# Invariant Ricci-flat Kähler metrics on tangent bundles of compact symmetric spaces

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# Introduction

- P.M. Gadea, J.C. González-Dávila, I.V. Mykytyuk, Invariant Ricci-flat Kähler metrics on tangent bundles of compact symmetric spaces, arXiv: 1905.04308, (2019), preprint.
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We give a new technique to determine explicitly all invariant Ricci-flat Kähler structures on the tangent bundle of compact symmetric spaces of any rank, not only for rank one. For rank one, we find new examples of Ricci-flat Kahler metrics.

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# Polarizations and Kähler structures

Let $J$ be an almost complex structure on a $2n$-dimensional manifold $M$ ($J^2 = -Id$). The complex $\pm i$-eigenspaces of $J$ on $T^{\mathbb{C}}M$ can be expressed as

$$T^{(1,0)}M = \{z = u - iJu \mid u \in TM\}, \quad T^{(0,1)}M = \{z = u + iJu \mid u \in TM\}.$$

- ***J* defines a complex subbundle** $F(J) = T^{(1,0)}M = \{z = u - iJu \mid u \in TM\} \subset T^{\mathbb{C}}M$ s. t.
$$T^{\mathbb{C}}M = F(J) \oplus \overline{F(J)}.$$

The converse holds.

**Existence of almost complex structures**

Let $F$ be a complex subbundle of $T^{\mathbb{C}}M$ such that $T^{\mathbb{C}}M = F \oplus \overline{F}$. Then there exists a unique almost complex structure $J$ on $M$ s. t.

$$F = F(J) = \{z = u - iJu \mid u \in TM\}.$$

Moreover, $F$ is involutive ($[F, F] \subset F$) if and only if $J$ is integrable.

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On an almost Hermitian manifold $(M, J, g)$ $(g(JX, JY) = g(X, Y))$, the *fundamental* 2-form $\omega$ is given by

$$\omega(X, Y) = -g(JX, Y), \quad X, Y \in \mathfrak{X}(M).$$

Then, $g(X, Y) = \omega(JX, Y)$.

- If $d\omega = 0$, $(M, J, g)$ is called *almost Kähler*.
- If, moreover $J$ is integrable, it is called *Kähler*.

- $F \subset T^{\mathbb{C}}M$ is said to be *integrable* if $F \cap \overline{F}$ has constant rank and the subbundles $F$ and $F + \overline{F}$ are involutive. ($F(J)$ is integrable if and only if it is involutive).

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# Polarizations and Kähler structures

Fix a non-degenerate 2-form $\omega$ on a $2n$-dimensional manifold $M$ :

- $F \subset T^{\mathbb{C}}M$ is said to be *Lagrangian* if $\omega(F, F) = 0$ and $\dim_{\mathbb{C}} F = n$.
- A *polarization* of $M$ is an integrable complex subbundle $F$ which is Lagrangian.
- A polarization $F$ is said to be *positive-definite* if the Hermitian form

$$h(u, v) = i\omega(u, \overline{v}), \quad u, v \in T^{\mathbb{C}}M,$$

is positive-definite on $F$.

**Equivalent Kähler condition**

Let $(M, \omega)$ be a symplectic manifold and let $J$ be an almost complex structure on $M$. The pair $(J, g = \omega(J\cdot, \cdot))$ is a Kähler structure on $M$ if and only if the subbundle $F(J)$ is a positive-definite polarization.

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# The canonical complex structure on $T(G/K)$

## The tangent bundle $T(G/K)$

Let $M = G/K$ where $G$ is a compact, connected Lie group and $K$ is closed subgroup of $G$. Then there exists a positive-definite $\mathrm{Ad}(G)$-invariant form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$.

- **Reductive decomposition:** $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ $(\mathrm{Ad}(K)\mathfrak{m} \subset \mathfrak{m})$.
- $(M = G/K, g)$ is a Riemannian homogeneous manifold determined by $\langle \cdot, \cdot \rangle_{\mathfrak{m}}$.

Consider $G \times \mathfrak{m}$ with two actions:

$$l_a : (g, w) \mapsto (ag, w), \quad r_k : (g, w) \mapsto (gk, \mathrm{Ad}(k^{-1})(w)),$$

where $a, g \in G$ and $k \in K$.

- The projection $\pi : G \times \mathfrak{m} \to G \times_K \mathfrak{m}$, $(g, w) \mapsto [(g, w)]$, is $G$-equivariant.
- The mapping $\phi : G \times_K \mathfrak{m} \to T(G/K)$, $[(g, w)] \mapsto (\tau_g)_{*o} w$, is a $G$-equivariant diffeomorphism.

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## Complexifications of Lie groups

Any compact Lie group $G$ admits, up to isomorphisms, a unique complexification $G^{\mathbb{C}}$ which is given by $G^{\mathbb{C}} = G\exp(i\mathfrak{g})$.

- $G^{\mathbb{C}}/K^{\mathbb{C}}$ is a complex homogeneous manifold and $p_h : G^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}$ is a holomorphic mapping. Moreover, $G^{\mathbb{C}} = G\exp(i\mathfrak{m})\exp(i\mathfrak{k})$.
- The complex vector fields $\xi_h^{\mathrm{r}} = \xi^{\mathrm{r}} - i(I\xi)^{\mathrm{r}}$, $\xi \in \mathfrak{g}$, $I\xi = i\xi$, determine a complex **involutive subbundle** of $T^{\mathbb{C}}G^{\mathbb{C}}$.
- **The $G^{\mathbb{C}}$-invariant canonical complex structure $J_c^K$:** $(p_h)_*(\xi_h^{\mathrm{r}}) = (p_h)_*(\xi^{\mathrm{r}}) - i(p_h)_*(I\xi)^{\mathrm{r}}$
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- The complex vector fields $\xi_h^{\mathfrak{r}} = \xi^{\mathfrak{r}} - i(I\xi)^{\mathfrak{r}}$, $\xi \in \mathfrak{g}$, $I\xi = i\xi$, determine a complex **involutive subbundle** of $T^{\mathbb{C}}G^{\mathbb{C}}$.
- **The $G^{\mathbb{C}}$-invariant canonical complex structure $J_c^K$:** $(p_h)_*(\xi_h^{\mathfrak{r}}) = (p_h)_*(\xi^{\mathfrak{r}}) - i(p_h)_*(I\xi)^{\mathfrak{r}}$
- **A relevant fact**: The mapping $f_K : G^{\mathbb{C}}/K^{\mathbb{C}} \to G\times_K \mathfrak{m}, \quad g\exp(iw)\exp(i\zeta)K^{\mathbb{C}} \mapsto [(g,w)]$, $(g,w,\zeta) \in G\times\mathfrak{m}\times\mathfrak{k}$, is a $G$-equivariant diffeomorphism. Then it determines a $G$-invariant complex structure $J_c^K$ on $T(G/K)$.

# Restricted roots on symmetric spaces of compact type

Let $G/K$ be a rank-$r$ symmetric space of compact type. Here, there exists an involutive automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ and $\mathfrak{k} = \{\xi \in \mathfrak{g} : \ \sigma(\xi) = \xi\}$ and $\mathfrak{m} = \{\xi \in \mathfrak{g} : \ \sigma(\xi) = -\xi\}$.

Let $\mathfrak{a} \subset \mathfrak{m}$ be some Cartan subspace of $\mathfrak{m}$. Then $\dim \mathfrak{a} = r$ and there exists a Cartan subalgebra $\mathfrak{t}$ $\sigma$-invariante de $\mathfrak{g}$ such that $\mathfrak{a} \subset \mathfrak{t}$.

- $\mathfrak{t}^{\mathbb{C}}$ is a **Cartan subalgebra** of $\mathfrak{g}^{\mathbb{C}}$.
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$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha = \{\xi \in \mathfrak{g}^{\mathbb{C}} : \ [t, \xi] = \alpha(t)\xi, \ t \in \mathfrak{t}^{\mathbb{C}}\}.$$

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- $\lambda(\mathfrak{a}) \subset i\mathbb{R}$. Define $\lambda' \colon \mathfrak{a} \to \mathbb{R}$, $\lambda \in \Sigma^+$, such that $i\lambda' = \lambda$.
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$T^+(G/K) = \phi(G \times_K \mathfrak{m}^R)$ is an open dense subset of $T(G/K)$.

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If $G/K$ is a rank-one symmetric space, then $\dim \mathfrak{a} = 1$ ($\mathfrak{a} = \mathbb{R}X$) and

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## Some previous considerations

- Let $H$ be the subgroup of $K$ given by $H = \{k \in K : \ \mathrm{Ad}_k u = u, \ \text{for all } u \in \mathfrak{a}\}$.
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## The Ricci form

On a Riemannian Kähler manifold $(M, g, J)$,

$$\mathrm{Ric}(g)(X, Y) = \mathrm{Ric}(X, JY), \quad X, Y \in \mathfrak{X}(M),$$

is a 2-form, known as the **Ricci form** of $g$.

- Its complex extension can be expressed (locally) as $\mathrm{Ric}(g) = -i\partial\bar{\partial} \ln \det(\omega_{js})$.
- If $g = \omega(J_c^K \cdot, \cdot)$ is a $G$-invariant Kähler metric on $T(G/K)$ then $\mathrm{Ric}(g) = i\partial\bar{\partial} \ln \mathcal{S}$, where $\mathcal{S} : T(G/K) \to \mathbb{C}$ is a $G$-invariant function.
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- A two-form $\omega$ on $G/H \times W^+$ is a $G$-invariant symplectic structure if and only if $\tilde{\omega} = (\pi_H \times id)^*\omega$ satisfies the following three conditions:
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- $\omega$ satisfying (1) − (3), is a positive-definite polarization if and only if

  (4) $\tilde{\omega}(T_j, T_k) = 0$, $j, k = 1, \dots n$;
  (5) $i\tilde{\omega}(T, \overline{T}) > 0$ for each $T = \sum_{j=1}^{n} c_j T_j$, $(c_1, \dots, c_n) \in \mathbb{C}^n \setminus \{0\}$.

- If moreover, $G$ is semisimple and $\tilde{\omega}$ satisfies

  (6) $\det(\tilde{\omega}(T_j, \overline{T}_k)) = \mathrm{const}$ on $G \times W^+$,

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### Theorem

**(Main Theorem)** *Let $G/K$ be a Riemannian symmetric space of compact type. Each $G$-invariant Kähler metric $g$, associated with the canonical complex structure $J_c^K$ on $G/H \times W^+ \cong T^+(G/K)$, is determined by the Kähler form $\omega(\cdot,\cdot) = -g(J_c^K\cdot,\cdot)$ on $G/H \times W^+$ given by $(\pi_H \times \mathrm{id})^*\omega = d\widetilde{\theta}^{\mathbf{a}}$, where $\mathbf{a} : W^+ \to \mathfrak{g}$ is a smooth vector-function which is unique for each $\omega$, satisfying certain conditions equivalent to the previous conditions (2)–(5) and $\tilde{\theta}^{\mathbf{a}}$ is the $G$-invariant 1-form on $G \times W^+$*

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*for all $(g,x) \in G \times W^+$, $\xi \in \mathfrak{g}$ and $w \in \mathfrak{a}$.*
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## A first application of the Main Theorem

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$$f'(x) = \sqrt{C\sinh^2 x + c_Z^2 \sinh^2 x \cosh^{-2} x + C_1},$$

*for some real constants $C > 0$ and $C_1 \geqslant 0$.*
*The corresponding $G$-invariant Ricci-flat Kähler metric on $T^+\mathbb{S}$ is uniquely extendable to a smooth complete metric on $T\mathbb{S}^2$ if and only if $C_1 = 0$ (that is, $\lim_{x\to 0} f'(x) = 0$).*

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