Geometry of CR submanifolds

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Symmetry and shape
Celebrating the 60th birthday of Prof. J. Berndt
Santiago de Compostella
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One of the aims of submanifold geometry is to understand geometric invariants of submanifolds and to classify submanifolds according to given geometric data.

In Riemannian geometry, the structure of a submanifold is encoded in the second fundamental form.

We are interested in certain submanifolds, called contact CR-submanifolds, of $\mathbb{S}^7(1)$, which are (nearly) totally geodesic.
We study certain conditions on the structure F and on h of CR submanifolds of maximal CR dimension in complex space forms and we characterize several important classes of submanifolds in complex space forms.
We also show some results on CR submanifolds of the nearly Kähler six sphere.

Let $(\overline{M}, \overline{g})$ be an $(n+p)$–dimensional Riemannian manifold with Levi Civita connection $\overline{\nabla}$

and let M be an n–dimensional submanifold of $\overline{M}$ with the immersion $\imath$ of M into $\overline{M}$,
whose metric g is induced from $\bar{g}$ in such a way that

$$g(X, Y) = \overline{g}(\imath X, \imath Y), \quad X, Y \in T(M).$$

We denote by $T(M)$ and $T^{\perp}(M)$ the tangent bundle of M and the normal bundle of M, respectively.

Then, for all $X, Y \in T(M)$, we have

$$\overline{\nabla}_{\imath X} \imath Y \;\; = \;\; \imath \nabla_X Y + h(X, Y) \, ,$$

The tangent part defines the the Levi-Civita connection ∇ with respect to the induced Riemannian metric g,

The normal part h defines the **second fundamental form**, symmetric covariant tensor field of degree two with values in $T^{\perp}(M)$.

We have further, for all $\xi \in T^{\perp}(M)$

$$\overline{\nabla}_{\imath X}\xi \quad = \quad -\imath A_{\xi}X + D_X\xi \,,$$

It is a easy to check that A_{ξ} (the **shape operator** with respect to the normal ξ) is a linear mapping from the tangent bundle $T(M)$ into itself and that D defines a linear connection on the normal bundle $T^{\perp}(M)$. We call D the normal connection of M in $\overline{M}$.
h and A_{ξ} are related by

$$\overline{g}(h(X, Y), \xi) = g(A_{\xi}X, Y).$$

M. Djorić, M. Okumura, Certain condition on the second fundamental form of CR submanifolds of maximal CR dimension of complex hyperbolic space, *Ann. Glob. Anal. Geom.*, 39, (2011), 1-12.

J. Berndt, *Über untermannifaltigkeiten von komplexen Raumformen*, Dissertation, Universität zu Köln, 1989.

J. Berndt, J. C. Diaz-Ramos, *Real hypersurfaces with constant principal curvatures in complex hyperbolic space*, *J. London Math. Soc.*, (2) **74**, 778–798, (2006).

J. Berndt, J. C. Diaz-Ramos, *Real hypersurfaces with constant principal curvatures in the complex hyperbolic plane*, *Proc. Amer. Math. Soc.*, (10) **135**, 3349–3357, (2007).

Main Theorem *Let M be a complete n–dimensional CR submanifold of maximal CR dimension of a complex hyperbolic space $\mathbb{CH}^{\frac{n+p}{2}}$. If the condition*

$$h(FX, Y) - h(X, FY) = g(FX, Y)\eta, \ \eta \in T^{\perp}(M)$$

is satisfied, where F is the induced almost contact structure and h is the second fundamental form of M, respectively, then F is a contact structure and M is an invariant submanifold of $\tilde{M}$ by the almost contact structure $\tilde{F}$ of $\tilde{M}$, where $\tilde{M}$ is a geodesic hypersphere or a horosphere, or M is congruent to one of the following:

(i) a tube of radius $r > 0$ around a totally geodesic, totally real hyperbolic space form $H^{\frac{n+1}{2}}(-1)$;

(ii) a tube of radius $r > 0$ around a totally geodesic complex hyperbolic space form $\mathbb{CH}^{\frac{n-1}{2}}(-4)$;

(iii) a geodesic hypersphere of radius $r > 0$;

(iv) a horosphere;

(v) a tube over a complex submanifold of $\mathbb{CH}^{\frac{n+1}{2}}$.

Let $\overline{M}$ be an almost Hermitian manifold with the structure $(J, \bar{g})$.

J is the endomorphism of the tangent bundle $T(\overline{M})$ satisfying

$$J^2 = -I$$

$\bar{g}$ is the Riemannian metric of $\overline{M}$ satisfying the Hermitian condition

$$\bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}), \quad \bar{X}, \bar{Y} \in T(\overline{M}).$$

The fundamental 2-form, (Kähler form) Ω of M is defined by

$$\Omega(X, Y) = g(JX, Y)$$

for all vector fields X and Y on M.

If a complex manifold (M, J) with Hermitian metric g satisfies $d\Omega = 0$, then (M, J) is called a **Kähler manifold**.
A necessary and sufficient condition that a complex manifold (M, J) with Hermitian metric is a Kähler manifold is $\nabla_X J = 0$ for any $X \in T(M)$.
Here ∇ is the *Levi-Civita connection* with respect to the Hermitian metric g.

Let M' be a real hypersurface of $\overline{M}$
and let ξ be the unit normal local field to M'.
Then

$$\begin{aligned} J\imath_1 X' &= \imath_1 F'X' + u'(X')\xi, \\ J\xi &= -\imath_1 U', \end{aligned}$$

where F' is a skew symmetric endomorphism acting on $T(M')$, $U' \in T(M')$, u' is a one form on M'.

Y.Tashiro,
On contact structure of hypersurfaces in complex manifold I,
Tôhoku Math. J., **15**, 62–78, (1963).

By iterating J on $i_1 X'$ and on ξ, we easily see

$$\begin{aligned} F'^2 X' &= -X' + u'(X')U', \\ g'(U', X') = u'(X'), &\quad u'(U') = 1, \\ u'(F'X') = 0, &\quad F'U' = 0. \end{aligned}$$

Thus the real hypersurface M' is equipped with an almost contact structure (F', u', U'), naturally induced by the almost Hermitian structure on $\overline{M}$.

CR submanifolds of maximal CR dimension

$$\mathcal{H}_x(M) = T_x(M) \cap JT_x(M)$$

is called the holomorphic tangent space of M.

$\mathcal{H}_x(M)$ is the maximal J-invariant subspace of $T_x(M)$.

$$n - p \leq dim_{\mathbb{R}} \mathcal{H}_x(M) \leq n$$

M is called the

Cauchy-Riemann submanifold

or briefly CR submanifold if $\mathcal{H}_x$ has constant dimension for any $x \in M$.

R. Nirenberg and R.O. Wells, Jr., *Approximation theorems on differentiable submanifolds of a complex manifold*, *Trans. Amer. Math. Soc.* **142**, 15–35, (1965).

Examples (CR submanifolds of a complex manifold)

- *J-invariant submanifolds.* $J\imath T_x(M) \subset \imath T_x(M)$,

$$H_x(M) = T_x(M)\,, \quad \dim_{\mathbf{R}} H_x(M) = n$$

.

- *Real hypersurfaces.*

$$\dim_{\mathbb{R}} H_x(M) = n - 1.$$

- *Totally real submanifolds.*

$$H_x(M) = \{0\} \quad \text{holds at every point} \quad x \in M.$$

A submanifold M of $\overline{M}$ is called a CR submanifold if there exist distributions $\mathcal{H}$ and $\mathcal{H}^{\perp}$ of constant dimension such that $\mathcal{H} \oplus \mathcal{H}^{\perp} = TM$, $J\mathcal{H} = \mathcal{H}$, $J\mathcal{H}^{\perp} \subset T^{\perp}M$.

A. Bejancu, *CR-submanifolds of a Kähler manifold I*, *Proc. Amer. Math. Soc.*, **69**, 135–142, (1978).

Let M^n be a CR submanifold of maximal CR dimension

$$\boxed{\dim_{\mathbb{R}}(JT_x(M) \cap T_x(M)) = n - 1}$$

at each point x of M

Then it follows that M is odd–dimensional and that there exists a unit vector field ξ normal to M such that

$$\boxed{JT_x(M) \subset T_x(M) \oplus \operatorname{span}\{\xi_x\}}$$

for any $x \in M$

Examples

- real hypersurfaces of almost Hermitian manifolds $\overline{M}$;
- real hypersurfaces M of complex submanifolds M' of almost Hermitian manifolds $\overline{M}$;
- odd-dimensional F'-invariant submanifolds M of real hypersurfaces M' of almost Hermitian manifolds $\overline{M}$, where F' is an almost contact metric structure naturally induced by the almost Hermitian structure on $\overline{M}$.

Defining a skew–symmetric $(1,1)$-tensor F from the tangential projection of J by

$$J\imath X = \imath FX + u(X)\xi,$$

for any $X \in T(M)$, the Hermitian property of $\bar{g}$ implies that the subbundle $T_1^{\perp}(M) = \{\eta \in T^{\perp}(M) | \overline{g}(\eta, \xi) = 0\}$ is J-invariant, from which it follows

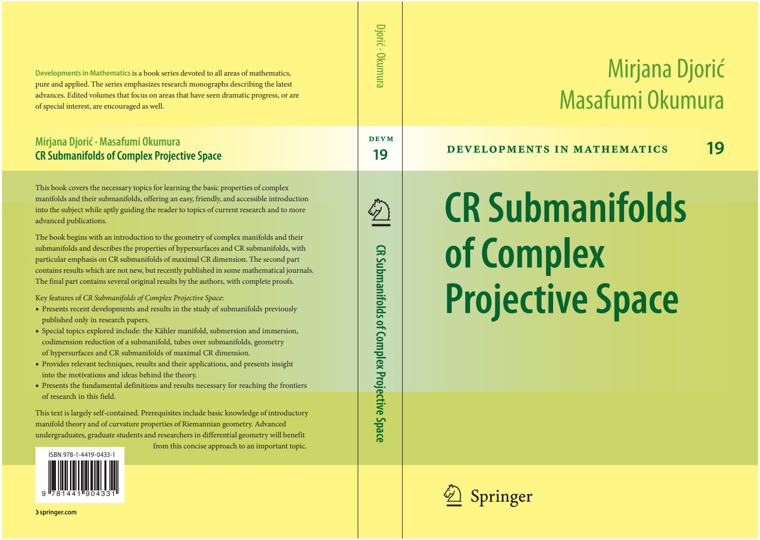
$$J\xi = -\imath U, \quad g(U, X) = u(X), \quad U \in T(M).$$

Here, U is a tangent vector field, u is one form on M. Also, from now on we denote the orthonormal basis of $T^{\perp}(M)$ by $\xi, \xi_1, \dots, \xi_q, \xi_{1^*}, \dots, \xi_{q^*}$, where $\xi_{a^*} = J\xi_a$ and $q = \frac{p-1}{2}$.

$$\begin{aligned} F^2X &= -X + u(X)U, \\ FU &= 0, \\ g(U,X) &= u(X) \end{aligned}$$

(F, u, U, g) defines an almost contact metric structure on M

M. Djorić, M. Okumura,
CR submanifolds of complex projective space,
Develop. in Math. **19**, Springer, (2009).



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Mirjana Djorić · Masafumi Okumura
CR Submanifolds of Complex Projective Space

This book covers the necessary topics for learning the basic properties of complex manifolds and their submanifolds, offering an easy, friendly, and accessible introduction into the subject while aptly guiding the reader to topics of current research and to more advanced publications.

The book begins with an introduction to the geometry of complex manifolds and their submanifolds and describes the properties of hypersurfaces and CR submanifolds, with particular emphasis on CR submanifolds of maximal CR dimension. The second part contains results which are not new, but recently published in some mathematical journals. The final part contains several original results by the authors, with complete proofs.

Key features of *CR Submanifolds of Complex Projective Space*:
- Presents recent developments and results in the study of submanifolds previously published only in research papers.
- Special topics explored include: the Kähler manifold, submersion and immersion, codimension reduction of a submanifold, tubes over submanifolds, geometry of hypersurfaces and CR submanifolds of maximal CR dimension.
- Provides relevant techniques, results and their applications, and presents insight into the motivations and ideas behind the theory.
- Presents the fundamental definitions and results necessary for reaching the frontiers of research in this field.

This text is largely self-contained. Prerequisites include basic knowledge of introductory manifold theory and of curvature properties of Riemannian geometry. Advanced undergraduates, graduate students and researchers in differential geometry will benefit from this concise approach to an important topic.

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DEVELOPMENTS IN MATHEMATICS 19

CR Submanifolds of Complex Projective Space

Springer

The first half of the text covers the basic material about the geometry of submanifolds of complex manifolds. Special topics that are explored include the (almost) complex structure, Kähler manifold, submersion and immersion, and the structure equations of a submanifold.

The second part of the text deals with real hypersurfaces and CR submanifolds, with particular emphasis on CR submanifolds of maximal CR dimension in complex projective space.

- eigenvalues of the shape operator of CR submanifolds of maximal CR dimension
- Levi form of CR submanifolds of maximal CR dimension
- CR submanifolds of maximal CR dimension satisfying the condition $h(FX, Y) + h(X, FY) = 0$
- contact CR submanifolds of maximal CR dimension $h(FX, Y) - h(X, FY) = g(FX, Y)\eta$
- invariant submanifolds of real hypersurfaces of complex space forms
- the scalar curvature of CR submanifolds of maximal CR dimension

Theorem

- $\overline{M} = \mathbb{C}^{\frac{n+k}{2}}$, *then* M *is isometric to* $\mathbb{E}^n$, $\mathbb{S}^n$ *or* $\mathbb{S}^{2p+1} \times \mathbb{E}^{n-2p-1}$;
- $\overline{M} = \mathbb{CP}^{\frac{n+k}{2}}$, *then* M *is isometric to* $M^C_{p,q}$, *for some* p, q *satisfying* $2p + 2q = n - 1$;
- $\overline{M} = \mathbb{CH}^{\frac{n+k}{2}}$, *then* M *is isometric to* M^*_n *or* $M^H_{p,q}(r)$, *for some* p, q *satisfying* $2p + 2q = n - 1$.

M. Djorić, M. Okumura,
Certain CR submanifolds of maximal CR dimension of complex space forms,
Differential Geometry and its Applications, 26/2, 208-217, (2008).

M. Djorić, M. Okumura,
Normal curvature of CR submanifolds of maximal CR dimension of the complex projective space, Acta Math. Hungar. (2018) 156 (1):82-90

Theorem

Let M be an n-dimensional CR submanifold of CR dimension $\frac{n-1}{2}$ of a complex projective space. If the distinguished normal vector field ξ is parallel with respect to the normal connection, the normal curvature of M can never vanish.

Namely, there do not exist CR submanifolds M^n of maximal CR dimension of a complex projective space $\mathbf{P}^{\frac{n+p}{2}}(\mathbf{C})$ with **flat normal connection** D of M, when the distinguished normal vector field is parallel with respect to D.

$R^\perp$ is the curvature tensor associated with the normal connection D (also called the normal curvature of M in $\overline{M}$), i.e.

$$R^\perp_{X\,Y}\xi_a = D_X D_Y \xi_a - D_Y D_X \xi_a - D_{[X,Y]}\xi_a.$$

If the normal curvature $R^\perp$ of M in $\overline{M}$ vanishes identically, we say that the normal connection of M is flat.

It is well known that an odd-dimensional sphere is a circle bundle over the complex projective space.

For an n-dimensional submanifold M of the real $(n+p)$-dimensional complex projective space $\mathbf{P}^{\frac{n+p}{2}}(\mathbf{C})$, let $\pi^{-1}(M)$ be the circle bundle over M which is compatible with the Hopf map

$$\pi : \mathbf{S}^{n+p+1} \to \mathbf{P}^{\frac{n+p}{2}}(\mathbf{C}).$$

Then $\pi^{-1}(M)$ is a submanifold of $\mathbf{S}^{n+p+1}$.

If the normal connection of $\pi^{-1}(M)$ in $\mathbf{S}^{n+p+1}$ is flat, we say that the normal connection of M is lift-flat, or L-flat.

$$\begin{array}{ccc} \pi^{-1}(M) & \xrightarrow{\imath_1'} & S^{n+p+1} \\ \pi\big\downarrow & & \big\downarrow\pi \\ M^n & \xrightarrow[\imath_1]{} & P^{\frac{n+p}{2}}(\mathbf{C}) \end{array}$$

Theorem

Let M be a real n-dimensional CR submanifold of maximal CR dimension of the complex projective space $\mathbf{P}^{\frac{n+p}{2}}(\mathbf{C})$. If the normal connection of M in $\mathbf{P}^{\frac{n+p}{2}}(\mathbf{C})$ is lift-flat and the distinguished normal vector field ξ is parallel with respect to the normal connection, then there exists a totally geodesic complex projective subspace $\mathbf{P}^{\frac{n+1}{2}}(\mathbf{C})$ of $\mathbf{P}^{\frac{n+p}{2}}(\mathbf{C})$ such that M is a real hypersurface of $\mathbf{P}^{\frac{n+1}{2}}(\mathbf{C})$.

CR submanifolds in $\mathbb{S}^6$

A nearly Kähler manifold is an almost Hermitian manifold (M, g, J) for which the tensor ∇J is skew-symmetric:

$$(\nabla_X J)Y + (\nabla_Y J)X = 0, \; X, Y \in TM.$$

These manifolds were intensively studied by A. Gray in

A. Gray, Nearly Kähler manifolds, J. Diff. Geom. 4 (1970), 283–309.

The first example was introduced on $\mathbb{S}^6$ by Fukami and Ishihara in

T. Fukami, S. Ishihara, Almost Hermitian structure on $\mathbb{S}^6$, Tohoku Math. J. (2), Volume 7, Number 3 (1955), 151-156.

A well known example is the nearly Kähler 6-dimensional sphere, whose complex structure J can be defined in terms of the vector cross product on $\mathbb{R}^7$.

The case of 6-dimensional nearly Kähler manifolds is of particular importance because of several results:

-the structure theorem of Nagy

P-A. Nagy, On nearly-Kähler geometry, Ann. Global Anal. Geom. 22 (2002), no. 2, 167–178.

asserts that a nearly Kähler manifold of arbitrary dimension may be expressed as the Riemannian product of nearly Kähler manifolds of dimension 6;

- Butruille in

J.-B. Butruille, Homogeneous nearly Kähler manifolds, in: Handbook of Pseudo-Riemannian Geometry and Supersymmetry, 399–423, RMA Lect. Math. Theor. Phys. 16, Eur. Math. Soc., Zürich, 2010.

showed that the only nearly Kähler homogeneous manifolds of dimension 6 are the compact spaces $\mathbb{S}^6$, $S^3 \times S^3$, $\mathbb{C}P^3$ and the flag manifold of $\mathbb{C}^3$, $SU(3)/U(1) \times U(1)$ (where the last three are not endowed with the standard metric);

M. Djorić and L. Vrancken, *Three-dimensional minimal CR submanifolds in $\mathbb{S}^6$ satisfying Chen's equality*, J. Geom. Phys., **56** (2006) 11, 2279–2288.

Theorem

Let M be a 3-dimensional minimal CR submanifold in S^6 satisfying the Chen's equality. Then M is a totally real submanifold or locally M is congruent with the immersion

$$\begin{aligned} f(t,u,v) &= (\cos t\cos u\cos v, \sin t, \cos t\sin u\cos v,\\ &\qquad \cos t\cos u\sin v, 0, -\cos t\sin u\sin v, 0). \end{aligned}$$

We notice that this immersion can also be described algebraically by the equations

$$x_5 = 0 = x_7, \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_6^2 = 1, \quad x_3 x_4 + x_1 x_6 = 0,$$

from which we see that it can be seen as a hypersurface lying in a totally geodesic $S^4(1)$.

In

B.-Y. Chen, *Some pinching and classification theorems for minimal submanifolds*, Archiv Math. (Basel) **60** (1993), 568–578.

Chen introduced a new invariant, nowadays called $\delta(2)$, for a Riemannian manifold M. More precisely, this invariant is given by:

$$\delta(2)(p) = \tau(p) - (\inf K)(p),$$

where
$(\inf K)(p) = \inf \left\{ K(\pi) \,|\, \pi \text{ is a 2-dimensional subspace of } T_pM \right\}$.
Here $K(\pi)$ is the sectional curvature of π and $\tau(p) = \sum_{i<j} K(e_i \wedge e_j)$ denotes the scalar curvature defined in terms of an orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space T_pM of M at p.

Later, in

B.-Y. Chen, *Pseudo-Riemannian Geometry, δ-invariants and Applications*, Word Scientific, Hackensack, NJ, 2011.

Chen introduced many other curvature invariants.

One of the aims of introducing these invariants is to use them to obtain a lower bound for the length of the mean curvature vector for an immersion in a real space form $\widetilde{M}(c)$.

A submanifold is called an *ideal submanifold*, for that curvature invariant, if it realizes equality at every point.

For a submanifold M^n in a Riemannian manifold $\widetilde{M}(c)$ of constant sectional curvature c, the following basic inequality involving the intrinsic invariant $\delta(2)$ and the length of the mean curvature vector $H = \frac{1}{n}\text{trace } h$ was first established in

B.-Y. Chen, *Some pinching and classification theorems for minimal submanifolds*, Archiv Math. (Basel) **60** (1993), 568–578.

$$\delta(2) \leq \frac{n^2(n-2)}{2(n-1)}||H||^2 + \frac{1}{2}(n-2)(n+1)c.$$

Almost contact metric manifolds

A differentiable manifold $\tilde{M}^{2m+1}$ is said to have an **almost contact structure** if it admits a (non-vanishing) vector field ξ (the so-called **characteristic vector field**), a one-form η and a $(1,1)$-tensor field φ (frequently considered as a field of endomorphisms on the tangent spaces at all points) satisfying

$$\eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi,$$

where I denotes the field of identity transformations of the tangent spaces at all points. These conditions imply

- $\varphi\xi = 0$
- $\eta \circ \varphi = 0$,
- endomorphism φ has rank $2m$ at every point in M.

A manifold $\tilde{M}$, equipped with an almost contact structure (ξ, η, φ) is called an **almost contact manifold** and will be denoted by $(\tilde{M}, \xi, \eta, \varphi)$.

Suppose that $\tilde{M}^{2m+1}$ is a manifold carrying an almost contact structure. A Riemannian metric $\tilde{g}$ on $\tilde{M}$ satisfying

$$\tilde{g}(\varphi X, \varphi Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y)$$

for all vector fields X and Y is called *compatible* with (or *associated* to) the almost contact structure, and $(\xi, \eta, \varphi, \tilde{g})$ is said to be an **almost contact metric structure** on M.
φ is skew-symmetric with respect to $\tilde{g}$ and ξ is unitary.

$(\xi, \eta, \varphi, \tilde{g})$ is called a **contact metric structure** and $\tilde{M}(\xi, \eta, \varphi, \tilde{g})$ is a **contact metric manifold** if

$$d\eta(X, Y) = \widetilde{g}(\varphi X, Y)$$

$\tilde{M}^{2m+1}(\xi, \eta, \varphi, \tilde{g})$ is **Sasakian** if

$$(\widetilde{\nabla}_X \varphi)Y = -\widetilde{g}(X, Y)\xi + \eta(Y)X, \quad X, Y \in \chi(\widetilde{M})$$

Contact CR-submanifolds. The odd dimensional analogue of CR-submanifolds in (almost) Kählerian manifolds is the concept of contact CR-submanifolds in Sasakian manifolds.
Namely, a submanifold M in the Sasakian manifold $(\widetilde{M}, \varphi, \xi, \eta, \tilde{g})$ carrying a φ-invariant distribution $\mathcal{D}$, i.e.

$$\varphi_p \mathcal{D}_p \subseteq \mathcal{D}_p,$$

for any $p \in M$, such that the orthogonal complement $\mathcal{D}^{\perp}$ of $\mathcal{D}$ in $T(M)$ is φ-anti-invariant, i.e.

$$\varphi_p \mathcal{D}_p^{\perp} \subseteq T_p^{\perp} M,$$

for any $p \in M$, is called a **contact CR-submanifold.**

This notion was used by A. Bejancu and N. Papaghiuc in

> A. Bejancu and N. Papaghiuc, *Semi-invariant submanifolds of a Sasakian manifold,*
> An. Şt. Univ. Al. I. Cuza Iasi, Matem. **1** (1981), 163–170.

using the terminology **semi-invariant submanifold**.
It is customary to require that ξ is tangent to M rather than normal, which is too restrictive, since Prop. 1.1 p.43 in
K. Yano and M. Kon, *CR submanifolds of Kaehlerian and Sasakian manifolds*, Progress in Math., vol. 30, Birkhauser, 1983.
implies that M must be φ-anti-invariant. Oblique position of ξ leads to highly complicated embedding equations.

The Sasakian structure on $\mathbb{S}^{2m+1}(1)$.

It is well-known that the $(2m+1)$-dimensional unit sphere

$$\mathbb{S}^{2m+1}(1) = \{\mathbf{p} \in \mathbb{R}^{2m+2} : \langle \mathbf{p}, \mathbf{p} \rangle = 1\}$$

where $\langle \ , \ \rangle$ is the usual scalar product in $\mathbb{R}^{2m+2}$, carries a natural Sasakian structure induced from the canonical complex structure of $\mathbb{R}^{2m+2}$.

Namely, identifying $\mathbb{R}^{2m+2}$ with $\mathbb{C}^{m+1}$, with J denoting the multiplication with the imaginary unit $i = \sqrt{-1}$, on $\mathbb{R}^{2m+2}$, since at any point $\mathbf{p} \in \mathbb{S}^{2m+1}(1)$, the outward unit normal to sphere coincides with the position vector $\mathbf{p}$, we put $\xi = J\mathbf{p}$ to be the characteristic vector field.

For X tangent to $\mathbb{S}^{2m+1}$, JX fails in general to be tangent and we decompose it into the tangent and the normal part, respectively

$$JX = \varphi X - \eta(X)\mathbf{p}.$$

Thus, $\mathbb{S}^{2m+1}(1)$ is equipped with an almost contact structure (φ, η, ξ). Together with the induced metric, this structure is Sasakian.

Let M be a contact CR-submanifold of $\mathbb{S}^7(1)$.

$$T(M) = H(M) \oplus E(M) \oplus \mathbb{R}\xi,$$

where

$$\varphi H(M) = H(M), \qquad \varphi E(M) \subseteq T^{\perp}M,$$

$$T^{\perp}(M) = \varphi E(M) \oplus \nu(M)$$

We have:

$$\mathbf{s + q + r = 3}$$

where $2s = \dim(H(M))$, $q = \dim(E(M))$, $2r = \dim(\nu(M))$.
Then:

I. $s = q = r = 1$, hence $\dim(M) = 4$

II. $s = 1,\ q = 2,\ r = 0$ hence $\dim(M) = 5$

III. $s = 2,\ q = 1,\ r = 0$ hence M is a hypersurface in $\mathbb{S}^7$

It is straightforward to show that a proper contact CR submanifold can never be totally geodesic.

A contact CR submanifold is called **nearly totally geodesic** if M is simultaneously $H(M)$-totally geodesic and $E(M)$-totally geodesic, namely if

$$h(H(M), H(M)) = 0 \quad \& \quad h(E(M), E(M)) = 0.$$

Problem. Find all proper nearly totally geodesic contact CR submanifolds in $\mathbb{S}^7$.

M. Djorić, M.I. Munteanu, L. Vrancken, *Four-dimensional contact CR-submanifolds in $S^7(1)$*, Math. Nachr. **290** (16) (2017), 2585–2596.

Theorem

Let M be a 4-dimensional nearly totally geodesic *contact CR-submanifold in* $\mathbb{S}^7$*. Then M is locally congruent with one of the following immersions:*

1. $$F(u,v,s,t) = \Big(\cos s \sin t\ \mathrm{e}^{i\lambda u}, \cos t \sin v\ \mathrm{e}^{i\mu u}, -\sin s \sin t\ \mathrm{e}^{i\lambda u}, \cos t \cos v\ \mathrm{e}^{i\mu u}\Big)$$
2. $F : \mathbb{S}^3 \times \mathbb{R} \longrightarrow \mathbb{R}^8,\ F(y,t) = (\cos t\ y, \sin t\ y), \quad ||y|| = 1$
3. $$F(u,v,s,t) = \left(e^{i(s+v)} \cos t \cos u, e^{-i(s-v)} \sin t, e^{i(s+v)} \cos t \sin u, 0\right)$$

M. Djorić, M.I. Munteanu,
Five-dimensional contact CR-submanifolds in $S^7(1)$, in progress.

M. Djorić, M.I. Munteanu, *On certain contact CR-submanifolds in S^7*, to appear in Contemporary Mathematics AMS (2020).

We constructed several examples of four-dimensional and five-dimensional contact CR-submanifolds of product and warped product type of seven-dimensional unit sphere, which are nearly totally geodesic, minimal and which satisfy the equality sign in some Chen type inequalities.

Theorem

Let $M = \mathbb{S}^3 \times \mathbb{S}^2$ *be the contact CR-submanifold (of warped product type) in* $\mathbb{S}^7$ *defined by the isometric immersion*

$$F : M = \mathbb{S}^3 \times \mathbb{S}^2 \longrightarrow \mathbb{S}^7$$
$$F(x_1, y_1, x_2, y_2; u, v, w) = (x_1 u, y_1 u, x_1 v, y_1 v, x_1 w, y_1 w, x_2, y_2).$$

Then

- (i) M *is nearly totally geodesic;*
- (ii) M *is minimal;*
- (iii) M *satisfies the equality in the Chen type inequality*

$$||h||^2 \geq 2p\left[||\nabla \ln f||^2 - \Delta \ln f + \frac{c+3}{2}\, s + 1\right];$$

- (iv) M *satisfies the equality in the Chen type inequality*

$$||h||^2 \geq 2p\left(||\nabla \ln f||^2 + 1\right).$$

Remarks:

1. In order to have an isometric immersion we need to consider on M the warped metric

$$g_M = g_{\mathbb{S}^3} + f^2 g_{\mathbb{S}^2}, \text{ where } f : D \subset \mathbb{S}^3 \to \mathbb{R}, \quad f(x_1, y_1, x_2, y_2) = \sqrt{x_1^2 + y_1^2}.$$

2. $M = N_1 \times_f N_2$ is a contact CR warped product of a Sasakian space form $\widetilde{M}^{2m+1}(c)$, if M is a contact CR-submanifold in $\widetilde{M}$, such that N_1 is φ-invariant and tangent to ξ, while N_2 is φ-anti-invariant.

Let us remark that $\dim(N_1) = 2s + 1$ and $\dim(N_2) = p$, $c = 1$.

3.

$$||h||^2 \geq 2p \left[||\nabla \ln f||^2 - \Delta \ln f + \frac{c+3}{2}\, s + 1 \right].$$

Here f is the warping function which has to satisfy $\xi(f) = 0$ and Δf is the Laplacian of f defined by

$$\Delta f = - \operatorname{div} \nabla f = \sum_{j=1}^{k} \left\{ (\nabla_{e_j} e_j) f - e_j e_j(f) \right\},$$

where ∇f is the gradient of f and $\{e_1, \dots, e_k\}$ is an orthonormal frame on M.

Finally, let us consider the immersion

$$F : M = \mathbb{S}^3 \times \mathbb{S}^1 \longrightarrow \mathbb{S}^7$$
$$F(x_1, y_1, x_2, y_2; u, v) = (x_1 u, y_1 u, x_1 u, y_1 v, x_2, y_2, 0, 0),$$

with the warped metric

$$g_M = g_{\mathbb{S}^3} + f^2 g_{\mathbb{S}^1}, \text{ where } f : D \subset \mathbb{S}^3 \to \mathbb{R}, \quad f(x_1, y_1, x_2, y_2) = \sqrt{x_1^2 + y_1^2}.$$

- F is an isometric immersion;
- $M = \mathbb{S}^3 \times \mathbb{S}^1$ is the contact CR-submanifold (of warped product type) in $\mathbb{S}^7$ defined by the isometric immersion F;
- M is nearly totally geodesic;
- M is minimal;
- M satisfies the equality in the two Chen type inequalities (as in the previous theorem);
- M is a $\delta(2)$-ideal in $\mathbb{S}^7$.