The normal holonomy group of complex submanifolds

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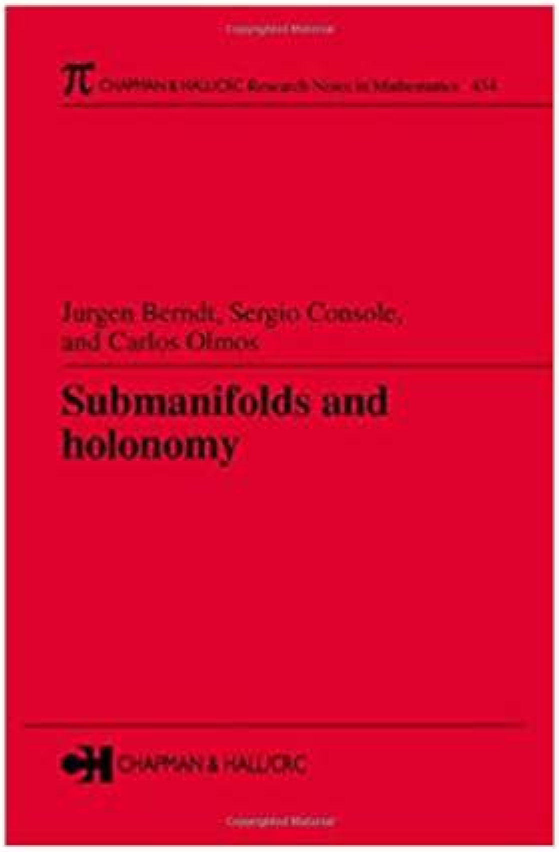
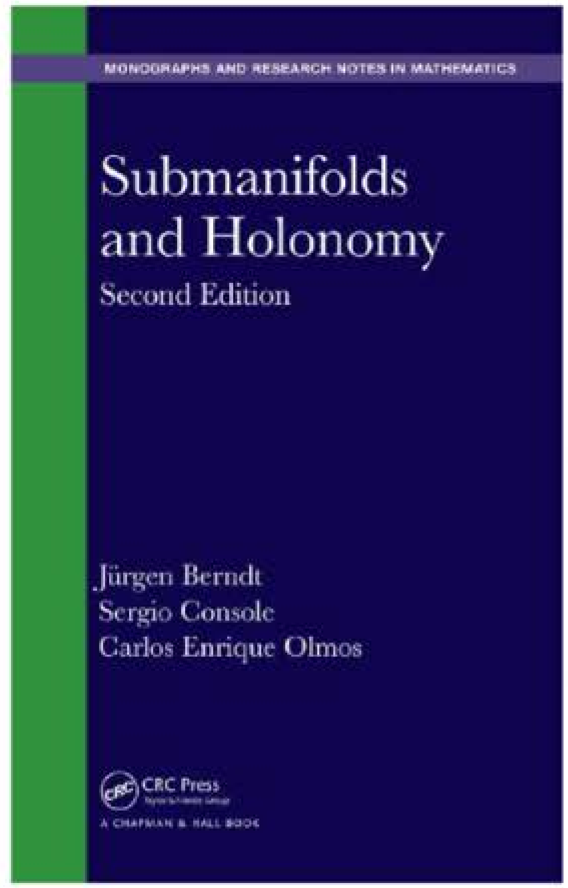
Abstract: This talk is going to be a survey talk of results, ideas and questions about the normal holonomy group of complex submanifolds of complex space forms.





International Conference CURVATURE IN GEOMETRY in honour of Professor Lieven Vanhecke, Lecce (Italy) was held June 11-14, 2003

Jurgen2003

1 Olmos' Normal Holonomy Theorem (NHT). [O90].

Theorem 3.1. *Let M^n be an immersed submanifold of a Riemannian manifold Q^N of constant curvature. Let $p \in M$ and let Φ^* be the restricted holonomy group of the normal connection at p. Then Φ^* is compact, there exists a unique (up to order) orthogonal decomposition of the normal space at p, $N(M)_p = \mathbf{V}_0 \oplus \cdots \oplus \mathbf{V}_k$, into Φ^*-invariant subspaces, and there exist $\Phi_0, \ldots, \Phi_k$ normal Lie subgroups of Φ^* such that:*

(i) $\Phi^* = \Phi_0 \times \cdots \times \Phi_k$ *(direct product).*
(ii) Φ_i *acts trivially on* $\mathbf{V}_j$ *if* $i \neq j$.
(iii) $\Phi_0 = \{1\}$ *and, if* $i \geq 1$, Φ_i *acts irreducibly on* $\mathbf{V}_i$ *as the isotropy representation of a simple Riemannian symmetric space.*

$M \subset \mathbb{R}^n$

$T\mathbb{R}^N = TM \oplus NM$

$D_X \xi = -A_\xi(X) + \nabla^\perp_X \xi$

Constant Principal curvature.

The above theorem plays a central role in the theory of isoparametric submanifolds or more in general in the theory of submanifolds with constant principal curvatures [BCO16].

An important Theorem of Thorbergsson shows that a full an irreducible isoparametric submanifold of $\mathbb{R}^n$ of codimension greater than 2 is an orbit of an s-representation.

Isoparametric

$\Updownarrow$

$K.p = M \subset \mathbb{R}^N \approx T_0 S$ C.P.C + Flat normal bundle.

$M \subset \mathbb{C}^N = \mathbb{R}^{2N}$

$J(T_pM) = T_pM$

2 NHT $\mathbb{C}^n$: The extrinsic De Rham splitting; [D00]

Theorem 1.1 *A complex isometric full immersion of a simply connected complete Kähler manifold $f : M \to \mathbb{C}^N$ is irreducible, up a totally geodesic factor, if and only if the normal holonomy group acts irreducibly.*

$\Phi^* \curvearrowright N = \nu_1 \oplus \nu_2$ ξ_1, ξ_2

$M = M_1 \times M_2$

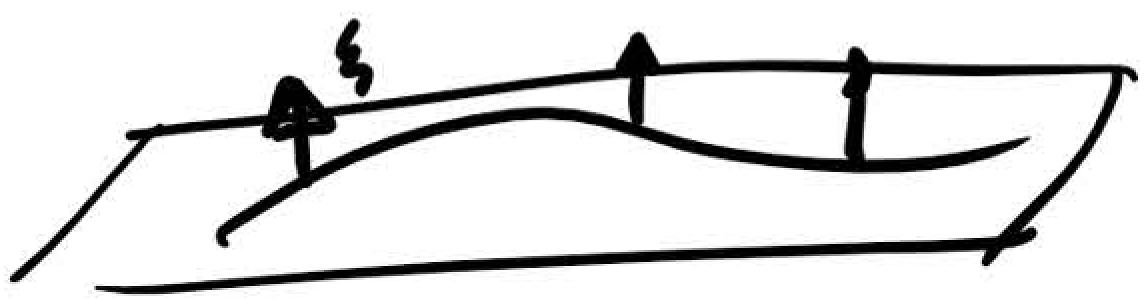
$\downarrow \quad \downarrow$

$\mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$

$\xi \in \nu_0 \leftarrow \Phi^*$

$\nabla^\perp \xi \equiv 0$

$\Longrightarrow$ Reduce codim



$A_\xi = 0$

ξ const on Eucl

$\langle R^\perp_{X,Y}\xi, \eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle$

$0 = \langle [A_\xi, A_{J\xi}]X, Y\rangle$

OSS: $\Delta \subset \mathbb{C} \xrightarrow{f \text{ holomorph}} \mathbb{C}^N$

$\Rightarrow f(\Delta) \subset S_p(r)$

$\Rightarrow$ f is const.

$A_\xi \cdot A_{J\xi} - A_{J\xi} \cdot A_\xi = 0$

$A_\xi \cdot J A_\xi - J A_\xi^2 = 0$

$-2J A_\xi^2 = 0$

$A_\xi^2 \equiv 0$

$A_{\xi_1} \cdot A_{\xi_2} \equiv 0$

$\bigcap_{\xi \in \nu_1} \operatorname{Ker} A_\xi = \mathcal{D}_1$

$\bigcap_{\xi \in \nu_2} \operatorname{Ker} A_\xi = \mathcal{D}_2$

$\to \mathcal{D}_1 \cap \mathcal{D}_2 = \{0\}$

$0 = \alpha(\mathcal{D}_1, \mathcal{D}_2)$ + tot...

3 NHT $\mathbb{CP}^n$: work with D.V. Alekseevsky; [AD04]

Motivations ?

It turns out that Olmos NHT is not true if the submanifold is not full.

THEOREM 1. *Let $\mathbb{S}^n_c = \mathbb{C}P^n, \mathbb{C}^n, \mathbb{C}H^n$ be the complex space form of holomorphic constant sectional curvature c and $M \subset \mathbb{S}^n_c$ be a Kähler submanifold. If the normal holonomy group $\mathrm{Hol}_p(\nabla^\perp) \subset \mathrm{SO}(N_p(M))$ of M at a point $p \in M$ acts irreducibly on the normal space $N_p(M)$ then $\mathrm{Hol}_p(\nabla^\perp)$ is linearly isomorphic to the isotropy group of an irreducible Hermitian symmetric space, that is, one of the groups in Table 1. In particular, this is true if $M \subset \mathbb{C}^n$ is a locally irreducible Kähler submanifold of $\mathbb{C}^n$.*

TABLE 1. *Isotropy representations $K \hookrightarrow \mathrm{SO}(V)$ of compact irreducible Hermitian symmetric spaces G/K.*

G/K	K	V
$\mathrm{Gr}_p(\mathbb{C}^{p+q}) := \mathrm{SU}(p+q)/S(U(p)\times U(q))$	$S(U(p)\times U(q))$	$\mathbb{C}^p \otimes \mathbb{C}^q$
$\mathrm{SO}(2n)/U(n)$	$U(n)$	$\Lambda^2(\mathbb{C}^n)$
$\mathrm{Gr}_2(\mathbb{R}^{n+2}) := \mathrm{SO}(n+2)/\mathrm{SO}(2)\times\mathrm{SO}(n)$	$\mathrm{SO}(2)\times\mathrm{SO}(n)$	$\mathbb{R}^2\otimes\mathbb{R}^n$
$\mathrm{Sp}(n)/U(n)$	$U(n)$	$S^2\mathbb{C}^n$
$E_6/T^1\cdot\mathrm{Spin}_{10}$	$T^1\cdot\mathrm{Spin}_{10}$	$\mathbb{C}^{16}$
$E_7/T^1\cdot E_6$	$T^1\cdot E_6$	$\mathbb{C}^{27}$

KÄHLER - EINSTEIN subm of $\mathbb{CP}^n$

$\mathbb{CP}^n \subset \mathbb{CP}^{n+2}$

$\nu_p \mathbb{CP}^n \cong \mathbb{C}^2$

ϕ^* $e^{i\theta}$

$SO(2)$

Dominique Mullin

Theorem 12. *Let $(M^m, g) \subset \mathbb{C}P^n$ be a non-full and non-totally geodesic Kähler–Einstein submanifold with Ricci tensor $\mathrm{Ric}_M = k \cdot g$. Let $\mathbb{C}P^{\overline{m}} \subset \mathbb{C}P^n$ be the totally geodesic Kähler submanifold of $\mathbb{C}P^n$ such that M is full in $\mathbb{C}P^{\overline{m}}$. Then,*

$$\mu = \frac{\overline{m} - m}{\overline{m} + 1 - k/2}$$

where μ is the invariant of Definition 1.

ϕ^* compact $\iff$ $\mu \in \mathbb{Q}$

4 NHT $\mathbb{CP}^n$: works with S. Console and C. Olmos; [CD09, CDO11]

Parallel submanifolds of complex projective space 3

Table 1 Symmetric complex submanifolds $M \subset \mathbb{P}(T_{[K]}G/K)$

Hermitian symmetric space G/K	M as complex K-orbit	Normal holonomy	Remarks
$\dfrac{E_7}{T^1 \cdot E_6}$	$\dfrac{E_6}{T^1 \cdot Spin_{10}}$	$\dfrac{SO(12)}{T^1 \cdot SO(10)}$	
$\dfrac{E_6}{T^1 \cdot Spin_{10}}$	$\dfrac{SO(10)}{U(5)}$	$\dfrac{U(6)}{U(5)}$	
$\dfrac{Sp(n+1)}{U(n+1)}$	$\mathbb{C}P^n$	$\dfrac{Sp(n)}{U(n)}$	Veronese
$Gr_2^+(\mathbb{R}^{n+2}) := \dfrac{SO(n+2)}{T^1 \cdot SO(n)}$	$Gr_2^+(\mathbb{R}^n)$	$\dfrac{U(2)}{U(1)}$	Quadrics
$\dfrac{SO(2n)}{U(n)}$	$Gr_2(\mathbb{C}^n)$	$\dfrac{SO(2(n-2))}{U(n-2)}$	Plücker
$Gr_a(\mathbb{C}^{a+b}) := \dfrac{SU(a+b)}{S(U(a) \times U(b))}$	$\mathbb{C}P^{a-1} \times \mathbb{C}P^{b-1}$	$\dfrac{SU(a+b-2)}{S(U(a-1) \times U(b-1))}$	Segre

The space in the third column is the Hermitian symmetric space whose isotropy representation gives the normal holonomy action

Note: the two exceptional spaces and the quadrics Q_n, $n \notin \{1,2,3,4,6,10\}$ are not in the third column

4.1 An extrinsic Berger type theorem; [CDO11]

Theorem 2 *The normal holonomy group of a complete irreducible and full immersed complex submanifold of* $\mathbb{C}^n$ *acts transitively on the unit sphere of the normal space. Indeed,* $\Phi^{\perp} = U(k)$, *where* k *is the codimension of the submanifold.*

Theorem 1 *Let* M *be a full and complete complex projective submanifold of* $\mathbb{C}P^n$. *Then the following are equivalent:*

(1) *The normal holonomy is not transitive on the unit sphere of the normal space (i.e., different from* $U(k)$, $k = \mathrm{codim}(M)$, *since it is an s-representation).*
(2) M *is the complex orbit, in the complex projective space, of the isotropy representation of an irreducible Hermitian symmetric space of rank greater or equal to* 3.

$\mathcal{C}S_1 \subset \mathcal{C}S_2 \subset \cdots \subset \mathcal{C}S_r$ G/K

5 NHT $\mathbb{CP}^n$: work with F. Vittone. [DV17]

Bounded domain in $\mathbb{C}^n$ which Symmetric

Circle

Theorem 1. *Let $M \subset \mathbb{C}^n$ be a full and irreducible complex submanifold (non necessarily complete w.r.t. the induced metric of $\mathbb{C}^n$). Let $Hol^*(M, \nabla^\perp)$ be the restricted normal holonomy group of M. If the action of $Hol^*(M, \nabla^\perp)$ is non-transitive on the unit sphere of the normal space then there exists an irreducible bounded symmetric domain $D \subset \mathbb{C}^n$ (realized as a circled domain) such that M is an open subset of the smooth part of the Mok's characteristic cone $\mathcal{C}S^j(D)$ for $1 \le j < rank(D) - 1$.*

Conversely, for any irreducible bounded symmetric domain $D \subset \mathbb{C}^n$, the restricted normal holonomy group of an open subset of the smooth part of the cone $\mathcal{C}S^j(D)$ for $1 \le j < rank(D) - 1$ acts irreducibly but non-transitively on the unit sphere of each normal space.

DEFINITION 1

Let $1 \le k \le r(\Omega)$ and let $\mathcal{S}_{k,x}$ denote $\{[\xi]: \xi \in T_x(\Omega)$ and $1 \le r(\xi) \le k\}$. We call $\mathcal{S}_{k,x}(\Omega) \subset \mathbb{P}T_x(X)$ the k–th characteristic projective subvariety at $x \in \Omega$. The union $\mathcal{S}_k(\Omega) := \cup_{x \in \Omega} \mathcal{S}_{k,x} \subset \mathbb{P}T(\Omega)$ is called the k–th characteristic bundle over Ω. The quotient $\mathcal{S}_k(\Omega)/\Gamma$ (noting that $\mathcal{S}_k(\Omega)$ is invariant under the standard action of Γ), written $\mathcal{S}_k(X)$, is called the k–th characteristic bundle over X.

[Mok89, page 252]

6 Two open problems.

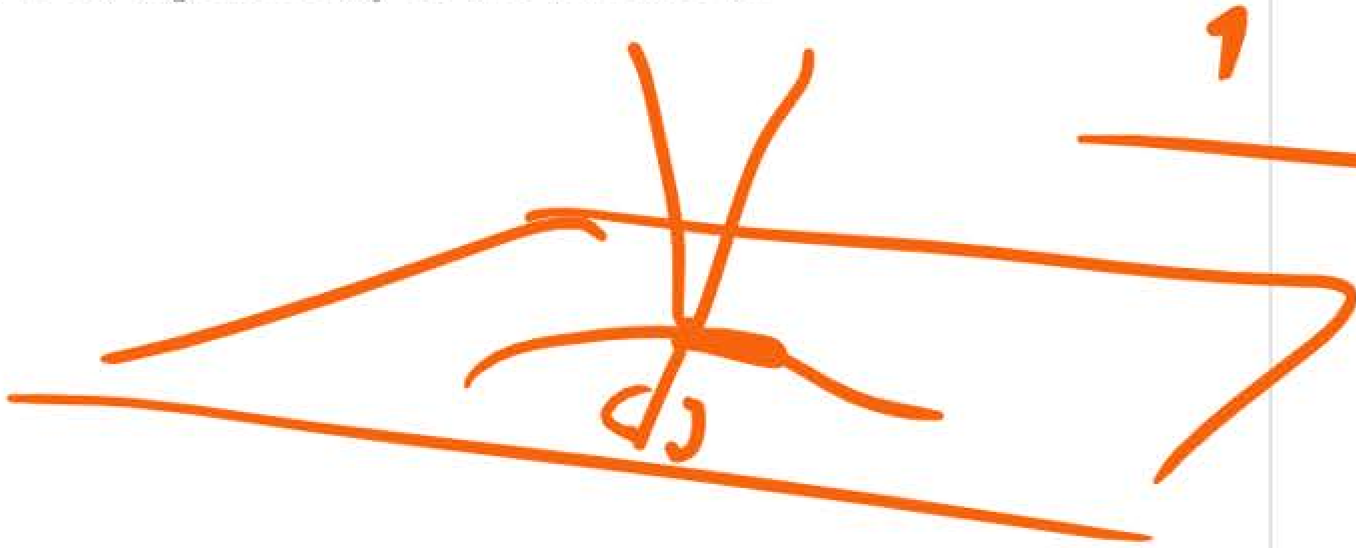
For complex submanifolds of $\mathbb{CH}^n$ the conjecture is that if the normal holonomy is irreducible but non transitive then must be the full unitary group of the normal space.

We remark that there are examples of full complex submanifolds of $\mathbb{CH}^n$ whose normal holonomy is not irreducible.

Φ^ν not irred

$\Phi^\nu = U(\nu)$

Is the invariant μ a rational number ? or equivalently is the normal holonomy a compact Lie group?



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