Totally geodesic submanifolds in the Riemannian symmetric spaces of rank 2

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Totally geodesic submanifolds in Riemannian manifolds

- **Totally geodesic** submanifolds in **symmetric** spaces **rank 2**.
- **Reminder.** A submanifold M' of a Riemannian manifold M is called **totally geodesic**, if
  - the **second fundamental form** h of $M' \hookrightarrow M$ **vanishes**,

  or equivalently, if
  - every **geodesic** of M' also is a geodesic in M.

  If M' is totally geodesic, then T_pM' is **curvature-invariant**, i.e. $R^M(T_pM', T_pM')T_pM' \subset T_pM'$.
- **Examples.**
  - $\mathbb{R}^k \subset \mathbb{R}^n$
  - $S^k \subset S^n$
  - $\mathbb{CP}^k \subset \mathbb{CP}^n$ $\quad$ $\mathbb{RP}^k \subset \mathbb{CP}^n$
  - $\mathbb{HP}^k \subset \mathbb{HP}^n$ $\quad$ $\mathbb{CP}^k \subset \mathbb{HP}^n$ $\quad$ $\mathbb{RP}^k \subset \mathbb{HP}^n$
- Murphy (2019): On a differentiable manifold M with $\dim(M) \geq 4$, **generic** Riemannian metrics on M **do not admit any totally geodesic submanifolds** of dimension ≥ 2.

The classification problem for totally geodesic submanifolds

- Today we are interested in the following **classification problem**:

  Given a Riemannian symmetric space M,
  find all totally geodesic submanifolds of M.

- Clearly, totally geodesic submanifolds in M come in **families** by the action of $I(M)$. In general, there exist **several** such families of totally geodesic submanifolds M'.
- **Classify** totally geodesic submanifolds in M?
  - Up to **congruence**?
  - Up to **(local) isometry**?
  - Up to **(local) homothety**?

Known classification results for totally geodesic submfds

- **All** totally geodesic submanifolds are known in
  - **Rank 1 symmetric spaces.**
    Spheres, projective spaces, Cayley plane. Wolf 1963.
  - **Rank 2 symmetric spaces.** *"We have to talk."*
  - No symmetric spaces of rank ≥ 3.
- **Specific types** of totally geodesic submanifolds have been classified in all (irreducible) symmetric spaces, for example:
  - **Reflective submanifolds.** They are connected components of the fixed point set of involutive isometries of M.
    Leung 1974/75.
  - **Complex submanifolds** (in Hermitian symmetric spaces).
    Ihara 1967.
  - **Maximal spheres.**
    Makiko Sumi Tanaka 1991.
  - **Subspaces of maximal rank.**
    Ikawa/Tasaki 2000, Zhu/Liang 2004.

Totally geodesic submanifolds in spaces of rank 2

- Chen/Nagano 1978: Classification to **local homothety**.
  - First application of **(M_+, M_-)-method** (polars/meridians).
  - No information about the **position** of the submanifolds.
  - **Missed** some "skew" maximal totally geodesic submanifolds:

$$S^2(\tfrac{1}{2}\sqrt{10}) \subset Q^3 = G_2^+(\mathbb{R}^5)\,, \qquad \mathbb{CP}^2 \subset G_2(\mathbb{C}^6)\,, \qquad \mathbb{HP}^2 \subset G_2(\mathbb{H}^7)\,,$$
$$S^3(\tfrac{1}{2}\sqrt{10}) \subset \mathrm{Sp}(2)\,, \qquad S^2(\tfrac{2}{3}\sqrt{21}) \subset \mathrm{G}_2/\mathrm{SO}(4)\,, \qquad S^3(\tfrac{2}{3}\sqrt{21}) \subset \mathrm{G}_2\,.$$

- Kimura/Tanaka 2008: Classification **global homothety**.
  - Refinement of the method by Chen/Nagano.
  - The above "skew" submanifolds are still missing.
- K∼ 2005–09: Classification up to **congruence**.
  - Postdoctoral Fellowship at the University College Cork (2006–08), under the guidance of **Jürgen Berndt**.
  - Different methods: **Root systems**.
  - Description of the **position** of submanifolds (tangent spaces/totally geodesic embeddings).
  - The missing "skew" totally geodesic submanifolds were **found**.

Totally geodesic submfds in Riemannian symmetric spaces

- Let $M = G/K$ be a **Riemannian symmetric space** with **symmetric triple** (G, K, σ) and **origin** $p_0 := eK \in M$.
- Every **connected** totally geodesic **(t.g.)** submanifold of M is contained in a **complete** one, **congruent** to one through p_0.
- Two **connected, complete**, t.g. submanifolds M', M'' through p_0 with $T_{p_0}M' = T_{p_0}M''$ are **identical**: $M' = M''$.
- A connected, complete submanifold M' of M with $p_0 \in M'$ is **t.g.** if and only if it is a **symmetric subspace**, i.e. if there exists a σ-invariant Lie subgroup G' of G so that $(G', G' \cap K, \sigma|G')$ is a **symmetric triple** for M'.
- $U \subset T_{p_0}M$ a **linear subspace**. There **exists a t.g. submanifold** $M' \subset M$ with $p_0 \in M'$ and $T_{p_0}M' = U$ if and only if U is **curvature invariant** (a **Lie triple system**), i.e. if $R_M(U, U)U \subset U$ (or $[[\mathfrak{m}', \mathfrak{m}'], \mathfrak{m}'] \subset \mathfrak{m}'$) holds.

Riemannian symmetric spaces of rank 2

- The task that is set before us is to **classify the Lie triple systems** of M, for every Riemannian symmetric space M of rank 2.
- The **simply connected, irreducible** Riemannian symmetric spaces M of **compact type** are the following:
  - The **2-Grassmannians**
    $Q^m = G_2^+(\mathbb{R}^{m+2})$, $G_2(\mathbb{C}^{m+2})$ and $G_2(\mathbb{H}^{m+2})$.
  - The **classical quotient spaces**
    $\mathrm{SU}(3)/\mathrm{SO}(3)$, $\mathrm{SU}(6)/\mathrm{Sp}(3)$ and $\mathrm{SO}(10)/\mathrm{U}(5)$.
  - The **exceptional spaces**
    $\mathrm{E}_6/(\mathrm{U}(1)\cdot\mathrm{Spin}(10))$, $\mathrm{E}_6/\mathrm{F}_4$ and $\mathrm{G}_2/\mathrm{SO}(4)$.
  - The **compact Lie groups** $\mathrm{SU}(3)$, $\mathrm{Sp}(2)$ and G_2.

Roots and root spaces

- Suppose that $M \cong (G, K, \sigma)$ is **of compact type**.
- Let $\mathfrak{g}$ be the **Lie algebra** of G, σ_L the linearisation of σ. Then $\mathfrak{k} = \mathrm{Eig}(\sigma_L, 1)$, $\mathfrak{m} = \mathrm{Eig}(\sigma_L, -1) \cong T_{p_0}M$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and $-[[u,v],w] \cong R_M(u,v)w$ for $u, v, w \in \mathfrak{m} \cong T_{p_0}M$.
- Choose a **Cartan subalgebra** $\mathfrak{a} \subset \mathfrak{m}$. For $\lambda \in \mathfrak{a}^* \setminus \{0\}$ we let

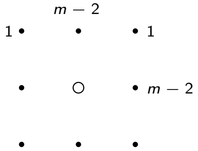
  $$\mathfrak{m}_\lambda = \left\{ X \in \mathfrak{m} \mid \forall H \in \mathfrak{a} : \mathrm{ad}(H)^2 X = -\lambda(H)^2 \cdot X \right\} .$$

  If $\mathfrak{m}_\lambda \neq \{0\}$, λ is called a **root** of M, and $\mathfrak{m}_\lambda$ is its **root space**. The set $\Delta \subset \mathfrak{a}^* \setminus \{0\}$ of all roots is the **root system**.
- We have $-\Delta = \Delta$. For $H_0 \in \mathfrak{a}$ with $\lambda(H_0) \neq 0$ for all $\lambda \in \Delta$, $\Delta_+ := \{\lambda \in \Delta \mid \lambda(H_0) > 0\}$ is the set of **positive roots** with respect to H_0. We have

  $$\mathfrak{m} = \mathfrak{a} \oplus \bigoplus_{\lambda \in \Delta_+} \mathfrak{m}_\lambda .$$

Example: The complex quadric

- The **complex quadric** $Q^m = G_2^+(\mathbb{R}^{m+2}) = \mathrm{SO}(m+2)/\mathrm{SO}(2) \times \mathrm{SO}(m)$ is a **Hermitian symmetric space of rank 2**.
- We can visualise the **root system** of Q^m with respect to a Cartan algebra $\mathfrak{a}$ by plotting $\alpha^\sharp \in \mathfrak{a}$ for $\alpha \in \Delta \subset \mathfrak{a}^*$:



How to describe Lie triple systems in root theory

- Let $\mathfrak{m}' \subset \mathfrak{m}$ be a **Lie triple system**, $\mathfrak{k}' = [\mathfrak{m}', \mathfrak{m}'] \subset \mathfrak{k}$ and $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{m}'$.
- There exists a **Cartan subalgebra** $\mathfrak{a}$ of $\mathfrak{m}$ such that $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{m}'$ is a Cartan subalgebra of $\mathfrak{m}'$. Let $\Delta' \subset (\mathfrak{a}')^*$ be the **root system** of $\mathfrak{m}'$ with respect to $\mathfrak{a}'$, and for $\alpha \in \Delta'$, let $\mathfrak{m}'_\alpha$ be the corresponding **root space**.
- Then we have

$$\mathfrak{a}' \subset \mathfrak{a}$$

$$\Delta' \subset \left\{ \lambda|\mathfrak{a}' \;\middle|\; \lambda \in \Delta, \lambda|\mathfrak{a}' \neq 0 \right\}$$

$$\forall \alpha \in \Delta' \;:\; \mathfrak{m}'_\alpha \subset \bigoplus_{\substack{\lambda \in \Delta \\ \lambda|\mathfrak{a}' = \alpha}} \mathfrak{m}_\lambda$$

$$\mathfrak{m}' = \mathfrak{a}' \oplus \bigoplus_{\alpha \in \Delta'_+} \mathfrak{m}'_\alpha \,.$$

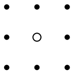
- In particular for $\mathrm{rk}(\mathfrak{m}') = \mathrm{rk}(\mathfrak{m})$:

$$\mathfrak{a}' = \mathfrak{a} \,, \quad \Delta' \subset \Delta \,, \quad \mathfrak{m}'_\alpha \subset \mathfrak{m}_\alpha \,.$$

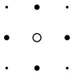
Lie triple systems of rank 2.

- Consider the case $\mathrm{rk}(\mathfrak{m}') = \mathrm{rk}(\mathfrak{m}) = 2$.
  Then $\mathfrak{a}' = \mathfrak{a}$, $\Delta' \subset \Delta$ and $\mathfrak{m}'_\alpha \subset \mathfrak{m}_\alpha$.
- The possibilities for $\mathfrak{m}'$ are **further restricted** by:
  - Δ' is **invariant** under its **Weyl group**.
  - $[[\mathfrak{m}'_\alpha, \mathfrak{m}'_\beta], \mathfrak{m}'_\gamma] \subset \bigoplus_{\alpha \pm \beta \pm \gamma \in \Delta'} \mathfrak{m}'_{\alpha \pm \beta \pm \gamma}$
- Need to **evaluate** the Lie bracket.
  - If G is a **classical** Lie group, do **matrix calculations** in $\mathfrak{g}$ (or something similar).
  - If G is an **exceptional** Lie group, consider the **root system of** $\mathfrak{g}^{\mathbb{C}}$. Use $\dim_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}_\lambda) = 1$ and $[X_\lambda, X_\mu] = c_{\lambda,\mu} \cdot X_{\lambda+\mu}$. The numbers $c_{\lambda,\mu}$ are determined up to sign from the root system, consistent choice of signs can be obtained. **Computer algebra** is useful. $\rightsquigarrow$ `http://satake.sourceforge.net`.
- In this way, one can **classify** the rank 2 Lie triple systems in every rank 2 symmetric space.

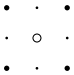
Lie triple systems of rank 2 in the complex quadric



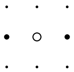
$M' = G_2^+(\mathbb{R}^{k+2})$
$3 \leq k < m$



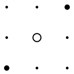
$M' = (S^k \times S^\ell)/\mathbb{Z}_2$
$k, \ell \geq 2;\ k + \ell \leq m$



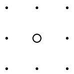
$M' = \mathbb{CP}^1 \times \mathbb{CP}^1 \cong G_2^+(\mathbb{R}^4)$



$M' = (S^k \times S^1)/\mathbb{Z}_2$
$2 \leq k \leq m - 1$



$M' = \mathbb{CP}^1 \times \mathbb{RP}^1$

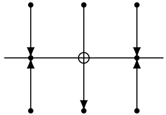


$M' = (S^1 \times S^1)/\mathbb{Z}_2$
(a maximal flat torus)

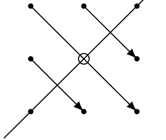
Lie triple systems of rank 1.

- Consider the case $\mathrm{rk}(\mathfrak{m}') = 1$ and $\mathrm{rk}(\mathfrak{m}) = 2$.
  Then $\mathfrak{a}'$ is a **line** in the **plane** $\mathfrak{a}$.
- Is every line $\mathfrak{a}' \subset \mathfrak{a}$ **possible**? Take $\alpha \in \Delta'$, then $\mathfrak{a}' = \mathbb{R}\alpha^\sharp$, and $\alpha = \lambda|\mathfrak{a}'$ for one or more $\lambda \in \Delta$.
  - We call α **elementary**, if there exists only one $\lambda \in \Delta$ with $\lambda|\mathfrak{a}' = \alpha$. In this case we have $\lambda|(\mathfrak{a}')^\perp = 0$, i.e. $\lambda^\sharp \in \mathfrak{a}'$.
  - We call α **composite**, if there exist (at least) two different $\lambda, \mu \in \Delta$ with $\lambda|\mathfrak{a}' = \alpha = \mu|\mathfrak{a}'$. Then $\mathfrak{a} \perp (\lambda^\sharp - \mu^\sharp)$.
- Therefore
  - **either** $\mathfrak{a}' = \mathbb{R}\lambda^\sharp$ for some $\lambda \in \Delta$,
  - **or** $\mathfrak{a}' = (\mathbb{R}(\lambda^\sharp - \mu^\sharp))^\perp$ for some $\lambda, \mu \in \Delta$, $\lambda \neq \mu$.
- It follows that for every space M, there exist only **finitely many** possible $\mathfrak{a}'$.
- Still have to **evaluate** $[[\mathfrak{m}'_{j\alpha}, \mathfrak{m}'_{k\alpha}], \mathfrak{m}'_{\ell\alpha}]$
  (for $j, k, \ell \in \{\pm 1, \pm 2\}$) to determine the possibilities for $\mathfrak{m}'_\alpha$ and $\mathfrak{m}'_{2\alpha}$.

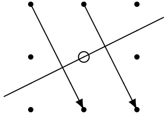
Rank 1 Lie triple systems in the complex quadric



$M' = S^k(1), \quad 1 \leq k \leq m$
$M' = G_2(\mathbb{R}^3) \cong S^2$



$M' = \mathbb{CP}^k, \quad 1 \leq k \leq \frac{n}{2}$
$M' = \mathbb{RP}^k, \quad 1 \leq k \leq \frac{n}{2}$



$M' = S^2(\frac{1}{2}\sqrt{10})$
in a special, "skew" position

The “skew” 2-sphere in Q^3

- We want to **embed** the **2-sphere** $M = \mathrm{SO}(3)/\mathrm{SO}(2)$ in $Q^3 = \mathrm{SO}(5)/\mathrm{SO}(2) \times \mathrm{SO}(3)$ as a **totally geodesic** submanifold (**symmetric subspace**).
- $V := \mathrm{End}_+^0(\mathbb{R}^3)$: symmetric, trace-free real (3×3)-matrices. The **Cartan representation** is the 5-dimensional irreducible, orthogonal, real representation

$$\mathrm{SO}(3) \times V \to V,\ (B, X) \mapsto BXB^t = BXB^{-1} .$$

  It acts on the complex quadric $Q^3 \cong G_2^+(V)$ via **isometries**.
- Let $Z_0 := \mathbb{R} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \mathbb{R} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in G_2^+(V)$.
- It turns out that the **orbit** M of the action of $\mathrm{SO}(3)$ on $G_2^+(V)$ through Z_0 is **totally geodesic**, and **isometric** to S^2. It is neither a complex nor a totally real submanifold of $Q(V, \beta)$, and is therefore the **totally geodesic 2-sphere that we seek**.