# The index of a symmetric space

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# Introduction.

This talk is based on joint works with *Jürgen Berndt*.

A submanifold $\Sigma$ of a Riemannian manifold $M$ is said to be totally geodesic if every geodesic in $\Sigma$ is also a geodesic in $M$.

The existence and classification of totally geodesic submanifolds are two fundamental problems in submanifold geometry.

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The totally geodesic submanifolds in Riemannian symmetric spaces of rank one were classified by *Joseph Wolf* in 1963.

It is remarkable that the classification of totally geodesic submanifolds in Riemannian symmetric spaces of higher rank is a very complicated and essentially unsolved problem.

*Élie Cartan* already noticed an algebraic characterization of totally geodesic submanifolds in terms of Lie triple systems.

Although a Lie triple system is an elementary algebraic object, explicit calculations with them can be tremendously complicated.

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A rather well-known result states that an irreducible Riemannian symmetric space which admits a totally geodesic hypersurface must be a space of constant curvature. As far as we know, the first proof of this fact was given by *N. Iwahori* in 1965.

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In 1980, *A. L. Onishchik* introduced the index $i(M)$ of a Riemannian symmetric space $M$ as the minimal codimension of a totally geodesic submanifold of $M$. Onishchik gave an alternative proof for Iwahori's result and also classified the irreducible Riemannian symmetric spaces with index 2.

In this lecture we would like to present a new approach to the index based, essentially, on geometric tools. By means of this approach we were able to determine the index of all symmetric spaces, with the exception of three families of classical type (on which we are still working).

Our point of view also allows us to determine the maximal totally geodesic submanifolds (of symmetric spaces) that are non-semisimple.

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# Rank and index.

The starting point for dealing with the index of a symmetric space is the inequality in the following main result:

**Theorem** (Berndt-O.)

*Let $M$ be an irreducible Riemannian symmetric space. Then*

$$rk(M) \leq i(M).$$

*Moreover, the equality holds if and only if, up to duality, $M = SL_{k+1}/SO_{k+1}$ or $M = G_k^*(\mathbb{R}^{n+k}) = SO^o_{k+n}/SO_kSO_n$*

We prove the inequality $\mathrm{rank}(M) \leq i(M)$ by showing the following: *if $\Sigma$ is a totally geodesic submanifold of a symmetric space $M$, then there exists a maximal flat $F$ of $M$ that intersects $\Sigma$ transversally.*

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Let us now fix some notation.

$$M = G/K$$

denotes a simply connected symmetric space with Cartan decompostion $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ at $p = [e]$, $\mathfrak{p} \simeq T_pM$,

$$\Sigma \subset M$$

denotes a complete totally geodesic submanifold with $p \in \Sigma$,

$$G^\Sigma$$

is the group of *glide transformations* of $\Sigma$, i.e.,

$$\mathrm{Lie}(G^\Sigma) = T_p\Sigma \oplus [T_p\Sigma, T_p\Sigma]$$

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$$\tilde{\rho}(g) = (dg)_{|\nu_p \Sigma}$$

The slice representation $\rho : G_p^{\Sigma} \to \mathrm{O}(\nu_p \Sigma)$ is

$$\rho = \tilde{\rho}_{|G_p^{\Sigma}}$$

If $M$ is of the non-compact type then $G_p^{\Sigma}$ is connected, since $\Sigma$ is simply connected and so $\rho : G_p^{\Sigma} \to \mathrm{SO}(\nu_p \Sigma)$.

*For our purposes, from duality, we may assume that $M$ is of the non-compact type.*

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# Reflective totally geodesic submanifolds.

A totally geodesic submanifold $\Sigma$ of a symmetric space $M$ is called *reflective* if the exponential of the normal space $\exp_p(\nu_p\Sigma)$ is also a totally geodesic submanifold of $M$.

Equivalently, $\Sigma$ is reflective if $T_p\Sigma$ and $\nu_p\Sigma$ are both Lie triple systems.

One has that $\Sigma$ is reflective if and only if the reflection of $M$ in $\Sigma$ is an isometry.

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# Sufficient criteria for reflectivity.

By making use of the Slice Lemma, we proved the following results:

**Proposition (Berndt-O.; Berndt-O.-Rodríguez)**

*Let $M = G/K$ be an irreducible Riemannian symmetric space with $\mathrm{rk}(M) \geq 2$, where $(G, K)$ is an effective Riemannian symmetric pair. Let $\Sigma$ be a semisimple totally geodesic submanifold of $M$ with $p = [e] \in \Sigma$. Then $\Sigma$ is reflective if and only if the kernel of the full slice representation $\tilde{\rho} : \tilde{G}_p^{\Sigma} \to O(\nu_p \Sigma)$ is non-trivial.*

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# Non-semisimple totally geodesic submanifolds.

If $\Sigma \subset M = G/K$ is totally geodesic and $p \in \Sigma$, then $T_p\Sigma$ is a Lie triple system of $\mathfrak{p}$. Assume that $\Sigma$ is not semisimple. Then there exists $0 \neq v \in T_p\Sigma$ such that $[v, T_p\Sigma] = 0$. Therefore $T_p\Sigma$ must be contained in the Lie triple system

$$\mathcal{C}(v) = \{z \in \mathfrak{p} : [v, z] = 0\},$$

the centralizer of $v$ in $\mathfrak{p}$. One has that $\mathcal{C}(v)$ coincides with the normal space $\nu_v(K.v)$ of the isotropy orbit. If $\Sigma$ is maximal, then the orbit $K.v$ must be most singular and hence

$$T_p\Sigma \cap v^\perp$$

is semisimple.

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If $\Sigma \subset M = G/K$ is totally geodesic and $p \in \Sigma$, then $T_p\Sigma$ is a Lie triple system of $\mathfrak{p}$. Assume that $\Sigma$ is not semisimple. Then there exists $0 \neq v \in T_p\Sigma$ such that $[v, T_p\Sigma] = 0$. Therefore $T_p\Sigma$ must be contained in the Lie triple system

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**Theorem (Berndt-O.)**

*Let $\Sigma$ be a non-semimple totally geodesic submanifold of a symmetric space $M = G/K$, $p = [e] \in \Sigma$. Then $\Sigma$ is a maximal totally geodesic submanifold of $M$ if and only if $T_p\Sigma$ coincides with the normal space to an extrinsic symmetric orbit $K.v$ (and so it is reflective).*

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# Fixed vectors of the slice representation.

**Theorem (Berndt-O.-Rodríguez)**

*Let $M = G/K$ be an irreducible, simply connected, Riemannian symmetric space with $\mathrm{rk}(M) \geq 2$, where $G = I(M)^o$, $K = G_p$ and $p \in M$. Let $\Sigma = G'/K'$ be a (proper) totally geodesic submanifold of $M$ with $p \in \Sigma$ and $\dim(\Sigma) \geq \frac{1}{2}\dim(M)$, where $G'$ is the subgroup of $G$ consisting of the glide transformations of $\Sigma$ and $K' = (G')_p$. Let $\rho$ be the slice representation of $(K')^o$ on $SO(\nu_o\Sigma)$. Then the following statements are equivalent:*

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# The index conjecture.

Let us define the *reflective index* of a symmetric space $M$ by

$$i_r = min\ \{\mathrm{codim}(\Sigma) : \Sigma \subsetneq M \text{ is totally geodesic and reflective}\}$$

The reflective index can be computed from Leung's classification of reflective totally geodesic submanifolds of symmetric spaces. Clearly, $i(M) \leq i_r(M)$. Therefore

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In all known examples $i(M) = i_r(M)$, except for the space $M = \mathrm{G}_2^2/\mathrm{SO}_4$, or its compact dual. In this case the index is 3 but the reflective index is 4. We conjecture that this is the only exception. Namely,

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**Conjecture.** *The index of an irreducible symmetric space $M$, which is different from $G_2/\mathrm{SO}(4)$ and its symmetric dual, coincides with its reflective index $i_r(M)$.*

We proved this conjecture for many classical families and for all symmetric spaces of group type. Moreover, we also determined the index, verifying the conjecture, for all exceptional symmetric spaces.
The conjecture remains open only for the following three series of classical symmetric spaces and their symmetric dual spaces:

($i$) $M = SO_{2k+2}/U_{k+1}$ for $k \geq 5$. Conjecture: $i(M) = 2k$.
($ii$) $M = SU_{2k+2}/Sp_{k+1}$ for $k \geq 3$. Conjecture: $i(M) = 4k$.
($iii$) $M = Sp_{2k+l}/Sp_k Sp_{k+l}$ for $l \geq 0$ and $k \geq \text{máx}\{3, l+2\}$. Conjecture: $i(M) = 4k$.

We have some strategy, for dealing with these families, that hopefully would prove the index conjecture. But this is still a work in progress.

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# *Happy birthday Jürgen!!*

*Thank you for the generous and great influence of your ideas and work!*

Moitas grazas pola atención e gocen da gran hospitalidade galega!!