Weakly Einstein Hypersurfaces in Spaces of Constant Curvature

(joint work with Ruy Tojeiro de Figueirido Jr)

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Symmetry and Shape 2019

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Motivation

Einstein metrics are critical metrics for the Hilbert-Einstein functional, $\mathcal{E} : g \mapsto \int_M \tau dv_g$, restricted to constant volume metrics

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$\underline{dimM = 2}$: Gauss-Bonnet Theorem

$$\int_M \tau dv_g = 2\pi\chi(M)$$

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$\underline{dimM = 2}$: Gauss-Bonnet Theorem

$$\int_M \tau dv_g = 2\pi\chi(M)$$

Universal 2-dimensional curvature identity

$$\rho = \frac{\tau}{2} g$$

A 4-dimensional curvature identity

Quadratic curvature functional

$$\mathcal{F}_{a,b,c} : g \mapsto \int_M \{a||R||^2 - 4b||\rho||^2 + c\tau^2\} dv_g$$

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$\underline{dimM = 4:}$ Chern-Gauss-Bonnet Theorem

$$\int_M \{||R||^2 - 4||\rho||^2 + \tau^2\} dv_g = 32\pi^2 \chi(M)$$

A 4-dimensional curvature identity

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4-dimensional curvature identity

$$\left(\check{R} - \frac{\|R\|^2}{4} g\right) + \tau \left(\rho - \frac{\tau}{4} g\right) - 2\left(\check{\rho} - \frac{\|\rho\|^2}{4} g\right) - 2\left(R[\rho] - \frac{\|\rho\|^2}{4} g\right) = 0$$

$$\check{R}_{ij} = R_{iabc} R_j^{abc}, \quad \check{\rho}_{ij} = \rho_{ia}\rho_j^a, \quad R[\rho]_{ij} = R_{iabj}\rho^{ab}$$

M. Berger, Quelques formules de variation pour une structure riemannienne, *Ann. Sci. Éc. Norm. Super. (4)* **3** (1970), 285–294

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Weakly Einstein Conditions

$$\left(\check{R} - \frac{\|R\|^2}{4} g\right) + \tau\left(\rho - \frac{\tau}{4} g\right) - 2\left(\check{\rho} - \frac{\|\rho\|^2}{4} g\right) - 2\left(R[\rho] - \frac{\|\rho\|^2}{4} g\right) = 0$$

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Definition

A non–Einstein Riemannian manifold (M, g) is said to be

1. $\check{R}$-Einstein if $\check{R} = \frac{\|R\|^2}{n} g$.
2. $\check{\rho}$-Einstein if $\check{\rho} = \frac{\|\rho\|^2}{n} g$.
3. $R[\rho]$-Einstein if $R[\rho] = \frac{\|\rho\|^2}{n} g$.

4-dimensional examples

Y. Euh, J. Park, and K. Sekigawa

$M = M_1(c) \times M_2(-c)$ is $\check{R}$-Einstein, $\check{\rho}$-Einstein and $R[\rho]$-Einstein.

Y. Euh, J. Park, and K. Sekigawa, A curvature identity on a 4-dimensional Riemannian manifold, *Result. Math.* **63** (2013), 107–114.

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Locally Conformally Flat $\check{R}$-Einstein Manifolds

Let M a four dimensional locally conformally flat Riemannian manifold. Then M is $\check{R}$-Einstein if and only if it has vanishing scalar curvature.

An examples is

- $\mathbb{R} \times_f N(c)$, with $f(t)^2 = t^2 - 1, t, 1 - t^2$ if $c = 1, 0, -1$ respectively.

E. García-Río, A. Haji-Badali, ———, and M.E. Vázquez-Abal Locally conformally flat weakly-Einstein manifolds, *Arch. Math. (Basel)* **111** (2018), 549–559.

$\check{R}$-Einstein Hypersurfaces in a space form $\mathbb{Q}_c^5$

$$R^M(X,Y,Z,V) = cR^0(X,Y,Z,V) + \langle SY,Z\rangle\langle SX,V\rangle - \langle SX,Z\rangle\langle SY,V\rangle.$$

$$R^0(X,Y,Z,V) = \{\langle Y,Z\rangle\langle X,V\rangle - \langle X,Z\rangle\langle Y,V\rangle\}$$

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Algebraic Structure of the Shape Operator

$$\mathcal{S}^4 - (\|\mathcal{S}\|^2 - 2c)\mathcal{S}^2 - (2ncH)\mathcal{S} - \frac{1}{n}\{\|\mathcal{S}^2\|^2 - (\|\mathcal{S}\|^2 - 2c)\|\mathcal{S}\|^2 - 2c(nH)^2\}\,\mathrm{Id} = 0,$$

where $H = \frac{1}{n}\mathrm{tr}S$ is the mean curvature.

$\check{R}$-Einstein Hypersurfaces in $\mathbb{R}^{n+1}$

Theorem (Intrinsic Characterization)

A hypersurface in $\mathbb{R}^{n+1}$ is $\check{R}$-Einstein if and only if it is a warped product $\mathbb{R} \times_f \mathbb{S}^{n-1}$ with $f(t)^2 = t^2 - 1$.

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Locally conformally flat weakly-Einstein manifolds, *Arch. Math. (Basel)* **111** (2018), 549–559.

Theorem (Extrinsic Characterization)

A hypersurface in $\mathbb{R}^{n+1}$ is $\check{R}$-Einstein if and only if it is a rotation hypersurface over a plane catenary.

Two different principal curvatures in $\mathbb{S}^{n+1}$, $\mathbb{H}^{n+1}$

$$\dim V_\lambda \geq \dim V_\mu \geq 2$$

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In $\mathbb{S}^{n+1}$ we have

$$\mathbb{S}^m\left(\sin^{-2}\theta\right) \times \mathbb{S}^{n-m}\left(\cos^{-2}\theta\right), \text{ with } \tan^4\theta = \frac{m-1}{n-m-1}.$$

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In $\mathbb{H}^{n+1}$ we have

$$\mathbb{S}^m\left(\sinh^{-2}\theta\right) \times \mathbb{H}^{n-m}\left(\cosh^{-2}\theta\right), \text{ with } \tanh^4\theta = \frac{m-1}{n-m-1.}$$

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dim $V_\lambda = n-1$, dim $V_\mu = 1$
M is locally conformally flat

$$x_1'(s)^2 + x_1(s)x_1''(s) - 1 = 0,$$

S. Nishikawa, Y. Maeda, Conformally flat hypersurfaces in a conformally flat Riemannian manifold, *Tohoku Math. J.* **26** (1974), 159–168.

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dim $V_\lambda = n-1$, dim $V_\mu = 1$
M is locally conformally flat

$$x_1'(s)^2 + x_1(s)x_1''(s) - 1 = 0,$$

In $\mathbb{S}^{n+1}$, M is a rotation hypersurface over a curve $\alpha(t) = (x_1(t), x_2(t), x_3(t))$ with

Parametrization

$$x_2(s) = (1 - x_1^2)^{\frac{1}{2}} \sin\phi(s),$$
$$x_3(s) = (1 - x_1^2)^{\frac{1}{2}} \cos\phi(s),$$
$$\phi(s) = \int_0^s \frac{\sqrt{1 - x_1^2 - x_1'^2}}{1 - x_1^2} d\sigma.$$

M. do Carmo, M. Dajczaer, Rotation Hypersurface in Spaces of Constant Curvature, *Trans. Amer. Math. Soc* **277** (1983), 685–709.

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Parametrization (Parallels in a Lorentzian space)

$$x_2(s) = (1 + x_1^2)^{\frac{1}{2}} \sinh \phi(s),$$

$$x_3(s) = (1 + x_1^2)^{\frac{1}{2}} \cosh \phi(s),$$

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Parametrization (Parallels in a Riemannian space)

$$x_2(s) = (-1 + x_1^2)^{\frac{1}{2}} \sin\phi(s),$$

$$x_3(s) = (-1 + x_1^2)^{\frac{1}{2}} \cos\phi(s),$$

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$$x_1'(s)^2 + x_1(s)x_1''(s) + 1 = 0,$$

Parametrization (Parallels in a degenerate space)

$$x_3'(s)x_1(s) - x_1'(s)x_3(s) = \sqrt{x_1(s) - x_1'(s)}$$

$$x_3(s) = x_1 \int_0^s \frac{\sqrt{x_1 - x_1'}}{x_1^2} d\sigma.$$

Three different principal curvatures in $\mathbb{S}^5$ and $\mathbb{H}^5$

Algebraic characterization for S

$$2ncH + (\lambda_\alpha + \lambda_\beta)\left(||S||^2 - 2c - \lambda_\alpha^2 - \lambda_\beta^2\right) = 0.$$

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In $\mathbb{S}^5$, solutions are

$$S = \operatorname{diag}\left[0, 0, \frac{-2}{\gamma}, \gamma\right] \quad \text{and} \quad S = \operatorname{diag}\left[\lambda, \lambda, \frac{-1+\sqrt{1-\lambda^4}}{\lambda}, \frac{-1-\sqrt{1-\lambda^4}}{\lambda}\right].$$

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Three different principal curvatures in $\mathbb{S}^5$ and $\mathbb{H}^5$

Algebraic Structure:

$$S = \text{diag}\left[0, 0, \frac{\pm 2}{\gamma}, \gamma\right]$$

Lemma 1

There does not exist a hypersurface in $\mathbb{S}^5$ (respectively $\mathbb{H}^5$) with 0 as a principal curvature of multiplicity two and two others simple principal curvatures μ and γ satisfying $\mu\gamma = -2$ (respectively $\mu\gamma = 2$).

Three different principal curvatures in $\mathbb{S}^5$ and $\mathbb{H}^5$

Algebraic Structure:

$$S = \operatorname{diag}\left[\lambda, \lambda, \frac{\mp 1 + \sqrt{1-\lambda^4}}{\lambda}, \frac{\mp 1 - \sqrt{1-\lambda^4}}{\lambda}\right]$$

Lemma 2

Let $M^n \hookrightarrow \mathbb{Q}_c^{n+1}$ be a hypersurface with principal curvatures λ, μ_1 and μ_2 where λ has multiplicity $r \geq 2$ and μ_1 and μ_2 are simple. Assume that $\mu_i = \mu_i(\lambda)$ for $i = 1, 2$. Then M is a rotation hypersurface over an umbilical-free surface $L^2 \hookrightarrow \mathbb{Q}_c^3$.

Three different principal curvatures in $\mathbb{S}^5$ and $\mathbb{H}^5$

Algebraic Structure:

$$S = \operatorname{diag}\left[\lambda, \lambda, \frac{\mp 1 + \sqrt{1-\lambda^4}}{\lambda}, \frac{\mp 1 - \sqrt{1-\lambda^4}}{\lambda}\right]$$

Corollary

Let $M^4 \hookrightarrow \mathbb{Q}^5_{\pm 1}$ a hypersurface with principal curvatures λ, μ and γ where λ has multiplicity two and μ and γ are simple and depends on λ. Assume that

$$\mu\gamma = \lambda^2 \quad \text{and} \quad \mu + \gamma = \mp\frac{2}{\lambda}.$$

Then M is a rotation hypersurface over an umbilic-free surface $g : L^2 \hookrightarrow \mathbb{Q}^3_c$, $g = (g_1, g_2, g_3, g_4)$, such that the following conditions hold:

1. g_1 is a harmonic function on L^2.
2. $\pm 1 - ||\operatorname{grad} g_1||^2 = K g_1^2$, where K is the Gaussian curvature of L^2.

Four different principal curvatures in $\mathbb{S}^5$

$$\mathcal{S}^4 - (\|\mathcal{S}\|^2 - 2c)\mathcal{S}^2 - (2ncH)\mathcal{S} - \frac{1}{n}\{\|\mathcal{S}^2\|^2 - (\|\mathcal{S}\|^2 - 2c)\|\mathcal{S}\|^2 - 2c(nH)^2\}\,\mathrm{Id} = 0,$$

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$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0 = nH$$

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$$\lambda_1 = \sqrt{\frac{(||S||^2 - 2) + \sqrt{||S^2||^2 - 2(||S||^2 - 2)}}{2}} = -\lambda_2$$

$$\lambda_3 = \sqrt{\frac{(||S||^2 - 2) - \sqrt{||S^2||^2 - 2(||S||^2 - 2)}}{2}} = -\lambda_4,$$

and so

$$||S||^2 = 2\lambda_1^2 + 2\lambda_3^2 = 2(||S||^2 - 2),$$

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$$||S||^2 = 2\lambda_1^2 + 2\lambda_3^2 = 2(||S||^2 - 2),$$

$$||S||^2 = 4$$

Four different principal curvatures in $\mathbb{S}^5$

Theorem

The only minimal submanifold into the sphere with $||S||^2 = 4$ is the product $\mathbb{S}^m\left(\frac{\sqrt{m}}{2}\right) \times \mathbb{S}^{4-m}\left(\frac{\sqrt{4-m}}{2}\right)$.

S-S. Chern, M. do Carmo, S. Kobayashi, Minimal Submanifolds of a Sphere with Second Fundamental Form of Constant Length, *Springer-Verlag, Berlin and New York* (1970), 59–75.

Four different principal curvatures in $\mathbb{S}^5$

Theorem

The only minimal submanifold into the sphere with $||S||^2 = 4$ is the product $\mathbb{S}^m\left(\frac{\sqrt{m}}{2}\right) \times \mathbb{S}^{4-m}\left(\frac{\sqrt{4-m}}{2}\right)$.

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There are no examples of $\check{R}$-Einstein hypersurfaces in $\mathbb{S}^5$ with four different principal curvatures.

Four different principal curvatures in $\mathbb{H}^5$

Lemma

Let M be a minimal $\check{R}$-Einstein hypersurface in $\mathbb{H}^{n+1}$. Then M has two different principal curvatures λ and μ, where $\mu = -\lambda$.

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$$(\lambda_\alpha - \lambda_\beta)(\lambda_\alpha + \lambda_\beta)\{\|S\|^2 + 2 - (\lambda_\alpha^2 + \lambda_\beta^2)\} = 0,$$

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$$(\lambda_\alpha - \lambda_\beta)(\lambda_\alpha + \lambda_\beta)\{\|S\|^2 + 2 - (\lambda_\alpha^2 + \lambda_\beta^2)\} = 0,$$

$$2 + (m_\alpha - 1)\lambda_\alpha^2 + (m_\beta - 1)\lambda_\beta^2 + \sum_{i \neq \alpha, \beta}^{n} m_i \lambda_i^2.$$

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