Some New Myers-Type Theorems via m-Bakry-Émery Ricci Curvature

Homare TADANO
Tokyo University of Science, JAPAN
tadano@rs.tus.ac.jp

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Aim & Plan

1. Introduction : A Brief Review of

   — The Classical Bonnet-Myers Theorem

2. Previous Works and New Results :

   — m-Modified Ricci and m-Bakry-Émery Ricci Curvatures

   — Bonnet-Myers Type Theorems

   — New Compactness Theorems

Bonnet-Myers Theorem

Natural questions about a complete Riemannian manifold are

(1) When is (M, g) compact ?

(2) How large is $\mathrm{diam}(M, g)$?

Theorem (S. B. Myers 1942) If $\exists\, \lambda > 0$ s.t. $\mathrm{Ric}_g \geqslant \lambda g$

$$\Longrightarrow \quad (M, g) : \text{compact} \quad \& \quad \mathrm{diam}(M, g) \leqslant \pi \sqrt{\frac{n-1}{\lambda}}.$$

$\rightsquigarrow$ A topological obstruction to the existence of a metric with a positive Ricci curvature bound.

Modified Ricci and Bakry-Émery Ricci Curvatures

$V \in \mathfrak{X}(M)$: vector field, $f \in \mathcal{C}^\infty(M)$: smooth function.

Definition (D. Bakry and M. Émery 1985, M. Limoncu 2009)

$$\mathrm{Ric}_V := \mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g$$: modified Ricci curvature

$$\mathrm{Ric}_f := \mathrm{Ric}_g + \mathrm{Hess} f$$: Bakry-Émery Ricci curvature

Good substitutes of the Ricci curvature :

eigenvalue estimates, Li-Yau Harnack inequalities, ...

Question Is the Myers theorem true for Ric_V and Ric_f ?

Remark The shrinking Gaussian soliton is non-compact.

A Bonnet-Myers Type Theorem via Modified Ricci Curvature

Theorem (M. Fernández-Lópes and E. García-Río 2004, M. Limoncu 2009, — 2015, J.-Y. Wu 2017)

Suppose $\exists\, \lambda > 0$ s.t.

$$\mathrm{Ric}_V := \mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g \geqslant \lambda g.$$

If $|V| \leqslant k$ for $\exists\, k \geqslant 0$

$$\Longrightarrow \quad (M,g) \text{ : compact} \quad \& \quad \mathrm{diam}(M,g) \leqslant \frac{2k}{\lambda} + \pi\sqrt{\frac{n-1}{\lambda}}.$$

Remark $|V| \leqslant k$ may be improved to $|V|(x) \leqslant \exists\, \alpha r(x) + \exists\, \beta$, where $0 \leqslant \alpha < \lambda$ and $\beta \in \mathbb{R}$.

A Bonnet-Myers Type Theorem via Bakry-Émery Ricci Curvature

Remark A function $f \in \mathcal{C}^\infty(M)$ does **not** always satisfy

$$|\nabla f| \leqslant \exists\, k \iff |f| \leqslant \exists\, k.$$

Theorem (G. Wei and W. Wylie 2007, M. Limoncu 2009, — 2015)

Suppose $\exists\, \lambda > 0$ s.t.

$$\mathrm{Ric}_f := \mathrm{Ric}_g + \mathrm{Hess}\, f \geqslant \lambda g.$$

If $|f| \leqslant k$ for $\exists\, k \geqslant 0$

$$\Longrightarrow \quad (M, g) \ : \text{compact} \quad \& \quad \mathrm{diam}(M, g) \leqslant \frac{\pi}{\sqrt{\lambda}} \sqrt{\frac{8k}{\pi} + n - 1}.$$

A New Compactness Theorem via Bakry-Émery Ricci Curvature

Theorem (— 2018)

Suppose $|f| \leqslant k$ for $\exists\, k \geqslant 0$.

If $\exists\, p \in M$, $\exists\, r_0 > 0$, $\exists\, \ell \geqslant 2$ s.t.

$$\mathrm{Ric}_f(x) \geqslant (n + 4k - 1)\frac{C(r_0, \ell)}{(r_0 + r(x))^{\ell}} g(x)$$

for $\forall\, x \in M$, where

$$C(r_0, \ell) := \begin{cases} \frac{(\ell-1)^{\ell}}{(\ell-2)^{\ell-2}} r_0^{\ell-2} & \ell > 2, \\ 1 + \varepsilon,\ \forall \varepsilon > 0 & \ell = 2 \end{cases}$$

$\Longrightarrow$ (M, g) : compact.

Remark This theorem was proved by J. Wan (2017) via Ric_g.

m-Modified Ricci and
m-Bakry-Émery Ricci Curvatures

$V \in \mathfrak{X}(M)$: vector field, $f \in \mathcal{C}^\infty(M)$: smooth function,

$m \in \mathbb{R} \cup \{\pm\infty\}$.

Definition (D. Bakry and M. Émery 1985, M. Limoncu 2009)

$$\mathrm{Ric}_V^m := \mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g - \frac{1}{m-n} V^* \otimes V^* \quad (m \neq n), \quad \mathrm{Ric}_V^n := \mathrm{Ric}_g$$

: m-modified Ricci curvature

$$\mathrm{Ric}_f^m := \mathrm{Ric}_g + \mathrm{Hess} f - \frac{1}{m-n} df \otimes df \quad (m \neq n), \quad \mathrm{Ric}_f^n := \mathrm{Ric}_g$$

: m-Bakry-Émery Ricci curvature

(1) Good substitutes of the Ricci curvature.

(2) Important in Optimal Transport Theory by Lott-Sturm-Villani and in Perelman's entropy formula for the Ricci flow.

A Bonnet-Myers Type Theorem via m-Modified Ricci and m-Bakry-Émery Ricci Curvatures

Theorem (Z. Qian 1995, M. Limoncu 2009, K. Kuwada 2011)

Let $m \geqslant n$. If $\exists\, \lambda > 0$ s.t.

$$\mathrm{Ric}_V^m := \mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g - \frac{1}{m-n} V^* \otimes V^* \geqslant \lambda g.$$

$\Longrightarrow$ (M, g) : **compact** & $\mathrm{diam}(M, g) \leqslant \pi \sqrt{\dfrac{m-1}{\lambda}}.$

$\rightsquigarrow$ A **topological obstruction** to the existence of a metric with a **positive m-modified Ricci curvature bound**.

A New Compactness Theorem via m-Modified Ricci and m-Bakry-Émery Ricci Curvatures

Theorem (— 2018)

Let $m \geqslant n$. Suppose $\exists\, p \in M$, $\exists\, r_0 > 0$, $\exists\, \ell \geqslant 2$ s.t.

$$\mathrm{Ric}_V^m(x) \geqslant (m-1)\frac{C(r_0,\ell)}{(r_0+r(x))^\ell} g(x)$$

for $\forall\, x \in M$, where

$$C(r_0,\ell) := \begin{cases} \frac{(\ell-1)^\ell}{(\ell-2)^{\ell-2}} r_0^{\ell-2} & \ell > 2, \\ 1+\varepsilon,\ \forall \varepsilon > 0 & \ell = 2 \end{cases}$$

$\Longrightarrow \quad (M, g)$: **compact**.

Remark This theorem was proved by J. Wan (2017) via Ric_g.