Regularized mean curvature flow in a Hilbert space and its application to the Gauge theory

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Symmetry and shape
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1. Regularized mean curvature flow

Proper Fredholm submanifold

V : (separable) Hibert space

M : Hilbert manifold

$f : M \hookrightarrow V$: immersion

Definition(C.L. Terng, 1989)

$f : M \hookrightarrow V$: proper Fredholm

$$\underset{\text{def}}{\Longleftrightarrow} \begin{cases} \bullet\ \operatorname{codim} M < \infty \\ \bullet\ \exp^\perp|_{B^\perp{}_r(M)} \text{ : proper map } (\forall\, r > 0) \\ \bullet\ \exp^\perp_{*v} \text{ : Fredholm operator } (\forall\, v \in T^\perp M) \end{cases}$$

Properties of proper Fredholm submanifolds

$f : M \hookrightarrow V$: **proper Fredholm**

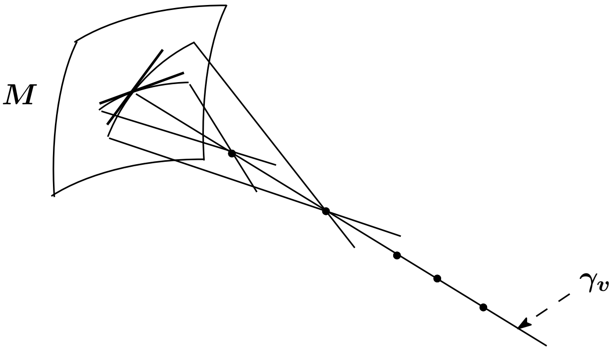
A_v : **the shape operator of** f **for** $v(\in T^\perp M)$

Fact

A_v : **compact operator**

The good focal structure of a proper Fredholm submnaiofold

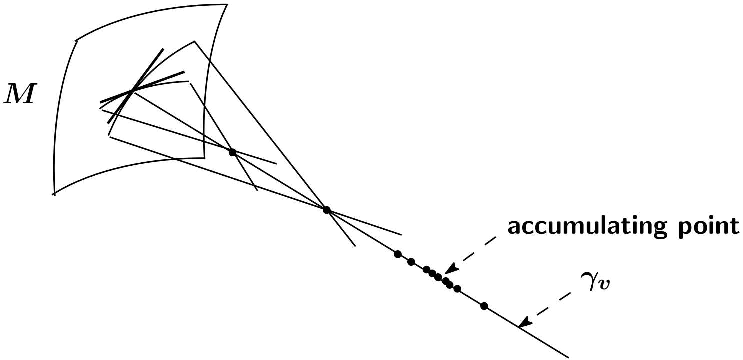
M : proper Fredholm submanifold-case



The set of all focal points of M along γ_v has no accumulating point and the multiplicity of each focal point is finite.

The focal structure of a general Hilbert submanifold

M : (general) Hilbert submanifold-case



The set of all focal points of M along γ_v is possible to have accumulating points and the multiplicity of each focal point is possible to be infinite.

Regularizable submanifolds

$f : M \hookrightarrow V$: **proper Fredholm**

Definition(Heintze-Liu-Olmos, 2006)

$f : M \hookrightarrow V$: **regularizable** $\underset{\text{def}}{\Longleftrightarrow}$

$$\begin{cases} \forall\, v \in T^{\perp}M, \\ \exists\, \mathrm{Tr}_r A_v\,(<\infty),\quad \exists\, \mathrm{Tr}(A_v^2)\,(<\infty) \\ \left(\begin{array}{l} \mathrm{Tr}_r A_v := \displaystyle\sum_{i=1}^{\infty}(\lambda_i+\mu_i) \\ (\mathrm{Spec}\, A_v = \{\mu_1 \le \mu_2 \le \cdot\cdot \le 0 \le \cdot\cdot \le \lambda_2 \le \lambda_1\}) \\ \mathrm{Tr}(A_v^2) := \displaystyle\sum_{i=1}^{\infty}\nu_i \\ (\mathrm{Spec}\, A_v^2 = \{\nu_1 \ge \nu_2 \ge \cdots > 0\}) \end{array} \right) \end{cases}$$

Regularized mean curvature vector (codimension 1-case)

$f : M \hookrightarrow V$: **regularizable hypersurface**

ξ : **a unit normal vector field of** f

Definition

$H^s := \mathrm{Tr}_r A_\xi$ **regularized mean curvature**

$H := \mathrm{Tr}_r A_\xi \cdot \xi$ **regularized mean curvature vector**

$\nexists$ Regularized mean curvature vector (codimension$\geq$ 2-case)

For a regularizable submanifold of codimension ≥ 2, its regularized mean curvature vector cannot be defined.

$$\mathrm{Tr}_r(A_{\xi_1+\xi_2}) \neq \mathrm{Tr}_r\, A_{\xi_1} + \mathrm{Tr}_r\, A_{\xi_2}$$

$\omega_u : T_u^{\perp}M \to \mathbb{R}$ $(\Leftrightarrow\ \omega_u(\xi) := \mathrm{Tr}_r\, A_\xi)$ **is not linear.**

Hence

$\nexists\, H_u \in T_u^{\perp}M$ s.t. $\langle H_u, \xi\rangle = \omega_u(\xi) \quad (\forall\, \xi \in T_u^{\perp}M)$.

$\not\exists$ Regularized mean curvature vector (codimension≥ 2-case)

<u>Remark</u> ω_u : linear ($\forall\, u \in M$) $\Rightarrow$ H is defined.

$\phi : H^0([0,1],\mathfrak{g}) \to G$: the parallel transport map
(G : compact semi-simple Lie group)

$\overline{M}$: compact submanifold in G

$\phi^{-1}(\overline{M})(\subset H^0([0,1],\mathfrak{g}))$ is a regularizable submanifold.
For $\phi^{-1}(\overline{M})$, ω_u is linear for any $u \in \phi^{-1}(\overline{M})$.
Hence its regularized mean curvature vector is defined.

Regularized mean curvature flow

$\{f_t : M \hookrightarrow V\}_{t\in[0,T)}$: C^∞-family of regularizable hypersurfaces

H_t : the regularized mean curvature vector of f_t

Definition

$\{f_t\}_{t\in[0,T)}$: **regularized mean curvature flow**

$$\underset{\text{def}}{\Longleftrightarrow} \frac{\partial F}{\partial t} = H_t (= (\triangle_t)_r f_t) \ (0 \leq t < T)$$

$$(F(x,t) := f_t(x) \ ((x,t) \in M \times [0,T)))$$

2. Collapsing theorem

Setting

V : (separable) Hilbert space

$\mathcal{G}$: Hilbert Lie group

$\mathcal{G} \curvearrowright V$: almost free isometric action satisfying

(MO) $\mathcal{G}$-orbits are minimal reg. submanifolds

$\left(\text{“minimal”} \underset{\text{def}}{\iff} \operatorname{Tr}_r A_\xi = 0 \ (\forall \xi \in T^\perp M) \right)$

$\phi : V \hookrightarrow V/\mathcal{G}$: the orbit map

g_N : the Riemannian orbi-metric of $N := V/\mathcal{G}$

s.t. $\begin{cases} \phi \text{ is a Riemannian orbi} - \text{submersion} \\ \text{of } (V, \langle\ ,\ \rangle) \text{ onto } (N, g_N) \end{cases}$

Example

Example

G/K : symmetirc space of compact type

$\mathfrak{g} := \mathrm{Lie}\, G$

$H^0([0,a],\mathfrak{g})$ (The space of all H^0-connections of $P_o := [0,a] \times G \to [0,a]$)

$H^1([0,a],G)$ (The group of all H^1-gauge transformations of P_o)

$$H^1([0,a],G) \curvearrowright H^0([0,a],\mathfrak{g})$$

$$:\underset{\text{def}}{\Longleftrightarrow} (\mathrm{g}\cdot u)(t) := \mathrm{Ad}(\mathrm{g}(t))(u(t)) - (R_{g(t)})_*^{-1}(g'(t))$$

$$(\mathrm{g} \in H^1([0,a],G),\ u \in H^0([0,a],\mathfrak{g}))$$

(This action is almost free and isometric.)

Example

$$P(G, \Gamma \times K) := \{\mathrm{g} \in H^1([0,a], G) \mid (\mathrm{g}(0), \mathrm{g}(a)) \in \Gamma \times K\}$$

(Γ : a finite subgroup of G)

Fact

- $P(G, \Gamma \times K) \curvearrowright H^0([0,a], \mathfrak{g})$ **is an almost free and isometric action s.t. the condition (MO).**
- $H^0([0,a], \mathfrak{g}) / P(G, \Gamma \times K) \cong \Gamma \backslash G / K$.

Setting (continued)

$\mathcal{G} \curvearrowright V$: **almost free isometric action satisfying**

(MO) $\mathcal{G}$**-orbits are minimal reg. submanifolds**

$f : M \hookrightarrow V$: **regularizable hypersurface**

s.t. $\begin{cases} f(M) \ : \ \mathcal{G}\text{-invariant} \\ \overline{M} := f(M)/\mathcal{G} \ : \ \text{compact} \end{cases}$

Setting (continued)

$(*_1)$ $\overline{M} \subset B_{\frac{\pi}{b}}(x_0)$ and $\exp_{x_0}|_{B^T_{\frac{\pi}{b}}(0)}$: injective

$(*_2)$ $b^2(1-\alpha)^{-2/n}(\omega_n^{-1} \cdot \mathrm{Vol}_{g_N}(\overline{M}))^{2/n} \leq 1$

$(0 < \alpha < 1)$

$\Bigg($ $b := \sqrt{\overline{K}}$ ($\overline{K}$: the max. sec. curv. of $N := V/\mathcal{G}$)

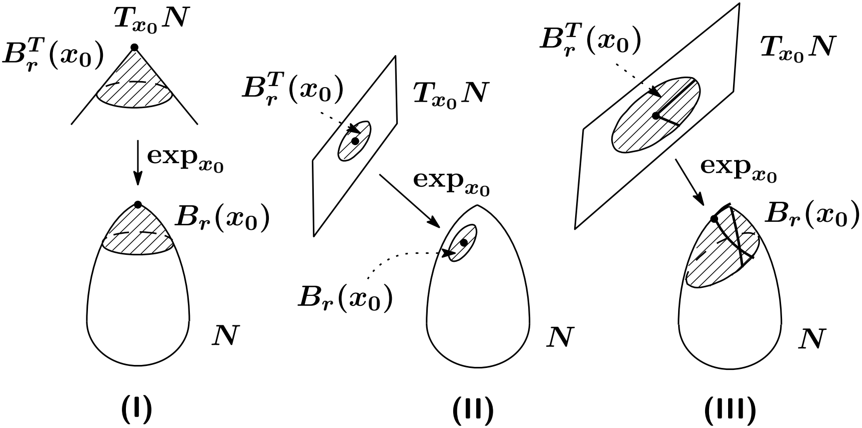
$B_{\frac{\pi}{b}}(x_0)$: the geodesic ball of radius $\frac{\pi}{b}$ centered at some point $x_0 \in N$

$B^T_{\frac{\pi}{b}}(0)$: the ball of radius $\frac{\pi}{b}$ centered at $0 \in T_{x_0}N$

ω_n : the volume of the Euclidean unit n-ball

$(n := \dim N - 1)$ $\Bigg)$

About the injectivity in $(*_1)$



(I),(II) : $\exp_{x_0}|_{B_r^T(x_0)}$: **injective**

(III) : $\exp_{x_0}|_{B_r^T(x_0)}$: **not injective**

Setting (continued)

$$(*_3) \quad (H^s)^2 h_{\mathcal{H}} > 2n^2 L g_{\mathcal{H}}$$

(horizontally convexity condition)

$$\left(\begin{array}{l}
g_{\mathcal{H}} \ : \ \text{the horizontal comp. of the induced metric on } M \\
h_{\mathcal{H}} \ : \ \text{the horizontal comp. of the second fund. form of } M \\
\mathcal{A}^{\phi}(\in \Gamma(\mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{V})) \underset{\text{def}}{\Leftrightarrow} \mathcal{A}^{\phi}_X Y := (\widetilde{\nabla}_X Y)_{\mathcal{V}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (X, Y \in \Gamma(\mathcal{H})) \\
L := \sup_{u \in V} \max_{(X_1, \cdots, X_5) \in (\mathcal{H}_1)^5_u} |\langle \mathcal{A}^{\phi}_{X_1}((\widetilde{\nabla}_{X_2} \mathcal{A}^{\phi})_{X_3} X_4), \ X_5 \rangle| \\
\qquad\qquad\qquad\qquad \left(\ (\mathcal{H}_1)_u := \{X \in \mathcal{H}_u \mid \ ||X|| = 1\} \ \right)
\end{array}\right)$$

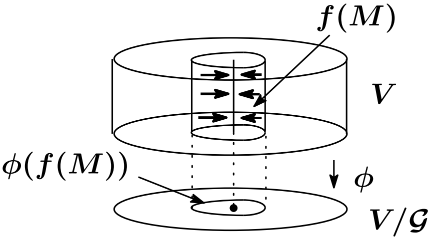
Collapsing theorem

$f(M)$: $\mathcal{G}$-invariant, $f(M)/\mathcal{G} = \phi(f(M))$: compact

$f(M)$ satisfies $(*_1)$, $(*_2)$, $(*_3)$

Theorem A(Collapsing theorem).

The reg. m.c.f. starting from $f(M)$ collapses to a $\mathcal{G}$-orbit in finite time.



3. Applications to the gauge theory

The space of H^0-connections of the principal bundle

$\pi : P \to B$: G-bundle

$$\left(\begin{array}{ll} B : & \text{compact Riemannian manifold} \\ G : & \text{compact semi-simple Lie group} \end{array} \right)$$

$\mathcal{A}_P^{H^0}$: the (affine) Hilbert space of all H^0-connections of P

$$\begin{array}{ccc} \mathcal{A}_P^{H^0} & \approx & T_{\omega_0}\mathcal{A}_P^{H^0} = \Omega_{\mathcal{T},1}^{H^0}(P, \mathfrak{g}) = \Gamma^{H^0}(T^*B \otimes \mathrm{Ad}(P)) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \omega & \longleftrightarrow & \widehat{A}(:= \omega - \omega_0) \end{array}$$

Holonomy map

$$c : [0, a] \to B \quad : \; C^\infty\text{-loop}$$

P_c^ω : the parallel translation along c with respect to ω

Definition

$$\mathrm{hol}_c \; : \; \mathcal{A}_P^{H^0} \to G \underset{\text{def}}{\Longleftrightarrow} P_c^\omega(u) = u \cdot \mathrm{hol}_c(\omega) \;\; (\forall\, u \in P_{c(0)})$$

Remark $\{\mathrm{hol}_c(\omega) \,|\, c \in \Omega_x^\infty(B)\}$ is the holonomy group of ω at x.

Construction of a map of $\mathcal{A}_P^{H^0}$ onto $H^0([0,a],\mathfrak{g})$

$$c:[0,a]\to B \quad : \text{unit speed } C^\infty\text{-loop}$$

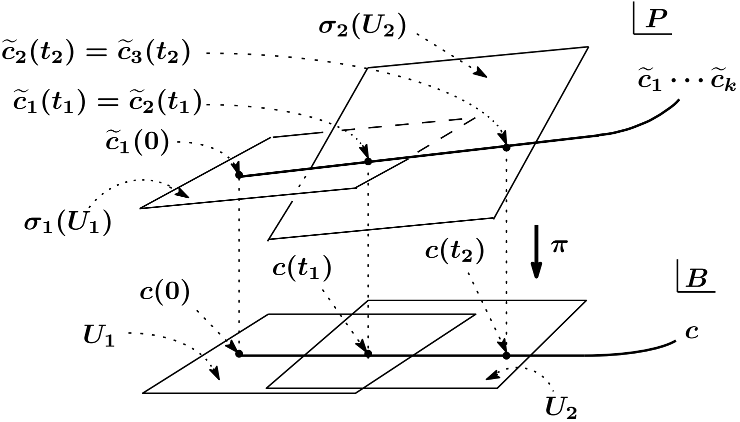
We take a division $0=t_0<t_1<t_2<\cdots<t_k=a$ of $[0,a]$ and a family $\{\varphi_i : P|_{U_i}\to U_i\times G\}_{i=1}^k$ of local trivializations of P satisfying the following condition:

$$\left\{\begin{array}{l} \bullet\ \ c([t_{i-1},t_i])\subset U_i\ \ (i=1,\cdots,k) \\ \bullet\ \ \widetilde{c}_1\cdots\widetilde{c}_k:[0,a]\to P \text{ is a } C^1\text{-loop} \\ \left(\begin{array}{l} \widetilde{c}_i \underset{\text{def}}{\Leftrightarrow} \widetilde{c}_i(t):=\varphi_i^{-1}(c(t),e)\ \ (t\in[t_{i-1},t_i]) \\ \widetilde{c}_1\cdots\widetilde{c}_k \underset{\text{def}}{\Leftrightarrow} \widetilde{c}_1\cdots\widetilde{c}_k\,|_{[t_{i-1},t_i]}=\widetilde{c}_i\ \ (i=1,\cdots,k) \end{array}\right) \end{array}\right.$$

Remark $\quad \widetilde{c}_i(t)=\sigma_i(c(t))$

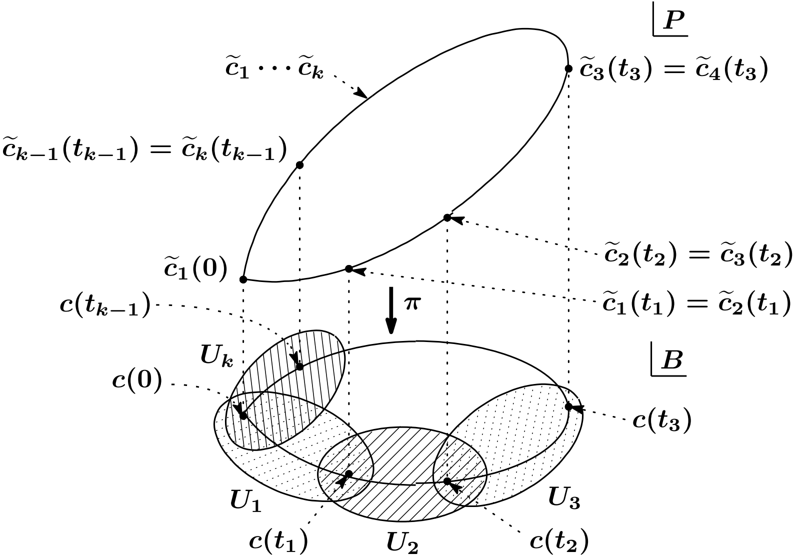
($\sigma_i:U_i\to P$: the section giving the local trivialization φ_i)

Construction of a map of $\mathcal{A}_P^{H^0}$ onto $H^0([0,a],\mathfrak{g})$



$$\sigma_i : U_i \to P \underset{\text{def}}{\Longleftrightarrow} \sigma_i(x) := \varphi_i^{-1}(x,e) \quad (x \in U)$$

Construction of a map of $\mathcal{A}_P^{H^0}$ onto $H^0([0,a],\mathfrak{g})$



Construction of a map of $\mathcal{A}_P^{H^0}$ onto $H^0([0,a],\mathfrak{g})$

$$c_i := c|_{[t_{i-1},t_i]}, \quad P_o^i := [t_{i-1},t_i] \times G \ (i = 1,\cdots,k)$$

$$\iota_{c_i} : c_i^* P \hookrightarrow P \underset{\text{def}}{\Longleftrightarrow} \iota_{c_i}(t,u) := u \ \ ((t,u) \in c_i^* P)$$

$$\varphi_i^{c_i} : c_i^* P \underset{\cong}{\rightarrow} P_o^i \underset{\text{def}}{\Longleftrightarrow} \varphi_i^{c_i}(t,u) := (t, \mathrm{pr}_2(\varphi_i(u)))$$

$$((t,u) \in c_i^* P)$$

Definition

$$\mu_{\varphi_i}^{c_i} : \mathcal{A}_P^{H^0} \rightarrow H^0([t_{i-1},t_i],\mathfrak{g})$$

$$\underset{\text{def}}{\Longleftrightarrow} \mu_{\varphi_i}^{c_i}(\omega)(t) := ((\iota_{c_i} \circ (\varphi_i^{c_i})^{-1})^* \widehat{A})_{(t,e)}(c_e'(t))$$

$$(\widehat{A} := \omega - \omega_0, \quad c_e(t) := (t,e) \ \ (t \in [0,a]))$$

Construction of a map of $\mathcal{A}_P^{H^0}$ onto $H^0([0,a],\mathfrak{g})$

Definition

$$\mu^{c_1,\cdots,c_k}_{\varphi_1,\cdots,\varphi_k} : \mathcal{A}_P^{H^0} \to H^0([0,a],\mathfrak{g})$$
$$\underset{\text{def}}{\Longleftrightarrow} \ \mu^{c_1,\cdots,c_k}_{\varphi_1,\cdots,\varphi_k}(\omega)|_{[t_{i-1},t_i]} = \mu^{c_i}_{\varphi_i}(\omega) \quad (\omega \in \mathcal{A}_P^{H^0})$$
$$(i = 1,\cdots,k)$$

Metrics of $\mathcal{A}_P^{H^0}$, $H^0([0,a],\mathfrak{g})$ and G

$$T_\bullet \mathcal{A}_P^{H^0} = \Gamma^{H^1}(T^*B \otimes \mathrm{Ad}(P))$$

$$\langle\ ,\ \rangle_{\mathcal{A}} : T_\bullet \mathcal{A}_P^{H^0} \times T_\bullet \mathcal{A}_P^{H^0} \to \mathbb{R}$$

$$\underset{\text{def}}{\Longleftrightarrow} \langle A_1, A_2 \rangle_{\mathcal{A}} := \int_{x\in M} \langle (A_1)_x, (A_2)_x \rangle_{B,\mathfrak{g}}\, dv_B$$

$$(A_1, A_2 \in T_\bullet \mathcal{A}_P^{H^0})$$

($\langle\ ,\ \rangle_{B,\mathfrak{g}}$: the fibre metric of $T^*B \otimes \mathrm{Ad}(P)$ induced from the metric of B and the Killing form $\langle\ ,\ \rangle_{\mathfrak{g}}$ of $\mathfrak{g}$)

Metrics of $\mathcal{A}_P^{H^0}$, $H^0([0,a],\mathfrak{g})$ and G

$$\langle\ ,\ \rangle_{\mathcal{P}} : H^0([0,a],\mathfrak{g}) \times H^0([0,a],\mathfrak{g}) \to \mathbb{R}$$

$$\underset{\text{def}}{\Longleftrightarrow} \langle u,v\rangle_{\mathcal{P}} := \int_0^a \langle u,v\rangle_{\mathfrak{g}}\, dv_M$$

$$(u,v \in H^0([0,a],\mathfrak{g}))$$

$\langle\ ,\ \rangle_G$: the bi-invariant metric induced from $\langle\ ,\ \rangle_{\mathfrak{g}}$

$$\langle\ ,\ \rangle_{G,a} := a\langle\ ,\ \rangle_G$$

Results for $\mu^{c_1\cdots c_k}_{\varphi_1\cdots\varphi_k}$

Proposition 3.1.

(i) $\mu^{c_1\cdots c_k}_{\varphi_1\cdots\varphi_k} : (\mathcal{A}_P^{H^0}, \langle\ ,\ \rangle_{\mathcal{A}}) \to (H^0([0,a],\mathfrak{g}), \langle\ ,\ \rangle_{\mathcal{P}})$ **is a Riemannian submersion with totally geodesic fibre.**

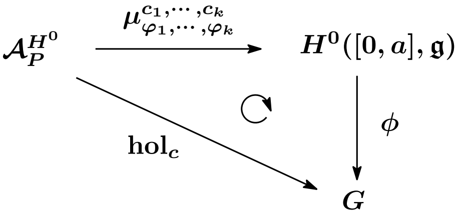
(ii) $\phi \circ \mu^{c_1\cdots c_k}_{\varphi_1\cdots\varphi_k} = \mathrm{hol}_c$.

$$\phi : H^0([0,a],\mathfrak{g}) \to G \quad \text{parallel transport map}$$

$$\underset{\text{def}}{\Longleftrightarrow} \ \phi(u) := g_u(a) \quad (u \in H^0([0,a],\mathfrak{g}))$$

$$\left(g_u \in H^1([0,a],G) \ \text{s.t.} \ \begin{cases} g_u(0) = e \\ (R_{g_u(t)})_*^{-1}(g_u'(t)) = u(t) \end{cases} \right)$$

Results for $\mu_{\varphi_1 \cdots \varphi_k}^{c_1 \cdots c_k}$



Results for hol_c

Theorem B.

$\mathrm{hol}_c : (\mathcal{A}_P^{H^0}, \langle\ ,\ \rangle_{\mathcal{A}}) \to (G, \langle\ ,\ \rangle_{G,a})$ **is a Riemannian submersion with minimal regularizable fibre.**

Theorem C.

$L(\subset G)$ **: equifocal** $\iff$ $\mathrm{hol}_c^{-1}(L)$ **: isoparametric**

- **The notion of an equifocal submanifold in symmetric spaces was introduced by C.L. Terng and G. Thorbergsson in 1995.**
- **The notion of an isoparametric submanifold in a Hilbert space was introduced by C.L. Terng in 1989.**

Holonomy concentration theorem

From Theorem A and Proposition 3.1, we obtain

Theorem D(Holonomy concentration theorem along r.m.c.f.)

$c : [0, a] \to B$: **unit speed C^∞-loop**

$\overline{M}$: **a strongly convex closed hypersurface in G satisfying $(*_1)$ and $(*_2)$**

Then the following statement (i),(ii) and (iii) hold :

(i) $\mathcal{B} := \mathrm{hol}_c^{-1}(\overline{M})$ **is a reg. hypersurface.**

(ii) **The reg. m.c.f. $\{\mathcal{B}_t\}_{t\in[0,T)}$ starting from $\mathcal{B}$ exists.**

(iii) **As $t \to T$, $\mathrm{hol}_c(\mathcal{B}_t)$ collapses to a one-point set.**

As $t \to T$, the holonomy elements of the connections belonging to $\mathcal{B}_t$ along c concentrate a point of G.

Recall of the conditions $(*_1)$ and $(*_2)$

$(*_1)$ $\overline{M} \subset B_{\frac{\pi}{b}}(x_0)$ and $\exp_{x_0}|_{B^T_{\frac{\pi}{b}}(0)}$: injective

$(*_2)$ $b^2(1-\alpha)^{-2/n}(\omega_n^{-1} \cdot \mathrm{Vol}_{g_N}(\overline{M}))^{2/n} \leq 1$

$(0 < \alpha < 1)$

$b := \sqrt{\overline{K}}$ ($\overline{K}$: the max. sec. curv. of $N := V/\mathcal{G}$)

$B_{\frac{\pi}{b}}(x_0)$: the geodesic ball of radius $\frac{\pi}{b}$ centered at some point $x_0 \in N$

$B^T_{\frac{\pi}{b}}(0)$: the ball of radius $\frac{\pi}{b}$ centered at $0 \in T_{x_0}N$

ω_n : the volume of the Euclidean unit n-ball

$(n := \dim N - 1)$

4. Future plan

Flow approach to the singular point of the moduli space of self-dual connections

$\pi : P \to B$: $SU(2)$-bundle $(\dim B = 4)$

$$\begin{array}{ccc} \mathcal{B} \subset \mathcal{A}_P^{H^l} & \xrightarrow{\mu^{c_1,\cdots,c_k}_{\varphi_1,\cdots,\varphi_k}} & H^l([0,a],\mathfrak{su}(2)) \\ & \underset{\mathrm{hol}_c}{\searrow} \quad \circlearrowright & \downarrow \phi \\ & & SU(2) \supset \overline{M} \end{array}$$

small geodesic sphere
center at e

$\mathcal{B} := \mathrm{hol}_c^{-1}(\overline{M})$: regularizable submanifold

$\exists \{\mathcal{B}_t\}_{t \in [0,T)}$: the regularized mean curvature flow s.t. $\mathcal{B}_0 = \mathcal{B}$

Flow approach to the singular point of the moduli space of self-dual connections

$$
\begin{array}{ccccccc}
 & & \mathcal{B}_t \cap \mathcal{SD}_P^{H^l} & & \mathcal{B}_t \cap \mathcal{YM}_P^{H^l} & & \mathcal{B}_t \\
 & & \cap & & \cap & & \cap \\
 & & \mathcal{SD}_P^{H^l} & \subset & \mathcal{YM}_P^{H^l} & \subset & \mathcal{A}_P^{H^l} \\
 & & \downarrow & & \downarrow & & \downarrow \\
(\mathcal{B}_t \cap \mathcal{SD}_P^{H^l})/\mathcal{G}_P^{H^{l+1}} & \subset & \mathcal{M}_P^{\mathcal{SD},l} & \subset & \mathcal{M}_P^{\mathcal{YM},l} & \subset & \mathcal{M}_P^l
\end{array}
$$

$$
\left(\begin{array}{c}
\mathcal{M}_P^l := \mathcal{A}_P^{H^l}/\mathcal{G}_P^{H^{l+1}},\ \mathcal{M}_P^{\mathcal{YM},l} := \mathcal{YM}_P^{H^l}/\mathcal{G}_P^{H^{l+1}} \\
\mathcal{M}_P^{\mathcal{SD},l} := \mathcal{SD}_P^{H^l}/\mathcal{G}_P^{H^{l+1}}
\end{array}\right)
$$

Flow approach to the singular point of the moduli space of self-dual connections

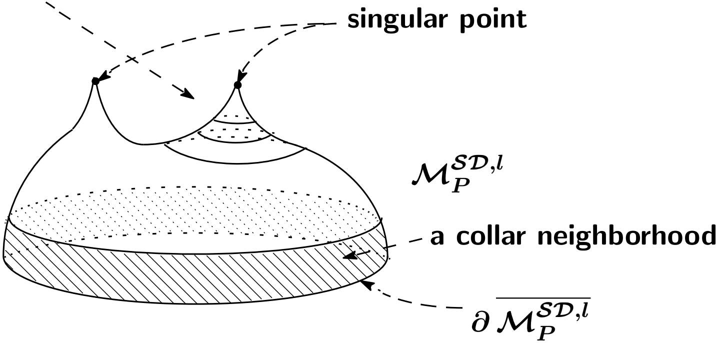
Question.

Can we find a unit speed C^∞-loop c such that

$$\{(\mathcal{B}_t \cap \mathcal{SD}_P^{H^l})/\mathcal{G}_P^{H^{l+1}}\}_{t\in[0,T)}$$

is a mean curvature flow collapsing to a singular point of $\mathcal{M}_P^{\mathcal{S}D,l}$?

Flow approach to the singular point of the moduli space of self-dual connections



We want to find a unit speed C^∞-loop c such that

$$\{(\mathcal{B}_t \cap \mathcal{SD}_P^{H^l})/\mathcal{G}_P^{H^{l+1}}\}_{t\in[0,T)}$$

is like this?

Why does this question arise?

Singular points of the moduli space are the gauge equivalence classes of reducible connections.

$$\mathcal{B}_t = \mathrm{hol}_c^{-1}(\overline{M}_t)$$

It is expected that, for a suitable loop c,

$$\overline{M}_t \to \{e\} \iff (\mathcal{B}_t \cap \mathcal{SD}_P^{H^l})/\mathcal{G}_P^{H^{l+1}} \to [\omega_{\mathrm{red}}] \ ?$$

In the case where $\overline{M}_t$ is the m.c.f. starting from a small geodesic sphere centered at e, $\overline{M}_t \to \{e\}$ and hence it is expected that, for a suitable loop c,

$$(\mathcal{B}_t \cap \mathcal{SD}_P^{H^l})/\mathcal{G}_P^{H^{l+1}} \to [\omega_{\mathrm{red}}].$$

Thank you for your attention!

Dear Professor Jürgen Berndt!

Congratulations on 60-th birthday!

With gratitude!

On the images of the Gauge orbits

$$\begin{array}{ccc} \mathcal{A}_P^{H^l} & \xrightarrow{\;\mathrm{g}\;} & \mathcal{A}_P^{H^l} \\ \mu^{c_1\cdots c_k}_{\varphi_1\cdots\varphi_k} \Big\downarrow & \circlearrowright & \Big\downarrow \mu^{c_1\cdots c_k}_{\varphi_1\cdots\varphi_k} \\ H^l([0,a],\mathfrak{g}) & \xrightarrow{\;\overline{\mathrm{g}}\;} & H^l([0,a],\mathfrak{g}) \\ \phi \Big\downarrow & \circlearrowright & \Big\downarrow \phi \\ G & \xrightarrow[\mathrm{Ad}(\overline{\mathrm{g}}(0))]{} & G \end{array}$$

$$\mathrm{g} \in \mathcal{G}_P^{H^{l+1}}, \quad \overline{\mathrm{g}} := \lambda^{c_1\cdots c_k}_{\varphi_1\cdots\varphi_k}(\mathrm{g}) \underset{\mathrm{def}}{:=} \widehat{\mathrm{g}} \circ \widetilde{c}_1 \cdots \widetilde{c}_k$$

Equivariance of the bridging map with the gauge action

$$
\begin{array}{ccc}
\mathcal{A}_P^{H^l} & \xrightarrow{\quad \mathrm{g} \quad} & \mathcal{A}_P^{H^l} \\
\Big\downarrow \mu^{c_1 \cdots c_k}_{\varphi_1 \cdots \varphi_k} & \circlearrowright & \Big\downarrow \mu^{c_1 \cdots c_k}_{\varphi_1 \cdots \varphi_k} \\
H^l([0,a],\mathfrak{g}) & \xrightarrow[\overline{\mathrm{g}} := \lambda^{c_1 \cdots c_k}_{\varphi_1 \cdots \varphi_k}(\mathrm{g})]{} & H^l([0,a],\mathfrak{g})
\end{array}
$$

$$\lambda^{c_1 \cdots c_k}_{\varphi_1 \cdots \varphi_k} : \mathcal{G}_P^{H^{l+1}} \to \Omega^{H^{l+1}}(G)$$

$$\left(\ \lambda^{c_1 \cdots c_k}_{\varphi_1 \cdots \varphi_k}((\mathcal{G}_P^{H^{l+1}})_{x_0}) \subset \Omega_e^{H^{l+1}}(G) \ \right)$$

- $\mu^{c_1 \cdots c_k}_{\varphi_1 \cdots \varphi_k}((\mathcal{G}_P^{H^{l+1}})_{x_0} \cdot \omega) \subset \Omega_e^{H^{l+1}}(G) \cdot \mu^{c_1 \cdots c_k}_{\varphi_1 \cdots \varphi_k}(\omega)$
- $\mu^{c_1 \cdots c_k}_{\varphi_1 \cdots \varphi_k}(\mathcal{G}_P^{H^{l+1}} \cdot \omega) \subset \Omega^{H^{l+1}}(G) \cdot \mu^{c_1 \cdots c_k}_{\varphi_1 \cdots \varphi_k}(\omega)$

On the images of the Gauge orbits

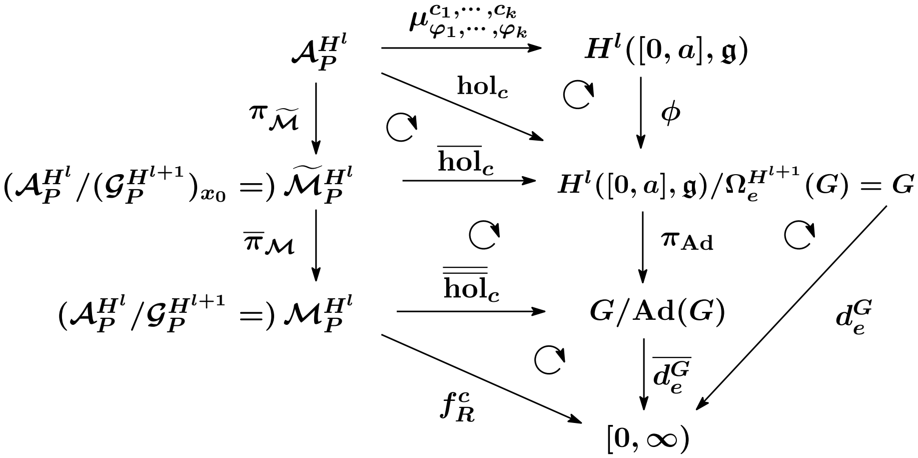
$$
\begin{array}{ccc}
H^l([0,a],\mathfrak{g}) & \xrightarrow{\ \overline{\mathbf{g}}\ } & H^l([0,a],\mathfrak{g}) \\
\phi \downarrow & \circlearrowright & \downarrow \phi \\
G & \xrightarrow[\mathrm{Ad}(\bar{g}(0))]{} & G
\end{array}
$$

- $\phi(\Omega_e^{H^{l+1}}(G)\cdot u) = \{\phi(u)\}$
- $\phi(\Omega^{H^{l+1}}(G)\cdot u) = \mathrm{Ad}(G)\cdot\phi(u)$

Hence

- $\mathrm{hol}_c((\mathcal{G}_P^{H^{l+1}})_{x_0}\cdot\omega) = \{\phi(\mu_{\varphi_1\cdots\varphi_k}^{c_1\cdots c_k}(\omega))\}$
- $\mathrm{hol}_c(\mathcal{G}_P^{H^{l+1}}\cdot\omega) \subset \mathrm{Ad}(G)\cdot\phi(\mu_{\varphi_1\cdots\varphi_k}^{c_1\cdots c_k}(\omega))$

An important function on the moduli space



An important function on the moduli space

$$\begin{array}{ccc}
\mathcal{SD}_P^{H^l} \subset \mathcal{A}_P^{H^l} & \xrightarrow{\mu^{c_1,\cdots,c_k}_{\varphi_1,\cdots,\varphi_k}} & H^l([0,a],\mathfrak{g}) \\
\downarrow \pi_{\widetilde{\mathcal{M}}} \quad \circlearrowright & \searrow \mathrm{hol}_c \quad \circlearrowright & \downarrow \phi \\
\widetilde{\mathcal{M}}_P^{\mathcal{SD},l} \subset \widetilde{\mathcal{M}}_P^{H^l} & \xrightarrow{\overline{\mathrm{hol}}_c} & H^l([0,a],\mathfrak{g})/\Omega_e^{H^{l+1}}(G) = G \\
\downarrow \overline{\pi}_{\mathcal{M}} & \circlearrowright & \downarrow d_e^G \\
\mathcal{M}_P^{\mathcal{SD},l} \subset \mathcal{M}_P^{H^l} & \xrightarrow{f_R^c} & [0,\infty)
\end{array}$$

$$(G = SU(2), \quad \mathfrak{g} = \mathfrak{su}(2))$$

Important fact

Fact

(i) $\mathcal{B}_t/\mathcal{G}_P^{H^{l+1}} = (f_R^c)^{-1}(r_t)\ \ (\exists\, r_t > 0)$.

(ii) $(\mathcal{B}_t \cap \mathcal{SD}_P^{H^l})/\mathcal{G}_P^{H^{l+1}} = (f_R^c)^{-1}(r_t) \cap \mathcal{M}_P^{\mathcal{SD},l}\ \ (\exists\, r_t > 0)$.

By using these facts, we will tackle the question.

Question.

Can we find a unit speed C^∞-loop c such that

$\{(\mathcal{B}_t \cap \mathcal{SD}_P^{H^l})/\mathcal{G}_P^{H^{l+1}}\}_{t\in[0,T)}$ **is a mean curvature flow?**

Groisser-Parker's result

B : compact oriented simply connected Riemannian 4-manifold whose intersection form is positive definite

$\pi : P \to B$: a $SU(2)$-bundle of instanton number $k \geq 1$

Theorem(Groisser-Parker)

(i) $(\mathcal{M}_P^{\mathcal{SD},l}, \langle\ ,\ \rangle)$ $(l \geq 2)$ is a $(8k-3)$-dim. singular Riemannian manifold with cone singularity.
(Cone points are the gauge equivalence classes of reducible connectons.)

Groisser-Parker's result

Theorem(Groisser-Parker) (continued)

(ii) A sufficiently small neighborhood U of a cone point p of $(\mathcal{M}_P^{\mathcal{SD},l}, \langle\ ,\ \rangle)$ is homeomorphic to the cone over $\mathbb{C}P^{4k-2}$, $\langle\ ,\ \rangle|_U$ is described as

$$\langle\ ,\ \rangle = dr^2 \cdot r^2(\mathrm{pr}^* g_0 + O(r^2)),$$

where r is the distance function from p, g_0 is the metric of $\mathbb{C}P^{4k-2}$ of constant holomorphic sectional curvature, pr is the projection of U onto $r^{-1}(\varepsilon)$ along $\mathrm{grad}\, r$.

Groisser-Parker's result

Theorem(Groisser-Parker)(continued^2)

(iii) In the case of $k = 1$, the boundary of the completion of $(\mathcal{M}_P^{\mathcal{SD}}, \langle\ ,\ \rangle)$ is homothetic to B and its sufficiently small neighborhood consists of the gauge equivalence classes of the connections such that the energy density concentrates at a point.

Groisser-Parker's result