Homogeneous and inhomogeneous isoparametric hypersurfaces in symmetric spaces of noncompact type

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Symmetry and Shape
Celebrating the 60th birthday of Prof. J. Berndt,
Santiago de Compostela

Main new results

Joint work with J. Carlos Díaz-Ramos and Miguel Domíguez-Vázquez

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  $\Longrightarrow$ Classification of cohomogeneity one actions on **symmetric spaces of rank one**

- Uncountably many **inhomogeneous isoparametric** families of hypersurfaces with **constant principal curvatures**

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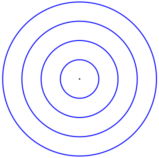

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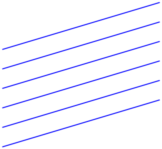
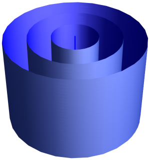
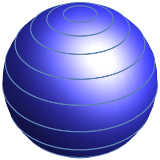
$A \cdot v = Av$



$\mathbb{R} \circlearrowright \mathbb{R}^2$
$t \cdot v = v + tw$



$\mathrm{SO}(2) \times \mathbb{R} \circlearrowright \mathbb{R}^3$
$(A, t) \cdot v = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} v + \begin{pmatrix} 0 \\ t \end{pmatrix}$



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Equivalent problem

Classify homogeneous hypersurfaces in $\overline{M}$ up to congruence.

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Question

What happens with homogeneous hypersurfaces in $\mathbb{H}H^n$, $n \geq 3$?

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One totally geodesic singular orbit [Berndt, Brück (2001)]

Tubes around tot. geodesic submanifolds P in $\mathbb{F}H^n$ are homogeneous iff

- in $\mathbb{R}H^n$: $P = \{\text{point}\}, \mathbb{R}H^1, \dots, \mathbb{R}H^{n-1}$
- in $\mathbb{C}H^n$: $P = \{\text{point}\}, \mathbb{C}H^1, \dots, \mathbb{C}H^{n-1}, \mathbb{R}H^n$
- in $\mathbb{H}H^n$: $P = \{\text{point}\}, \mathbb{H}H^1, \dots, \mathbb{H}H^{n-1}, \mathbb{C}H^n$
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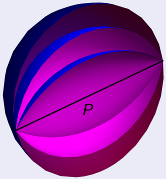

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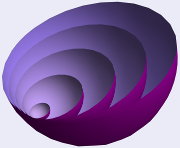
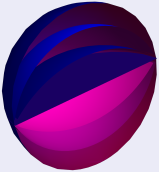
No singular orbits [Berndt, Tamaru (2003)]

Orbit equivalent to the action of:

- $N \rightsquigarrow$ horosphere foliation
- The connected subgroup of G with Lie algebra $\mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{z}$, where $\mathfrak{w}$ is a (real) hyperplane in $\mathfrak{v}$





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A non-totally geodesic singular orbit [Berndt, Tamaru (2007)]

$\mathfrak{w} \subsetneq \mathfrak{v}$ subspace $\Longrightarrow \mathfrak{s}_{\mathfrak{w}} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{z}$ is a Lie algebra

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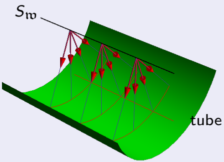

Cohomogeneity one actions on hyperbolic spaces

$\mathbb{F}H^n \cong G/K \overset{\text{isom.}}{\cong} AN$ symmetric space of noncompact type and rank one

In particular, $\mathfrak{a} \simeq \mathbb{R}$, $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ and $K_0 := N_K(\mathfrak{a})$.

$\overline{M}$	$\mathbb{R}H^n$	$\mathbb{C}H^n$	$\mathbb{H}H^n$	$\mathbb{O}H^2$
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$\mathfrak{v}$	$\mathbb{R}^{n-1}$	$\mathbb{C}^{n-1}$	$\mathbb{H}^{n-1}$	$\mathbb{O}$
$\dim \mathfrak{z}$	0	1	3	7
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A non-totally geodesic singular orbit [Berndt, Tamaru (2007)]

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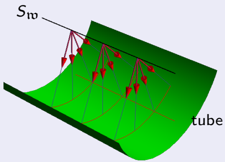

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Theorem [Berndt, Tamaru (2007)]

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The study of the last case was carried out for $\mathbb{R}H^n$, $\mathbb{C}H^n$, $\mathbb{H}H^2$ and $\mathbb{O}H^2$

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Classify real subspaces $\mathfrak{w} \subset \mathfrak{v} \cong \mathbb{H}^n$ such that $N_{K_0}(\mathfrak{w})$ acts transitively on the unit sphere of $\mathfrak{w}^\perp$, up to conjugation by $k \in K_0$.

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Quaternionic Kähler angle

$\mathfrak{J} \subset \mathrm{End}_{\mathbb{R}}(\mathbb{H}^n)$ quaternionic structure of $\mathbb{H}^n$
$\{J_1, J_2, J_3\}$ canonical basis of $\mathfrak{J}$: $J_i^2 = -\mathrm{Id}$, $J_i J_i^\top = \mathrm{Id}$, $J_i J_{i+1} = J_{i+2}$

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$$L_v \colon \mathfrak{J} \times \mathfrak{J} \to \mathbb{R}, \quad L_v(J, J') := \langle \pi J v, \pi J' v \rangle.$$

The **quaternionic Kähler angle** of v with respect to V is the triple $(\varphi_1, \varphi_2, \varphi_3)$, with $\varphi_1 \leq \varphi_2 \leq \varphi_3$, such that the eigenvalues of L_v are $\cos^2(\varphi_i)||v||^2$, $i = 1, 2, 3$.

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Proposition [Berndt, Brück (2001)]

$V \subset \mathbb{H}^n$ protohomogeneous $\Rightarrow$ V has constant quaternionic Kähler angle.

Some known results

There are subspaces V with constant quaternionic Kähler angle $(0, 0, 0)$, $(0, 0, \pi/2)$, $(0, \pi/2, \pi/2)$, $(\pi/2, \pi/2, \pi/2)$, $(\varphi, \pi/2, \pi/2)$, $(0, \varphi, \varphi)$...

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Not every triple arises as the constant quaternionic Kähler angle of a subspace V, e.g. $(0,0,\varphi)$, $\varphi \in (0,\pi/2)$

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Theorem [Díaz-Ramos, Domínguez-Vázquez (2013)]

The tubes around $S_{\mathfrak{w}}$ have constant principal curvatures if and only if $\mathfrak{w}^\perp \subset \mathfrak{v}$ has constant quaternionic Kähler angle.

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- If $k \geq 5$ is odd, then $\Phi(V) = (\pi/2, \pi/2, \pi/2)$.

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Question

What if we mix both types of 4-dimensional subspaces, V_+ and V_-?

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If $r_+, r_- \geq 1$, then V is a non-protohomogeneous subspace of $\mathbb{H}^n$ with constant quaternionic Kähler angle.

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$S_\mathfrak{w}$ and the tubes around it define an **inhomogeneous isoparametric** family of hypersurfaces with **constant principal curvatures** in $\mathbb{H}H^{n+1}$.