On product minimal Lagrangian submanifolds in complex space forms

Marilena Moruz
fwo Postdoctoral researcher
KU LEUVEN
FACULTY OF SCIENCE

Symmetry and shape
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Problem studied
jointly with X. Cheng, Z. Hu and L. Vrancken

Let

$$\psi : M^n \longrightarrow \tilde{M}^n(4\tilde{c})$$

be a minimal Lagrangian submanifold immersion into a complex space form, where

$$M^n = M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$$

and $M_1^{n_1}(c_1)$, $M_2^{n_2}(c_2)$

- ✓ are manifolds of real dimensions n_1, n_2 respectively: $n_1 + n_2 = n$,
- ✓ have each constant sectional curvature c_1 and c_2, respectively.

Motivation

Theorem 1 (N. Ejiri[1])

Let M be an n-dimensional, totally real, minimal submanifold of constant sectional curvature c, immersed in an n-dimensional complex space form. Then M is totally geodesic or flat ($c = 0$).

- ✓ there is a rich literature on minimal Lagrangian immersions of complex space forms
- ✓ the present problem represents a generalization of the classical result of N. Ejiri.

[1] N. Ejiri, *Totally real minimal immersions of n-dimensional real space forms into n-dimensional complex space forms*, Proc. Amer. Math. Soc. 84 (1982) 243–246.

Background

- **Kähler manifolds** are defined as the almost Hermitian manifolds for which the almost complex structure J is parallel with respect to the Levi-Civita connection ∇.
- A complex n-dimensional complete and simply connected Kähler manifold of constant holomorphic sectional curvature $4\tilde{c}$ is called **a complex space form.**
- Let $\tilde{M}^n(4\tilde{c})$ denote a complex space form.Then, if
  - $\tilde{c} > 0$: $\tilde{M}^n(4\tilde{c}) \equiv \mathbb{CP}^n$,
  - $\tilde{c} = 0$: $\tilde{M}^n(4\tilde{c}) \equiv \mathbb{C}^n$,
  - $\tilde{c} < 0$: $\tilde{M}^n(4\tilde{c}) \equiv \mathbb{CH}^n$.

Background

Let M be a submanifold of a Kähler manifold and let $X \in T_pM$.
Given the behaviour of J on tangent vectors, M can be:

- almost complex : JX tangent.
  $\star$The almost complex submanifolds must have even dimension.
- totally real : JX normal.
  $\star$If, additionally, the dimension of M is half the dimension of the ambient space then M is called *Lagrangian*.
- CR : $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$.

Main equations

- ✓ The formulas of Gauss and Weingarten write out as:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi,$$

- ✓ Properties of J:

$$\nabla^\perp_X JY = J\nabla_X Y, \quad A_{JX} Y = -Jh(X, Y) = A_{JY} X.$$

- ✓ The equations of Gauss, Codazzi and Ricci are

$$\begin{aligned} R(X, Y)Z &= \tilde{c}(\langle Y, Z\rangle X - \langle X, Z\rangle Y) + [A_{JX}, A_{JY}]Z, \\ (\nabla h)(X, Y, Z) &= (\nabla h)(Y, X, Z), \\ R^\perp(X, Y)JZ &= \tilde{c}(\langle Y, Z\rangle JX - \langle X, Z\rangle JY) + J[A_{JX}, A_{JY}]Z, \end{aligned}$$

A new equation – *The Tsinghua Principle*

due to Li Haizhong, Luc Vrancken and Wang Xianfeng (2013)

✓ need to have a tangential version of the Codazzi equation.
After appying the Tsinghua principle, we obtain in our case:

$$\begin{aligned}0 =& R(W,X)Jh(Y,Z) - Jh(Y,R(W,X)Z)+\\ & R(X,Y)Jh(W,Z) - Jh(W,R(X,Y)Z)+\\ & R(Y,W)Jh(X,Z) - Jh(X,R(Y,W)Z).\end{aligned}$$

✓ need to have an explicit expression for the curvature tensor:

$$R(X,Y)Z = c_1(\langle Y_1, Z_1\rangle X_1 - \langle X_1, Z_1\rangle Y_1) + c_2(\langle Y_2, Z_2\rangle X_2 - \langle X_2, Z_2\rangle Y_2),$$

where X_i, Y_i, Z_i are the projections of X, Y, Z on the i^{th} component of M^n, for $i = 1, 2$.

Theorem 2 (The main Theorem)

Let $\psi : M^n \longrightarrow \tilde{M}^n$ be a minimal Lagrangian submanifold into a complex space form. If $M^n = M_1^{n_1} \times M_2^{n_2}$, where $M_1^{n_1}$ and $M_2^{n_2}$ have constant sectional curvatures c_1 and c_2, then $c_1 c_2 = 0$. Moreover

1. $c_1 = c_2 = 0$. *M^n is equivalent to*
   - *the totally geodesic immersion in $\mathbb{C}^{n_1+n_2}$,*
   - *the Lagrangian flat torus in $\mathbb{CP}^{n_1+n_2}(4)$.*
2. $c_1 c_2 = 0$, $c_1^2 + c_2^2 \neq 0$. *We must have $\tilde{c} > 0$, so we may assume that the ambiant space is $\mathbb{CP}^{n_1+n_2}(4)$. We have that the lift of the immersion is congruent with*

$$\frac{1}{n+1}(e^{iu_1}, \ldots, e^{iu_{n_1}}, ae^{iu_{n_1+1}}y_1, \ldots, ae^{iu_{n_1+1}}y_{n_2+1}), \text{where}$$

1. $(y_1, y_2 \ldots, y_{n_2+1})$ *is the standard sphere $\mathbb{S}^{n_2} \subset \mathbb{R}^{n_2+1} \subset \mathbb{C}^{n_2+1}$,*
2. $a = \sqrt{n - n_1 + 1}$,
3. $u_{n_1+1} = -\frac{1}{a^2}(u_1 + \ldots + u_{n_1})$.

Case $c_1 = 0$ and $c_2 \neq 0$

Lemma 1

Let $\{X_i\}$ and $\{Y_j\}$, $i = 1, \ldots n_1$, $j = 1, \ldots n_2$ be orthonormal bases of $M_1^{n_1}$ and $M_2^{n_2}$, respectively. Then we have

$$A_{JX_i} Y_l = \mu(X_i) Y_l.$$

- It is straightforward to see that

$$\langle A_{JX_i} Y_k, X_j \rangle = 0 \text{ and } \langle A_{JX_i} Y_j, Y_k \rangle = \begin{cases} 0, & \text{if } j \neq k, \\ \mu(X_i), & \text{if } j = k. \end{cases}$$

Lemma 2 (Main Lemma)

There exist orthonormal frames of vector fields $\{X_i\}$, $\{Y_j\}$, $i = 1, \dots, n_1$, $j = 1, \dots, n_2$ *on* $M_1^{n_1}$ *and* $M_2^{n_2}$ *respectively, such that:*

$$\begin{aligned} &A_{JX_1}X_1 = \lambda_{11}X_1, \\ &A_{JX_i}X_i = \mu_1 X_1 + \dots + \mu_{i-1}X_{i-1} + \lambda_{ii}X_i, 1 < i \leq n_1, \\ &A_{JX_i}X_j = \mu_i X_j, 1 \leq i < j, \\ &A_{JX_i}Y_j = \mu_i Y_j, 1 \leq i \leq n_1, 1 \leq j \leq n_2 \end{aligned} \tag{1}$$

$$A_{JY_i}Y_j = \delta_{ij}(\mu_1 X_1 + \dots + \mu_{n_1}X_{n_1}), \tag{2}$$

where λ_{ii}, μ_i *are constant and satisfy*

$$\begin{aligned} &\lambda_{11} + (n-1)\mu_1 = 0, \\ &\lambda_{22} + (n-2)\mu_2 = 0, \\ &\dots \\ &\lambda_{n_1\, n_1} + (n - n_1)\mu_{n_1} = 0. \end{aligned} \tag{3}$$

Determine explicitly the Lagrangian immersion

Theorem 3 (H. Li, X.Wang)

Let $\psi : M \longrightarrow \mathbb{CP}^n(4)$ be a Lagrangian immersion. Then ψ is locally a Calabi product Lagrangian immersion of an $(n-1)$-dimensional Lagrangian immersion $\psi_1 : M_1 \longrightarrow \mathbb{CP}^{n-1}(4)$ and a point iff $\exists$ $\lambda_1, \lambda_2 \in \mathbb{R}$, $\exists$ $\mathcal{D}_1 = span\{E_1\}$ and $\mathcal{D}_2 = span\{E_2, \dots, E_n\}$ such that

$$\lambda_1 \neq 2\lambda_2 \text{ and } \begin{cases} h(E_1, E_1) = \lambda_1 JE_1, \\ h(E_1, E_i) = \lambda_2 JE_i, \ i = 2, \dots, n, \end{cases}$$

Moreover, $\psi : M \longrightarrow \mathbb{CP}^n(4)$ satisfies:

- *ψ is minimal iff ψ_1 is minimal. Locally $M = I \times M_1$ and $\psi = \Pi \circ \tilde{\psi}$*

$$\tilde{\psi}(t,p) = \left(\sqrt{\frac{n}{n+1}} e^{i\frac{1}{n+1}t} \tilde{\psi}_1(p), \sqrt{\frac{1}{n+1}} e^{-i\frac{n}{n+1}t}\right), \ (t,p) \in I \times M_1,$$

where Π is the Hopf fibration and $\tilde{\psi}_1 : M_1 \longrightarrow S^{2n-1}(1)$ is the horizontal lift of ψ_1.

Theorem 4 (H. Li, X.Wang)

Let $\psi : M \to \mathbb{CP}^n(4)$ *be a Lagrangian immersion.*
Suppose that*:*
$\exists\ \lambda_1, \lambda_2$ *local functions,*
$\exists\ \mathcal{D}_1 = span\{E_1\}$ *and* $\mathcal{D}_2 = span\{E_2 \ldots, E_n\}$ *orthogonal distributions s.t.*

$$\lambda_1 \neq 2\lambda_2 \text{ and } \begin{cases} h(E_1, E_1) = \lambda_1 JE_1, \\ h(E_1, E_i) = \lambda_2 JE_i, \ \ i = 2, \ldots, n, \end{cases}$$

Then M *has parallel second fundamental form if and only if* ψ *is locally a Calabi product Lagrangian immersion of a point and an* $(n-1)$*-dimensional Lagrangian immersion* $\psi_1 : M_1 \longrightarrow \mathbb{CP}^{n-1}(4)$ *which has parallel second fundamental form.*

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Apply Theorem 3 on M^n

- On $M^n = M_1 \times M_2$, consider $\mathcal{D}_1$ spanned by X_1 and $\mathcal{D}_2$ spanned by $\{X_2, \dots, X_{n_1}, Y_1, \dots, Y_{n_2}\}$.
- given the form of A_{JE_1} we may apply Theorem 2 (H. Li, X. Wang) $\Longrightarrow M^n$ is locally a Calabi product Lagrangian immersion of $\psi_1 : M_{11} \longrightarrow \mathbb{CP}^{n-1}(4)$ and a point, i.e. $M^n = I_1 \times M_{11}$.
- As ψ is minimal in our case, we get further that $\psi = \Pi \circ \tilde{\psi}$ for

$$\tilde{\psi}(t,p) = \Big(\sqrt{\frac{n}{n+1}} e^{i\frac{1}{n+1}t} \tilde{\psi}_1(p), \sqrt{\frac{1}{n+1}} e^{-i\frac{n}{n+1}t}\Big), \ (t,p) \in I_1 \times M_1,$$

where $\Pi : \mathbb{S}^{2n-1}(1) \longrightarrow \mathbb{CP}^{n-1}(4)$ is the Hopf fibration and $\tilde{\psi}_1 : M_1 \longrightarrow \mathbb{S}^{2n-1}(1)$ is the horizontal lift of ψ_1.

Apply Theorem 3 on M_{11}, where $M^n = I \times M_{11}$

- Consider next the immersion $\psi_1 : M_{11} \longrightarrow \mathbb{CP}^{n-1}(4)$.
- the restriction A^1_J of the shape operator A_J on $\{X_2, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2}\}$ (which spans T_pM_{11}) is defined as

$$\begin{aligned}
&A^1_{JX_2}X_2 = \lambda_{22}X_2,\\
&A^1_{JX_i}X_i = \mu_2X_2 + \ldots + \mu_{i-1}X_{i-1} + \lambda_{ii}X_i, 2 < i \leq n_1,\\
&A^1_{JX_i}X_j = \mu_iX_j, 2 \leq i < j,\\
&A^1_{JX_i}Y_j = \mu_iY_j, 2 \leq i \leq n_1, 1 \leq j \leq n_2,\\
&A^1_{JY_i}Y_j = \delta_{ij}(\mu_2X_2 + \ldots + \mu_{n_1}X_{n_1}),
\end{aligned} \tag{4}$$

- We may then apply Theorem 3 (H. Li, X. Wang) on M_{11}: $\mathcal{D}_1 \rightsquigarrow span\{X_2\}$, $\mathcal{D}_2 \rightsquigarrow span\{X_3, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2}\}$. $\Longrightarrow M_{11}$ is locally a Calabi product Lagrangian immersion of $\psi_2 : M_{12} \longrightarrow \mathbb{CP}^{n-2}(4)$ and a point: $M_{11} = I_2 \times M_{12}$, $I_2 \in \mathbb{R}$. Thus:

$$M^n = I_1 \times I_2 \times M_{12}$$

- As ψ_2 is minimal, we further apply Theorem 3 and we get for $\psi_1 = \Pi_1 \circ \tilde{\psi}_1$ that

$$\tilde{\psi}_1(t,p) = \Big(\sqrt{\frac{n-1}{n}} e^{i\frac{1}{n}t}\tilde{\psi}_2(p), \sqrt{\frac{1}{n}} e^{-i\frac{n-1}{n}t}\Big),$$

  where $(t,p) \in I_2 \times M_1$,
  $\Pi_1 : \mathbb{S}^{2n-3}(1) \longrightarrow \mathbb{CP}^{n-2}(4)$ is the Hopf fibration
  $\tilde{\psi}_2 : M_{12} \longrightarrow \mathbb{S}^{2n-3}(1)$ is the horizontal lift of ψ_2.
- Apply succesively Theorem 3 for n_1 times:
  M^n is locally a Calabi product Lagrangian immersion of n_1 points and an n_2-dimensional Lagrangian immersion

$$\psi_{n_1} : M_2^{n_2} \longrightarrow \mathbb{CP}^{n-n_1}(4),$$

  where $M_2^{n_2}$ is totally geodesic.

$$M^n = l_1 \times l_2 \times \ldots \times l_{n_1} \times M_2^{n_2},$$

for $l_1, \ldots, l_{n_1} \in \mathbb{R}$. Finally, for $q \in M_2^{n_2}$ and $t := (t_1, \ldots, t_{n_1})$ the parametrization of M^n is:

$$\psi(t,q) = \Big(\frac{\sqrt{n-(n_1-1)}}{\sqrt{n+1}} e^{iu_{n_1+1}} y_1, \frac{\sqrt{n-(n_1-1)}}{\sqrt{n+1}} e^{iu_{n_1+1}} y_2, \ldots, \frac{\sqrt{n-(n_1-1)}}{\sqrt{n+1}} e^{iu_{n_1+1}} y_{n_2+1}, \frac{1}{\sqrt{n+1}} e^{iu_1}, \ldots, \frac{1}{\sqrt{n+1}} e^{iu_{n_1}} \Big),$$

where $-(n-n_1+1)u_{n_1+1} = u_1 + u_2 + \ldots + u_{n_1}$ and

$$u_1 = -\frac{n}{n+1} t_1,$$

$$\ldots,$$

$$u_{n_1} = \frac{t_1}{n+1} + \frac{t_2}{n} + \ldots + \frac{t_{n_1-1}}{n-(n_1-2)+1} - \frac{n-(n_1-1)}{n-(n_1-1)+1} t_{n_1},$$

$$u_{n_1+1} = \frac{t_1}{n+1} + \frac{t_2}{n} + \ldots + \frac{t_{n_1-1}}{n-(n_1-2)+1} + \frac{t_{n_1}}{n-(n_1-1)+1}.$$

Recall

Theorem

Let $\psi : M^n \longrightarrow \tilde{M}^n$ be a minimal Lagrangian submanifold into a complex space form. If $M^n = M_1^{n_1} \times M_2^{n_2}$, where $M_1^{n_1}$ and $M_2^{n_2}$ have constant sectional curvatures c_1 and c_2, then $c_1 c_2 = 0$. Moreover

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$$\frac{1}{n+1}(e^{iu_1}, \ldots, e^{iu_{n_1}}, ae^{iu_{n_1+1}}y_1, \ldots, ae^{iu_{n_1+1}}y_{n_2+1}),$$

- *$(y_1, y_2 \ldots, y_{n_2+1})$ describes the standard sphere $\mathbb{S}^{n_2} \subset \mathbb{R}^{n_2+1} \subset \mathbb{C}^{n_2+1}$,*
- $a = \sqrt{n - n_1 + 1}$, $u_{n_1+1} = -\frac{1}{a^2}(u_1 + \ldots + u_{n_1})$.

Thank you!