Symmetries and non-negative curvature of vector bundles

Symmetry and shape, Santiago de Compostela

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Based on joint work with Manuel Amann and Marcus Zibrowius

Outline

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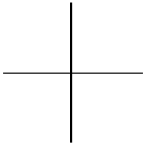

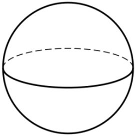

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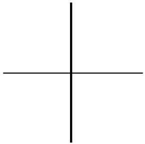

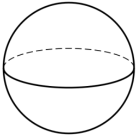

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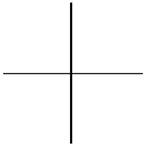

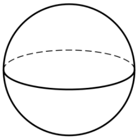

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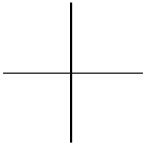

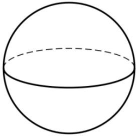

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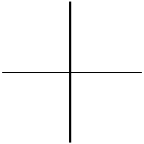

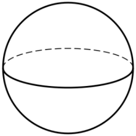

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2. Apply the methods (topological part)

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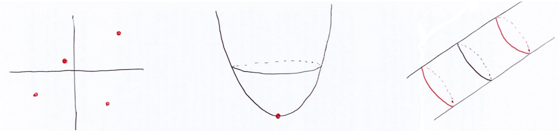

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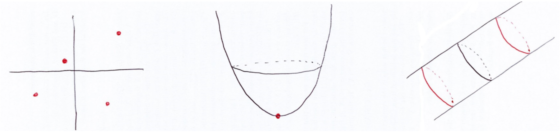

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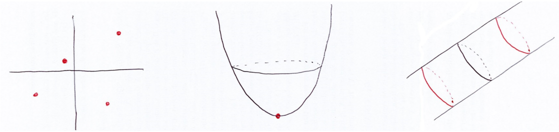

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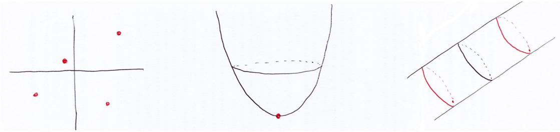

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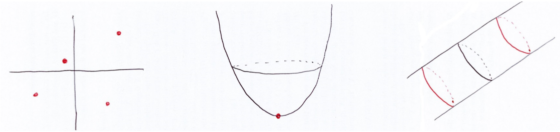

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Goal: Extend Rigas' result to other base manifolds (with a lot of symmetries).

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A G-vector bundle over a G-manifold is a v.b. $\pi : E \to M$, where E is a G-manifold, π is G-equivariant and $g : E_x \to E_{gx}$ is linear.

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$\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}$, every homotopy $\mathbb{RP}^5$, every $SO(4)$-principal bundle over $\mathbb{S}^4$,...

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- There exist G-manifolds M satisfying the following:

for every **complex** *vector bundle* $E \to M$, *there is an integer* k *such that* $E \oplus \mathbb{C}^k$ *is a* G*-vector bundle*.

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- There is a natural (FORGETFUL) map

$$F : K_G(M) \to K(M)$$

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$\Rightarrow$ Hodgkin's spectral sequence (1975).

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• Examples: all homogenous spaces with $sec > 0$ $(\mathbb{S}^n, \mathbb{CP}^n, \mathbb{HP}^n, ..)$

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- Examples: there is a cohomo 1 action by $SU(2)^{n+1}$ on

$$(\mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}) \times (\mathbb{S}^2)^n, \qquad n \geq 0$$

satisfying the hypotheses in Theorem 3. This manifold is not even homotopy equivalent to a homogeneous space.

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- Examples: the hypotheses now allow M's with $\chi(M) = 0$.

THANK YOU!