \frac{\int_{D_R} \left(1 - \frac{\lambda}{n} r^2\right) e^{\frac{\lambda}{2}\left(R^2 - r^2\right)} d\sigma}{\operatorname{Vol}(D_R)} < 0 \qquad (3)$$

  which is a contradiction.

- Hence, either $X(\Sigma) \subseteq S^{n+m-1}(R_0)$ for some radius $R_0 > 0$, or $\inf_\Sigma r = \sqrt{\frac{n}{\lambda}}$.
- In the first case, we have that $X : \Sigma \to S^{n+m-1}(R_0)$ will be a spherical immersion and, by Proposition 7, as Σ is a λ-self-shrinker, then X is minimal and $\lambda = \frac{n}{R^2{}_0}$, namely, $X : \Sigma \to S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$ is a minimal immersion.

Proof of Theorem 1

- In the second case, if $\inf_\Sigma r = \sqrt{\frac{n}{\lambda}}$, then $\sqrt{\frac{n}{\lambda}} \leq r(p)$ for all $p \in \Sigma$ and hence $1 - \frac{\lambda}{n} r^2(p) \leq 0 \; \forall p \in \Sigma$.
- Then by inequality (1) and Lemma 9 we have

$$0 \leq 1 - \frac{\int_{D_R} \|\vec{H}_\Sigma\|^2 d\sigma}{n\lambda \operatorname{Vol}(D_R)} = \frac{\int_{D_R} \left(1 - \frac{\lambda}{n} r^2\right) e^{\frac{\lambda}{2}\left(R^2 - r^2\right)} d\sigma}{\operatorname{Vol}(D_R)} \leq 0 \qquad (4)$$

- Therefore, $1 - \frac{\lambda}{n} r^2(p) = 0 \; \forall p \in \Sigma$, so $X(\Sigma) \subseteq S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$, and hence $X : \Sigma \to S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$ is a complete spherical immersion and a λ-self-shrinker. Then by Proposition 7, Σ is minimal in the sphere $S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$.
- Finally, as $X : \Sigma^n \to \mathbb{R}^{n+m}$ is proper, then $\Sigma = X^{-1}(S^{n+m-1}(\sqrt{\frac{n}{\lambda}}))$ is compact.

We are going to prove

Theorem 2

Let $X : \Sigma^n \to \mathbb{R}^{n+m}$, $(m \geq 2)$, be a complete and proper λ-self-shrinker, such that:

i) The sphere $S^{n+m-1}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$ does not separate $X(\Sigma)$

ii) The second fundamental form of Σ is bounded by

$$\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 < \frac{5}{3}\lambda$$

Then, Σ^n is isometric to $S^n\left(\sqrt{\frac{n}{\lambda}}\right)$ and $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 = \lambda$.

Proof of Theorem 2

- If the sphere $S^{n+m-1}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$ does not separate $X(\Sigma)$, then, applying Theorem 1,
- $X : (\Sigma, g) \to (S^{n+m-1}(\sqrt{\frac{n}{\lambda}}), g_{S^{n+m-1}(\sqrt{\frac{n}{\lambda}})})$ is a compact and minimal immersion,
- Hence, scaling the metric, $\widetilde{X} : (\Sigma, \frac{\lambda}{n} g) \to (S^{n+m-1}(1), g_{S^{n+m-1}(1)})$ realizes as a minimal immersion, with second fundamental form in the sphere satisfying

$$\left\| \widetilde{A}_{\Sigma}^{S^{n+m-1}(1)} \right\|^2 = \frac{n}{\lambda} \| A_{\Sigma}^{\mathbb{R}^{n+m}} \|^2 - n \tag{5}$$

- Hence, as by hypothesis $\| A_{\Sigma}^{\mathbb{R}^{n+m}} \|^2 < \frac{5}{3}\lambda$, then

$$\left\| \widetilde{A}_{\Sigma}^{S^{n+m-1}(1)} \right\|^2 < \frac{2n}{3} \tag{6}$$

Proof of Theorem 2

- Case I: Assume that $n \geq 1$ and $m = 2$. Apply following Theorem,

J. Simons-M.P. Do Carmo-S.S. Chern-S. Kobayashi Rigidity Theorem

Let $\varphi : (\Sigma^n, \widetilde{g}) \to (S^{n+1}(1), g_{S^{n+1}(1)})$ be a compact and minimal isometric immersion. Let us suppose that $\|\widetilde{A}_\Sigma^{S^{n+1}(1)}\|^2 \leq n$. Then

1. either $\|\widetilde{A}_\Sigma^{S^{n+1}(1)}\|^2 = 0$ and $(\Sigma^n, \widetilde{g})$ is isometric to $S^n(1)$
2. or $\|\widetilde{A}_\Sigma^{S^{n+1}(1)}\|^2 = n$. Then $(\Sigma^n, \widetilde{g})$ is isometric to a generalized Clifford torus $\Sigma^n = S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$ immersed as an hypersurface in $S^{n+1}(1)$.

Proof of Theorem 2

- Case II: Assume that $n \geq 1$ and $m \geq 3$. Apply following Theorem

A. M Li and J. Li, Archiv. Math. 58, (1992). Refinement of Simons' et al. Theorem

Let $\varphi : (\Sigma^n, \widetilde{g}) \to (S^{n+m-1}(1), g_{S^{n+m-1}(1)})$ be a compact and minimal isometric immersion, and $m \geq 3$. Let us suppose that $\|\widetilde{A}_\Sigma^{S^{n+m-1}(1)}\|^2 \leq \frac{2n}{3}$. Then,

1. either $\|\widetilde{A}_\Sigma^{S^{n+m-1}(1)}\|^2 = 0$ and $(\Sigma^n, \widetilde{g})$ is isometric to $S^n(1)$
2. or, (in case $n = 2$ and $m = 3$), $\|\widetilde{A}_\Sigma^{S^{n+m-1}(1)}\|^2 = \frac{4}{3}$ and $(\Sigma^2, \widetilde{g})$ is isometric to the Veronese surface $\Sigma^2 = \mathbb{R}P^2(\sqrt{3})$ in $S^4(1)$.

Thank you

Proof of our results. Lemma 9.

Proof of Lemma 9

- Consider $r^2 : \Sigma \to \mathbb{R}$, defined as $r^2(p) = \|X(p)\|^2$, where $r = dist_{\mathbb{R}^{n+m}}(\vec{0},)$. We have that $X = r\nabla^{\mathbb{R}^{n+m}} r$ and that $X^T = r\nabla^{\Sigma} r$
- Then, applying Lemma 8 to the radial function $F(r) = r^2$,

$$\Delta^{\Sigma} r^2 = 2n + 2\langle r\nabla^{\mathbb{R}^{n+m}} r, \vec{H}_{\Sigma}\rangle \tag{7}$$

- As $\langle r\nabla^{\mathbb{R}^{n+m}} r, \vec{H}_{\Sigma}\rangle = -\lambda\|X^{\perp}\| = -\frac{\|\vec{H}_{\Sigma}\|^2}{\lambda}$
- Then

$$\Delta^{\Sigma} r^2 = 2n - 2\frac{\|\vec{H}_{\Sigma}\|^2}{\lambda} \tag{8}$$

Proof of our results. Lemma 9.

Proof of Lemma 9

- Integrating on $D_R = X^{-1}(B_R^{n+m}(\vec{0}))$ equality above, we have

$$n\lambda \operatorname{Vol}(D_R) - \int_{D_R} \|\vec{H}_\Sigma\|^2 d\sigma = \frac{\lambda}{2} \int_{D_R} \Delta^\Sigma r^2 d\sigma \tag{9}$$

- Apply Divergence theorem (unitary normal to ∂D_R in Σ, pointed outward is $\mu = \frac{\nabla^\Sigma r}{\|\nabla^\Sigma r\|}$ and $X^T = r\nabla^\Sigma r$),

$$\begin{aligned}\int_{D_R} \Delta^\Sigma r^2 d\sigma &= \int_{\partial D_R} \langle \nabla^\Sigma r^2, \frac{\nabla^\Sigma r}{\|\nabla^\Sigma r\|} \rangle d\mu \\ &= \int_{\partial D_R} 2r\|\nabla^\Sigma r\| d\mu = 2\int_{\partial D_R} \|X^T\| d\mu\end{aligned} \tag{10}$$

Proof of our results. Lemma 9.

Proof of Lemma 9

- Then equation (9) becomes

$$
\begin{aligned}
n\lambda \operatorname{Vol}(D_R) - \int_{D_R} \|\vec{H}_\Sigma\|^2 d\sigma &= \lambda \int_{\partial D_R} \|X^T\| d\mu \\
&= \lambda \int_{\partial D_R} r \|\nabla^\Sigma r\| d\mu = \lambda R \int_{\partial D_R} \|\nabla^\Sigma r\| d\mu
\end{aligned} \tag{11}
$$

- Hence

$$
1 - \frac{\int_{D_R} \|\vec{H}_\Sigma\|^2 d\sigma}{n\lambda \operatorname{Vol}(D_R)} = \frac{R}{n \operatorname{Vol}(D_R)} \int_{\partial D_R} \|\nabla^\Sigma r\| d\mu \geq 0 \tag{12}
$$

Proof of our results. Lemma 9.

Proof of Lemma 9

- Applying the divergence theorem on D_R to the vector field $e^{-\frac{\lambda}{2}r^2}\nabla^\Sigma r^2$, we obtain

$$\int_{D_R} div^\Sigma \left(e^{-\frac{\lambda}{2}r^2}\nabla^\Sigma r^2\right) d\sigma = 2Re^{-\frac{\lambda}{2}R^2}\int_{\partial D_R} \|\nabla^\Sigma r\| d\mu. \qquad (13)$$

- Hence

$$\begin{aligned} 1 - \frac{\int_{D_R} \|\vec{H}_\Sigma\|^2 d\sigma}{n\lambda \operatorname{Vol}(D_R)} &= \frac{R}{n \operatorname{Vol}(D_R)} \int_{\partial D_R} \|\nabla^\Sigma r\| d\mu \\ &= \frac{e^{\frac{\lambda}{2}R^2}}{2n \operatorname{Vol}(D_R)} \int_{\partial D_R} div^\Sigma \left(e^{-\frac{\lambda}{2}r^2}\nabla^\Sigma r^2\right) d\sigma \end{aligned} \qquad (14)$$

Proof of our results. Lemma 9.

Proof of Lemma 9

- Finally, the proposition follows taking into account in equation above that

$$div^{\Sigma}\left(e^{-\frac{\lambda}{2}r^2}\nabla^{\Sigma}r^2\right) = 2e^{-\frac{\lambda}{2}r^2}\left(n-\lambda r^2\right) \tag{15}$$

The bound for the norm of second fundamental form in Theorem of Cao and Li is sharp

- The bound λ is sharp in the following sense:
- Consider the non-compact and proper 1-self-shrinker given by $\Sigma = \Gamma_{p,q} \times \mathbb{R} \subseteq \mathbb{R}^4$, where $\Gamma_{p,q} \subseteq \mathbb{R}^2$ is an Abresch-Langer curve.
- We have that

$$\|A_\Sigma^{\mathbb{R}^4}\|^2 = \|A_\Gamma^{\mathbb{R}^2}\|^2 = (k_g^\Gamma)^2$$

where k_g^Γ is the geodesic curvature ($=$ signed curvature) of the Abresch-Langer curve $\Gamma_{p,q} \subseteq \mathbb{R}^2$.

The bound for the norm of second fundamental form in Theorem of Cao and Li is sharp

- But the Abresch-Langer curves $\Gamma_{p,q}$ are contained in an annulus around the origin, and they are curves with rotation number p which touches each boundary of the annulus q times for each pair of mutally prime positive integers p, q such that $\frac{1}{2} < \frac{p}{q} < \frac{\sqrt{2}}{2}$
- As it is shown in the following picture

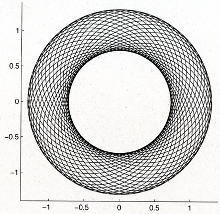


FIGURE 7. $A = 0$, $B = -1$ (shrinks) Abresch-Langer curve where the values of p and q are very high.

The bound for the norm of second fundamental form in Theorem of Cao and Li is sharp

- It has been shown, (see H. Halldorson, Trans. Amer. Math. Soc. 364, (2012)), that the signed curvature k_g^Γ of $\Gamma_{p,q}$:
    - Is an increasing function of the radius,
    - Never changes sign and
    - Takes its maximum and minimum at the same time as the radius, $k^\Gamma_{min} = r_{min}$ and $k^\Gamma_{max} = r_{max}$, where r_{min} and r_{max} are the inner and the outer radius of the annulus respectively.
    - Moreover, r_{min} take on every value in $(0, 1]$ and r_{max} take on every value in $[1, \infty)$

The bound for the norm of second fundamental form in Theorem of Cao and Li is sharp

- Hence, we can choose the values p and q in order to have:
  - $\|A_\Sigma^{\mathbb{R}^4}\|^2 = (k_g^\Gamma)^2 < \frac{5}{3}$
  - The inner radius satisfies $r_{min} < 1$, so the sphere $S^3(1)$ separates Σ
- In conclusion, if we consider bounds for $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2$ greater than λ, there are λ-self-shrinkers satisfying this bound which are not those identified by Cao and Li in their Theorem. Moreover, some of these self-shrinkers can be separated.