Left Invariant Structures on Statistical Manifolds

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- Information geometry, Fisher metric
- Manifolds of probability density functions-Amari, Lauritzen 80's
- Statistical manifolds, Holomorphic statistical manifolds-Kurose, 2004
- Statistical manifolds-generalization of Hessian manifolds
- Holomorphic statistical manifolds-generalization of Kaehler manifolds
- H. Matsuzoe, Statistical manifolds and affine differential geometry, 2010
- H. Furuhata, Hypersurfaces in statistical manifolds, 2009
- B. Opozda, Bochner's technique for statistical structures, 2015

FROM KÄHLER TO STATISTICAL MANIFOLDS

Kähler manifolds

(M, g)-a Riemannian manifold
∇-the Levi-Civita connection on M
(M, g) is called a complex manifold if it admits a complex structure J.
g is called Hermitian if

$$g(JX, JY) = g(X, Y).$$

$\Omega(X, Y) = g(JX, Y)$-Kähler form
(M, g) is called a Kähler manifold if

$$d\Omega = 0 \Leftrightarrow \nabla J = 0.$$

On statistical manifolds, we have a connection different than Levi-Civita one.

Definition

(Kurose) The triplet (M, ∇, g) is called a statistical manifold if:
(1) ∇ is torsion-free connection, and
(2) $(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)$ (Codazzi equation).

Definition

∇^*-the dual connection of ∇ wrt g iff

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z).$$

(M, ∇^*, g)-the dual statistical manifold of (M, ∇, g).

Definition

(Kurose) A statistical manifold $(\overline{M}, \overline{\nabla}, \overline{g})$ is said to be of constant curvature $c \in \mathbb{R}$ if

$$R^{\overline{\nabla}}(X,Y)Z = c\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\},$$

$R^{\overline{\nabla}}$-the curvature tensor of $\overline{\nabla}$.

A statistical structure $(\overline{\nabla}, \overline{g})$ of constant curvature 0 is called a **Hessian structure**.

H. Shima, The Geometry of Hessian Structures, World Sci. Publ., 2007

Example

Normal distributions. We denote by $l(x, \xi) = logp(x, \xi)$

$$M = \{p(x;\xi) | \xi = (\xi^1, \xi^2) = (\mu, \sigma),$$
$$p(x;\xi) = \frac{1}{\sqrt{2\pi(\xi^2)^2}} e^{-\frac{(x-\xi^1)^2}{(2\xi^2)^2}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\}$$

We regard that M is a manifold with local coordinates (μ, σ).

$$g_{ij} := \int_{-\infty}^{\infty} (\frac{\partial}{\partial \xi^i} logp(x;\xi))(\frac{\partial}{\partial \xi^j} logp(x;\xi)) p(x;\xi) dx$$
$$= E\Big[\frac{\partial l}{\partial \xi^i} \frac{\partial l}{\partial \xi^j}\Big] \quad (g = -\frac{1}{\sigma^2}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix})$$

-the Fisher information

$$C_{ijk} = E\Big[\frac{\partial l}{\partial \xi^i}\frac{\partial l}{\partial \xi^j}\frac{\partial l}{\partial \xi^k}\Big]$$

-the skewness or the cubic form

$$\Gamma_{ij,k} = E\Big[\frac{\partial^2 l}{\partial \xi^i \partial \xi^j}\frac{\partial l}{\partial \xi^k} - \frac{\partial l}{\partial \xi^i}\frac{\partial l}{\partial \xi^j}\frac{\partial l}{\partial \xi^k}\Big]$$
$$= \Gamma^0_{ij,k} - \frac{1}{2}C_{ijk}$$

∇^0-the Levi-Civita connection wrt g

$$\Gamma^*_{ij,k} = \Big[\frac{\partial^2 l}{\partial \xi^i \partial \xi^j}\frac{\partial l}{\partial \xi^k} + \frac{\partial l}{\partial \xi^i}\frac{\partial l}{\partial \xi^j}\frac{\partial l}{\partial \xi^k}\Big]$$
$$= \Gamma^0_{ij,k} + \frac{1}{2}C_{ijk}$$

(M, ∇, g) and (M, ∇^*, g) are statistical manifolds.

Definition

(Kurose) (M, g, J)- a Kaehler manifold, ∇-an affine connection of M

(M, ∇, g, J) is called a holomorphic statistical manifold if

(1) (M, g, ∇) is a statistical manifold, and

(2) $\omega := g(, J)$ is a ∇-parallel 2-form on M

HOW TO CONSTRUCT HOLOMORPHIC STATISTICAL MANIFOLDS?

If we define a connection ∇ as $\nabla := \nabla^{LC} + K$, where K is a (1, 2) tensor field satisfying

$$K(X,Y) = -K(Y,X) \tag{1}$$

$$g(K(X,Y),Z) = g(Y,K(X,Z)) \tag{2}$$

and

$$K(X,JY) = -JK(X,Y) \tag{3}$$

then (M,∇,g,J) is a holomorphic statistical manifold.

Kaehler manifolds:

$$\nabla^{LC} J = 0 \tag{4}$$

Holomorphic statistical manifolds:

$$\nabla J = J \nabla^{*} \tag{5}$$

BACKGROUND

S. Amari, Differential-geometrical methods in statistics, 1985
S. L. Lauritzen, Statistical manifolds, 1987
-$-\alpha$-connections
-first definition of statistical manifolds
(M, g)-a Riemannian manifold with Levi-Civita connection ∇^{LC}. C-(0,3)-tensor symmetric in its first two arguments

$$\nabla := \nabla^{LC} + \overline{C}, \tag{6}$$

$g(\overline{C}(X, Y), Z) = C(X, Y, Z)$.
(M, g, C)-statistical manifold
A conjugate symetric manifold: (M, g, ∇, ∇^*) if for conjugate connections $R(X, Y)Z = R^*(X, Y)Z$

LEFT INVARIANT STRUCTURES

Some new results on Lie groups

G-Lie group, g-its Lie algebra (vector fields invariant under left translations)
g-left invariant pseudo-Riemannian metric on G (left translations are isometries of (G, g))

$$< X, Y > := g(X, Y) \tag{7}$$

g(U, V)= const.

$$0 = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) \tag{8}$$

$$\nabla^\Sigma := \frac{1}{2}(\nabla + \nabla^*) \tag{9}$$

$$\Phi(X, Y, Z) := g((\nabla^\Sigma_X J)Y, Z) + g((\nabla^\Sigma_Y J)Z, X) + g((\nabla^\Sigma_Z J)X, Y) \tag{10}$$

Φ is skew-symmetric tensor

$$(\nabla_X J)Y = -(\nabla^*_X J)Y \tag{11}$$

$$(\nabla_{JX} J)Y = -J(\nabla_Y J)X \tag{12}$$

$$D(X,Y) := \nabla_{JX} Y + J \nabla_Y X \tag{13}$$

$$\Theta(X,Y,Z) =< D(X,Y), Z > + < D(Y,Z), X > + < D(Z,X), Y > \tag{14}$$

Equvalent conditions:

(1) (G, g, J) is holomorphic statistical manifold

(2)

$$\Theta(X,Y,Z) = -\Theta(X,Z,Y) and \Theta(JX,Y,Z) = \Theta(X,JY,Z)$$

THANK YOU!