Homogeneous submanifolds in complex space forms

Symmetry and shape

Celebrating the 60th birthday of Prof. J. Berndt

UNIÓN EUROPEA
FONDO EUROPEO DE DESARROLLO REGIONAL
"Una manera de hacer Europa"

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Contents

Homogeneous hypersurfaces

$\bar{M}$ Riemannian manifold, $\bar{\nabla}$ Levi-Civita connection

$M \subset \bar{M}$ hypersurface, ξ unit normal, ∇ Levi-Civita connection

M **homogeneous**	$\Longleftrightarrow$	$M = G \cdot o$, with $o \in M, G \subset I(\bar{M})$

G is said to act with **cohomogeneity one**

Problem.

- Classify homogeneous hypersurfaces (up to isometric congruence)
- Characterize homogeneous hypersurfaces in terms of geometric data

Complex space forms

Complex projective space

$(\mathbb{C}^{n+1}, i\cdot)$ $\langle v, w\rangle = \mathrm{Re}\Big(\sum_{i=0}^{n} \bar{v}_i w_i\Big)$

$S^{2n+1} = \{z \in \mathbb{C}^{n+1} \colon \langle z, z\rangle = 1\}$

$z \sim w \Leftrightarrow \exists \lambda \in \mathbb{C} : w = \lambda z$

$$\mathbb{C}P^n = S^{2n+1}/\sim$$

$\pi\colon S^{2n+1} \to \mathbb{C}P^n$ Hopf map

π Riemannian submersion

$\mathbb{C}P^n$ is a Kähler manifold with constant positive holomorphic sectional curvature

Complex hyperbolic space

$(\mathbb{C}^{1,n}, i\cdot)$ $\langle v, w\rangle = \mathrm{Re}\Big(-\bar{v}_0 w_0 + \sum_{i=1}^{n} \bar{v}_i w_i\Big)$

$H_1^{2n+1} = \{z \in \mathbb{C}^{1,n} \colon \langle z, z\rangle = -1\}$

$z \sim w \Leftrightarrow \exists \lambda \in \mathbb{C} : w = \lambda z$

$$\mathbb{C}H^n = H_1^{2n+1}/\sim$$

$\pi\colon H_1^{2n+1} \to \mathbb{C}H^n$ Hopf map

π semi-Riemannian submersion

$\mathbb{C}H^n$ is a Kähler manifold with constant negative holomorphic sectional curvature

Complex space forms

Complex projective space

$(\mathbb{C}^{n+1}, i\cdot) \quad \langle v, w\rangle = \mathrm{Re}\Big(\sum_{i=0}^{n} \bar{v}_i w_i\Big)$

$S^{2n+1} = \{z \in \mathbb{C}^{n+1} \colon \langle z, z\rangle = 1\}$

$$\mathbb{C}P^n = S^{2n+1}/\sim$$

$SU(n+1)$ acts transitively on $\mathbb{C}P^n$

$$\mathbb{C}P^n = \frac{SU(n+1)}{S(U(1)U(n))}$$

$\mathbb{C}P^n$ is a symmetric space of rank one and compact type

Complex hyperbolic space

$(\mathbb{C}^{1,n}, i\cdot) \quad \langle v, w\rangle = \mathrm{Re}\Big(-\bar{v}_0 w_0 + \sum_{i=1}^{n} \bar{v}_i w_i\Big)$

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$SU(1,n)$ acts transitively on $\mathbb{C}H^n$

$$\mathbb{C}H^n = \frac{SU(1,n)}{S(U(1)U(n))}$$

$\mathbb{C}H^n$ is a symmetric space of rank one and noncompact type

The complex hyperbolic space

$$\mathbb{C}H^n = \frac{SU(1,n)}{S(U(1)U(n))}$$

Iwasawa decomposition

$I^0(\mathbb{C}H^n)$= K $A\ N$ → acts simply transitively on $\mathbb{C}H^n$

$\mathbb{C}H^n \cong AN$ with left-invariant metric

K ↓ $S(U(1)U(n))$

AN ↓

$$\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$$

$\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$, $\mathfrak{g}_{2\alpha} \cong \mathbb{R}$

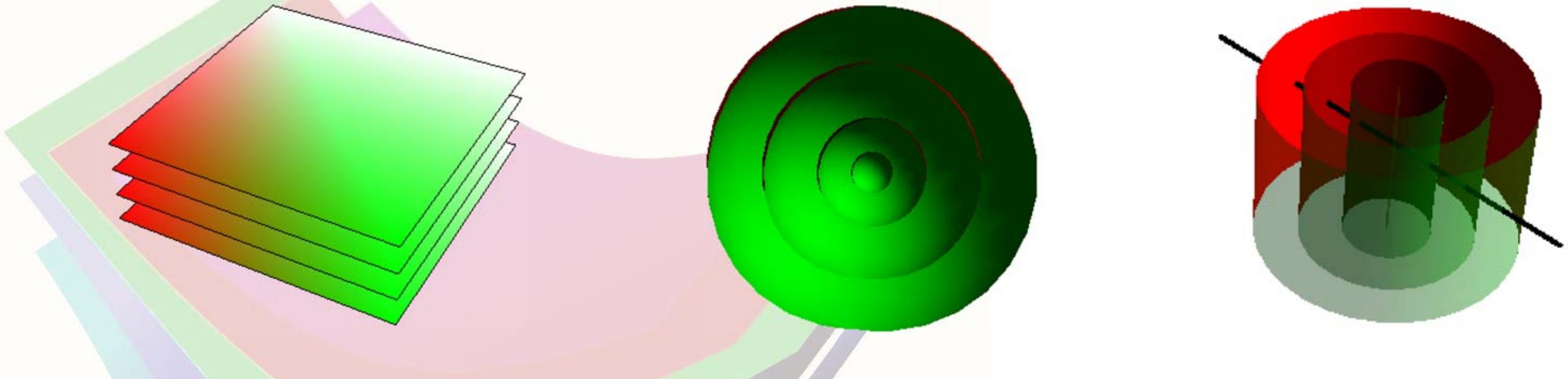
Heisenberg algebra

$$[U,V] = \langle JU, V\rangle Z$$

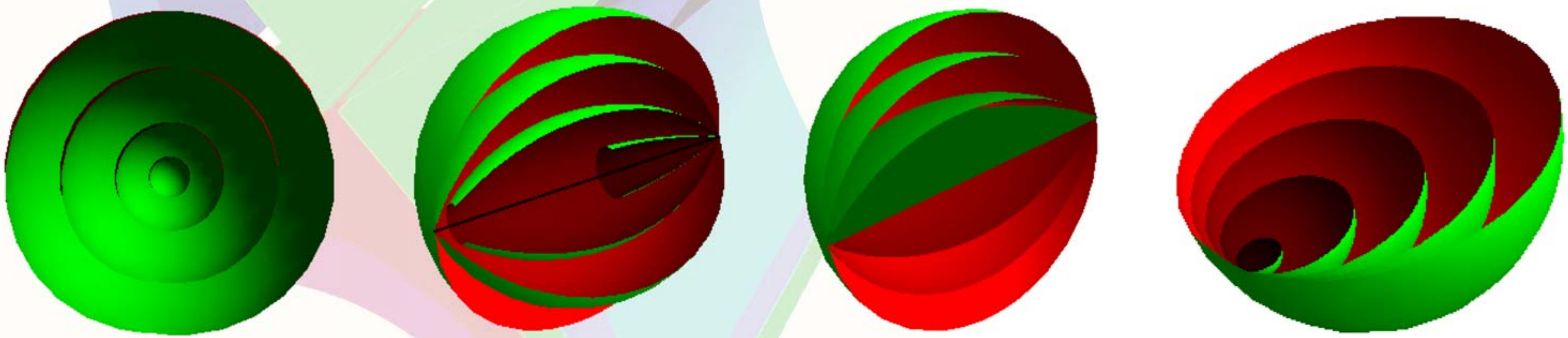
$$[A,U] = \frac{1}{2}U \qquad [A,Z] = Z$$

Homogeneous hypersurfaces in space forms

- Euclidean spaces $\mathbb{R}^n$ [Somigliana, Levi-Civita, Segre]:



- Real hyperbolics spaces $\mathbb{R}H^n$ [Cartan]:



- Spheres S^n [Hsiang, Lawson]:

  Isotropy representations of symmetric spaces of rank 2

Homogeneous hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{C}H^n$

Theorem. [Takagi] A homogeneous hypersurface in $\mathbb{C}P^n$ is a principal orbit of the quotient of the isotropy representation of a Hermitian symmetric space of rank two.

Theorem. [Berndt, Tamaru] Homogeneous hypersurfaces in $\mathbb{C}H^n$:

- tubes around totally geodesic $\mathbb{C}H^k$, $k \in \{0, \ldots, n-1\}$
- tubes around totally geodesic $\mathbb{R}H^n$
- horospheres
- ruled homogeneous minimal Lohnherr hypersurfaces W^{2n-1}, or their equidistant hypersurfaces
- tubes around ruled homogeneous minimal Berndt-Brück submanifolds W_φ^{2n-k}, for $k \in \{2, \ldots, n-1\}$, $\varphi \in (0, \pi/2]$
  (k even if $\varphi \neq \pi/2$)

Homogeneous hypersurfaces in $\mathbb{C}H^n$

Hopf examples

- Tubes around a totally geodesic $\mathbb{C}H^k$, $k \in \{0, \ldots, n-1\}$

  Group action: $S(U(1,k) \times U(n-k))$

  $g = 2$ if $k \in \{0, n-1\}$; $g = 3$ otherwise

- Tubes around a totally geodesic $\mathbb{R}H^n$

  Group action: $SO^0(1,n)$

  $g = 2$ if $r = \log(2+\sqrt{3})$; $g = 3$ otherwise

- Horospheres

  Group action: N

  $g = 2$

Non-Hopf examples

$V \subset \mathbb{C}^n$ has **constant Kähler angle** φ ⟺ $\angle(Jv, V) = \varphi, \forall v \in V \setminus \{0\}$

$\mathfrak{w} \subset \mathfrak{g}_\alpha$ such that $\mathfrak{w}^\perp$ is of constant Kähler angle φ, $k = \dim \mathfrak{w}^\perp$

$\mathfrak{s}_\mathfrak{w} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$

$S_\mathfrak{w}$ subgroup of AN whose Lie algebra is $\mathfrak{s}_\mathfrak{w}$

Theorem. [Berndt, Brück] Tubes around $W_\varphi^{2n-k} := W_\mathfrak{w} = S_\mathfrak{w} \cdot o$ are homogeneous

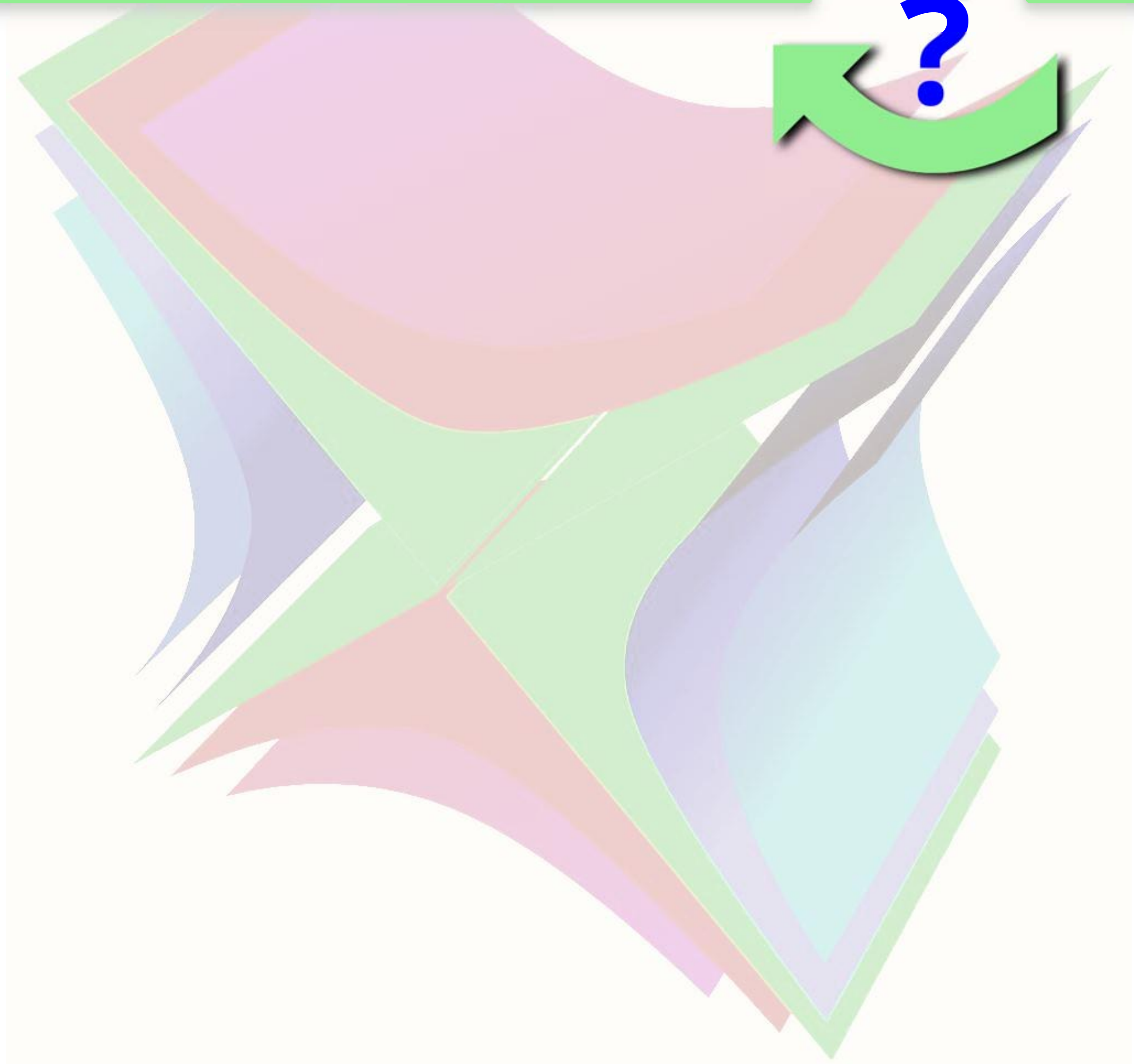
- If $\mathfrak{w}$ is a hyperplane, $W_\mathfrak{w}$ is the Lohnherr hypersurface ($g = 3$)
- If $\varphi = \pi/2$, then $g = 3$ if $r = \log(2 + \sqrt{3})$, otherwise $g = 4$
- If $\varphi \neq \pi/2$, then k is even; $g = 4$ if $k = 2$, otherwise $g = 5$

Characterization of homogeneous hypersurfaces

M homogeneous hypersurface

- M has constant principal curvatures
- M is isoparametric

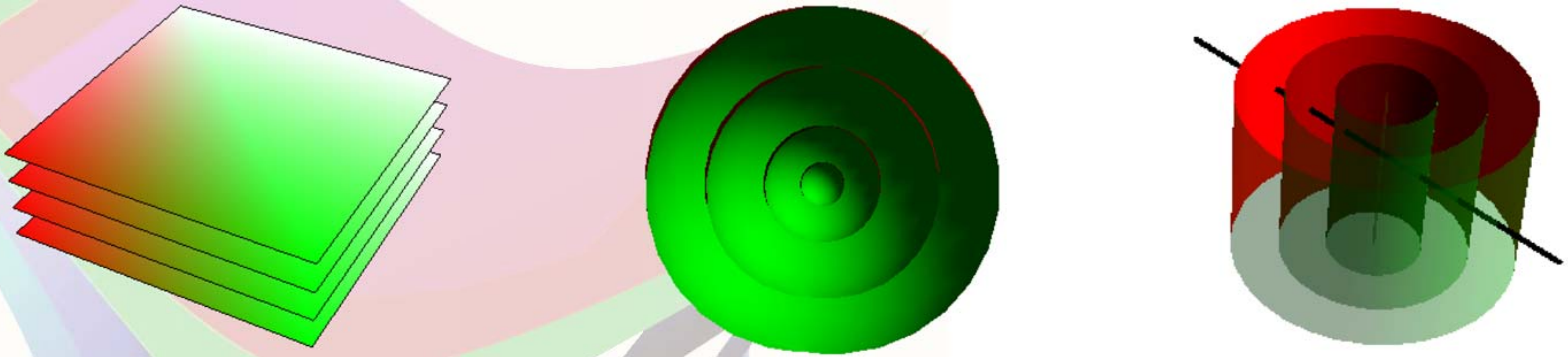
?



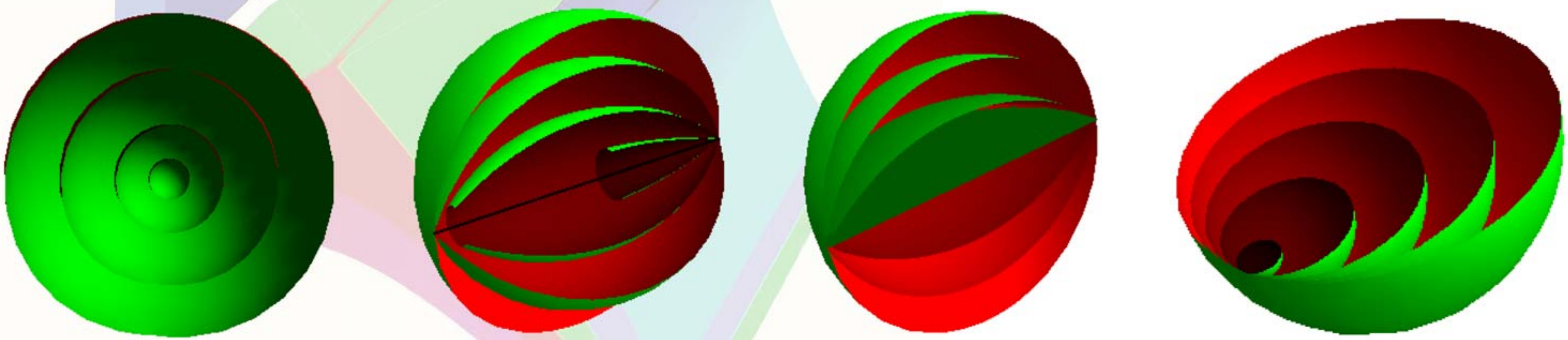
Characterization in real space forms

Theorem. [Cartan] Isoparametric $\Leftrightarrow$ constant principal curvatures

- Euclidean spaces $\mathbb{R}^n$:



- Real hyperbolics spaces $\mathbb{R}H^n$:



- Spheres S^n:

  There are inhomogeneous examples

Constant principal curvatures

M homogeneous hypersurface → M has constant principal curvatures

?

Shape operator: $\mathcal{S}X = -\bar{\nabla}_X \xi$

$\mathcal{S}$ self-adjoint
($\Rightarrow$ $\mathcal{S}$ diagonalizable) → **principal curvatures**: eigenvalues of $\mathcal{S}$

g: number of principal curvatures

$J\xi$: **Hopf vector field**

h: # of nontrivial projections of $J\xi$ onto principal curvature spaces

M is **Hopf** $\Leftrightarrow$ $J\xi$ is an eigenvector of $\mathcal{S}$ $\Leftrightarrow$ $h = 1$

Constant principal curvatures

M homogeneous hypersurface ⟹ M has constant principal curvatures

?

The answer is **YES** if:

$g = 1$ [Tashiro, Tachibana] No umbilical hypersurfaces in $\mathbb{C}H^n$

$g = 2$ [Montiel]

- tubes around totally geodesic $\mathbb{C}H^k$, $k \in \{0, n-1\}$
- tubes of radius $r = \log(2+\sqrt{3})$ around totally geodesic $\mathbb{R}H^n$
- horospheres

$g = 3$ [Berndt, Díaz-Ramos]

- tubes around totally geodesic $\mathbb{C}H^k$, $k \in \{1, \dots, n-2\}$
- tubes of radii $r \neq \log(2+\sqrt{3})$ around totally geodesic $\mathbb{R}H^n$
- ruled Lohnherr hypersurfaces $W^{2n-1}_{\pi/2}$, or their equidistant hypersurfaces
- tubes of radius $r = \log(2+\sqrt{3})$ around Berndt-Brück submanifolds $W^{2n-k}_{\pi/2}$, for $k \in \{2, \dots, n-1\}$

Constant principal curvatures

M homogeneous hypersurface $\Rightarrow$ M has constant principal curvatures

The answer is **YES** if:

$h = 1$ [Berndt]

- tubes around totally geodesic $\mathbb{C}H^k$, $k \in \{0, \dots, n-1\}$
- tubes around totally geodesic $\mathbb{R}H^n$
- horospheres

$h = 2$ [Díaz-Ramos, Domínguez-Vázquez]

- ruled Lohnherr hypersurfaces $W^{2n-1}_{\pi/2}$, or their equidistant hypersurfaces
- tubes around Berndt-Brück submanifolds $W^{2n-k}_{\pi/2}$, for $k \in \{2, \dots, n-1\}$

Isoparametric hypersurfaces

M homogeneous hypersurface ⇒ M is isoparametric

?

$f\colon \bar{M} \to \mathbb{R}$
isoparametric function
⇔
$\|\nabla f\|^2$ and Δf constant along the level sets of f

$M \subset \bar{M}$
isoparametric hypersurface
⇔
level set of codimension 1 of isoparametric function

[Cartan]

M isoparametric hypersurface ⇔ nearby parallel hypersurfaces of constant mean curvature

In real space forms: isoparametric $\Leftrightarrow$ constant principal curvatures

Isoparametric hypersurfaces in $\mathbb{C}H^n$

M homogeneous hypersurface → M is isoparametric

Inhomogeneous examples

$\mathfrak{w} \subset \mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$, $k = \dim \mathfrak{w}^\perp$

$\mathfrak{s}_\mathfrak{w} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$

$S_\mathfrak{w}$ subgroup of AN whose Lie algebra is $\mathfrak{s}_\mathfrak{w}$

Theorem. [Díaz-Ramos, Domínguez-Vázquez] Tubes around $W_\mathfrak{w} = S_\mathfrak{w} \cdot o$ are isoparametric

If $\mathfrak{w} \subset \mathfrak{g}_\alpha$, then $\mathfrak{w}^\perp = \bigoplus_{\varphi \in \Phi} \mathfrak{w}^\perp_\varphi$ is a sum of space of constant Kähler angle [Díaz-Ramos, Domínguez-Vázquez, Kollross]

Thus, $W_\mathfrak{w}$ is homogeneous if and only if $\mathfrak{w}^\perp$ has constant Kähler angle

Isoparametric hypersurfaces in $\mathbb{C}H^n$

Theorem. [Díaz-Ramos, Domínguez-Vázquez, Sanmartín-López] Isoparametric hypersurfaces in $\mathbb{C}H^n$:

- tubes around totally geodesic $\mathbb{C}H^k$, $k \in \{0, \dots, n-1\}$
- tubes around totally geodesic $\mathbb{R}H^n$
- horospheres
- ruled homogeneous minimal Lohnherr hypersurfaces $W^{2n-1}_{\pi/2}$, or their equidistant hypersurfaces
- tubes around a ruled homogeneous minimal Berndt-Brück submanifolds W^{2n-k}_{φ}, for $k \in \{2, \dots, n-1\}$, $\varphi \in (0, \pi/2]$ (k even if $\varphi \neq \pi/2$)
- tubes around ruled homogeneous minimal submanifolds $W_{\mathfrak{w}}$, for some proper real subspace $\mathfrak{w}$ of $\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$ such that $\mathfrak{w}^\perp$ has nonconstant Kähler angle

Characterization of homogeneous hypersurfaces

M homogeneous hypersurface

- M has constant principal curvatures
- M is isoparametric

?

Corollary. If M is a connected complete hypersurface in $\mathbb{C}H^2$ then the following statements are equivalent:

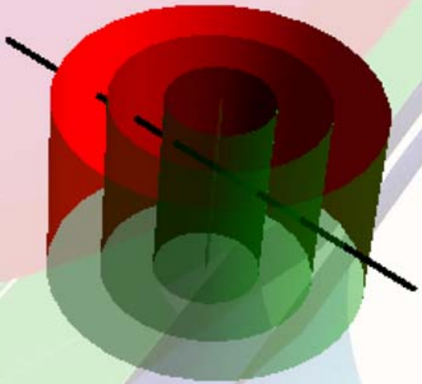
- M is homogeneous
- M has constant principal curvatures
- M is isoparametric

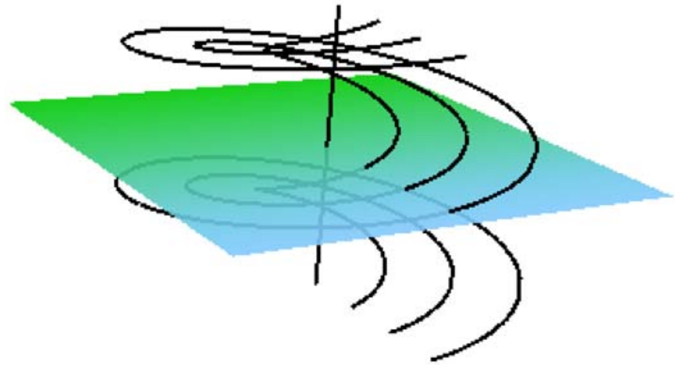
Corollary. If M is a connected complete hypersurface in $\mathbb{C}H^n$ then, M is homogeneous if and only if M is isoparametric and has constant principal curvatures

Polar actions

G acts **polarly** ⟷ there is a *section*

Section: submanifold that intersects *all* orbits of G *orthogonally*





A section is thought as a set of "canonical forms"

Example. $Sl(n,\mathbb{R})/SO(n)$ $\mathfrak{sl}(n,\mathbb{R}) = \mathfrak{so}(n) \oplus \{\text{symmetric matrices}\}$

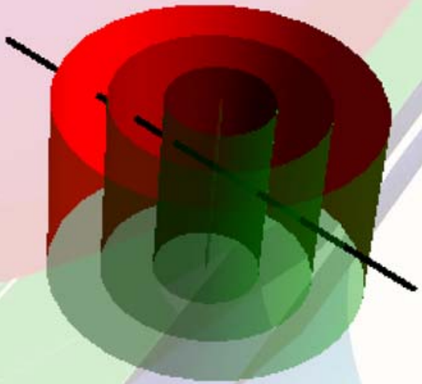
$SO(n)$ acts on symmetric matrices by conjugation

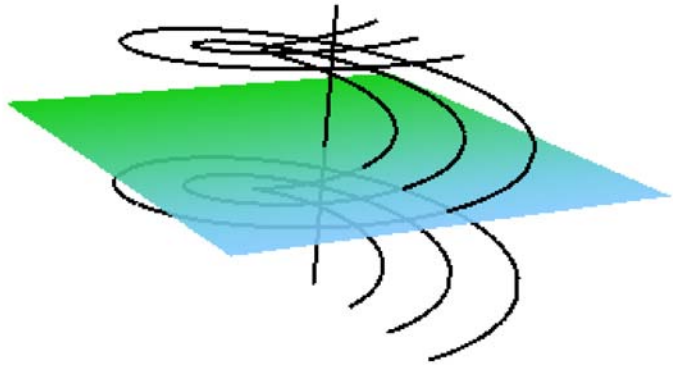
$\{\text{diagonal matrices}\}$ is a section

Polar actions

G acts **polarly** ⟺ there is a *section*

Section: submanifold that intersects *all* orbits of G *orthogonally*





Problem.

- Classify polar actions (up to isometric congruence)
- Characterize orbits of polar actions in terms of geometric data

Polar actions on real space forms

- Spheres S^n:

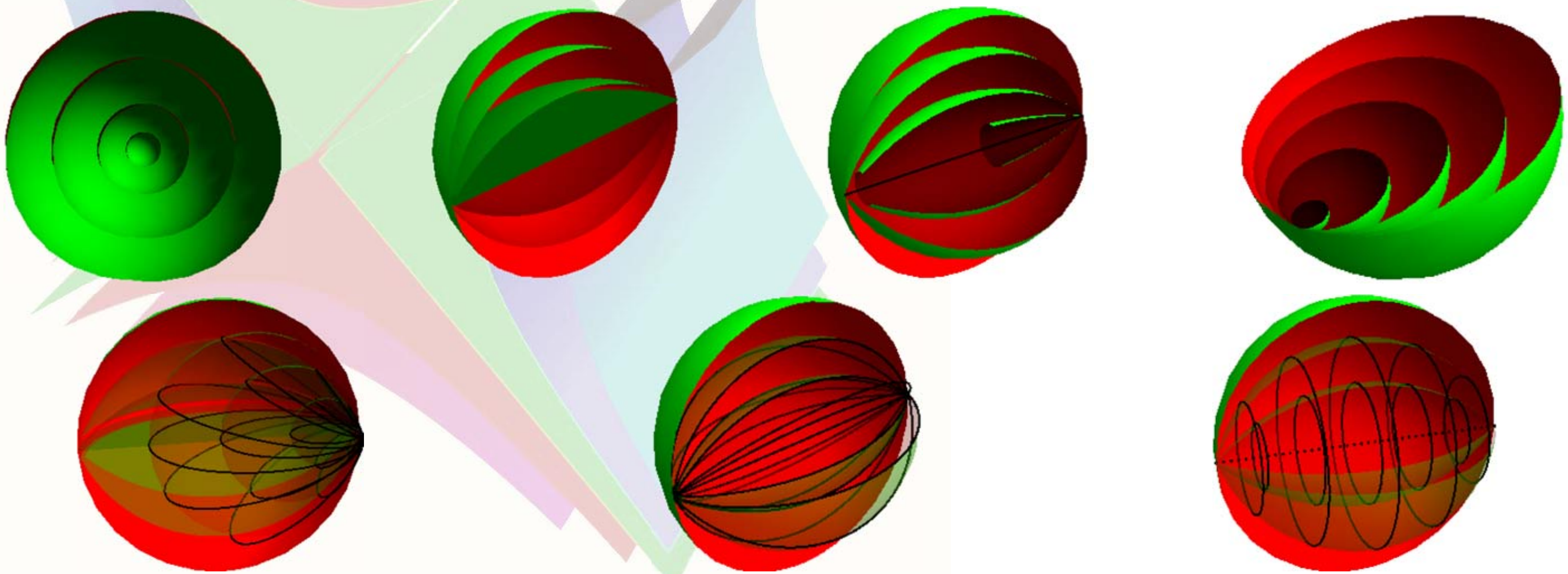
  [Dadok] Isotropy representations of symmetric spaces

- Euclidean spaces $\mathbb{R}^n$

  Isotropy representations of symmetric spaces $\times$ translations

- Real hyperbolic spaces $\mathbb{R}H^n$

  [Wu] $SO(1,k) \times K$ or $N \times K$, where K acts polarly on $\mathbb{R}^{n-k}$



Polar actions on $\mathbb{C}H^n$

Theorem. [Podestà, Thorbergsson] A polar action on $\mathbb{C}P^n$ is orbit equivalent to the a quotient of an isotropy representation of a Hermitian symmetric space

Theorem. [-, Domínguez-Vázquez, Kollross] Polar actions on $\mathbb{C}H^n$:

- $\mathfrak{h} = \mathfrak{q} \oplus \mathfrak{so}(1,k)$, $k \in \{0, \dots, n\}$
  $\mathfrak{q}$ subalgebra of $\mathfrak{u}(n-k)$
  Q acts polarly on $\mathbb{C}^{n-k}$ with totally real section
- $\mathfrak{h} = \mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$
  $\mathfrak{b}$ linear subspace of $\mathfrak{a}$, $\mathfrak{w}$ real subspace of $\mathfrak{g}_\alpha$,
  $\mathfrak{q}$ subalgebra of $\mathfrak{k}_0 = \mathfrak{n}_K(\mathfrak{a})$, $\mathfrak{q}$ normalizes $\mathfrak{w}$,
  Q acts polarly on $\mathfrak{g}_\alpha \ominus \mathfrak{w}$ with totally real section

Recall: $\mathfrak{w} = \bigoplus_{\varphi \in \Phi} \mathfrak{w}_\varphi$

Characterization of orbits of polar actions

[Heintze, Liu, Olmos]

$M \subset \bar{M}$ **isoparametric** ⟺

- normal bundle is flat
- parallel submanifolds have constant mean curvature in radial directions
- for any $p \in M$ there exists a *section* Σ_p through p (totally geodesic submanifold s.t. $T_p\Sigma_p = \nu_p M$)

Isoparametric in $\mathbb{C}H^2$ ⟺ principal orbit of polar action

[Terng]

$M \subset \bar{M}$ **isoparametric** ⟺

- normal bundle is flat
- eigenvalues of the shape operator with respect to any parallel normal vector field are constant

Terng-isoparametric in $\mathbb{C}H^2$ ⟺

- principal orbit of polar action
- Chen's surface
- circles