A topological lower bound for the energy of a unit vector field on a closed Euclidean hypersurface

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Energy functional

Let M^m be a compact oriented Riemannian manifold, $m \geq 2$, and let ∇ denote its Levi-Civita connection. The energy of a unit vector field on M is defined as the energy of the map $\vec{v} : M \to T^1M$, where T^1M denotes the unit tangent bundle equipped with the Sasaki metric,

$$E(\vec{v}) = \frac{1}{2}\int_M \|\nabla \vec{v}\|^2 + \frac{m}{2}\mathrm{vol}(M). \tag{1}$$

- G. Wiegmink, Total bending of vector fields on Riemannian manifolds, *Math. Ann.*, **303**, (1995) 325–344
- C. M. Wood, On the energy of a unit vector field, *Geom. Dedicata*, **64**, (1997) 319–330

Most symmetric unit vector fields

An important question regarding these functionals is whether one can find unit vector fields Minimizing them. It is expected that these vector fields have nice properties.

- F. G. B. Brito, Total bending of flows with mean curvature correction, *Diff. Geom. and its App.* **12**, (2000) 157–163

Most symmetric unit vector fields

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Theorem (Brito)

Hopf vector fields are the unique vector fields on $\mathbb{S}^3$ to minimize E.

- F. G. B. Brito, Total bending of flows with mean curvature correction, *Diff. Geom. and its App.* **12**, (2000) 157–163

Most symmetric unit vector fields

Gluck and Ziller proved that Hopf flows are also the unit vector fields of minimum volume, with respect to the following definition of volume,

$$\mathrm{vol}(\vec{v}) = \int_M \sqrt{\det(I + (\nabla \vec{v})(\nabla \vec{v})^*)},$$

where I is the identity and $(\nabla \vec{v})^*$ represents adjoint operator.

Theorem (Gluck and Ziller)

The unit vector fields of minimum volume on $\mathbb{S}^3$ are precisely the Hopf vector fields, and no others.

- H. Gluck and W. Ziller, On the volume of a unit field on the three-sphere, *Comment Math. Helv.* **61**, (1986) 177–192

Construction of harmonic and minimal unit vector fields

Theorem (Berndt, Vanhecke and Verhóczki)

Let M be a Riemannian symmetric space of compact or non-compact type, and let F be a reflective submanifold of M such that its codimension is greater than one and the rank of $F^{\perp}$ is equal to one. Then the radial unit vector field $\vec{v}$ associated to F is harmonic and minimal.

- Harmonic = critical point of energy
- Minimal = critical point of volume

- J. Berndt, L. Vanhecke and L. Verhóczki, Harmonic and minimal unit vector fields on Riemannian symmetric spaces, *Illinois Journ. of Math.* **47**, (2003) 1273–1286

Topological obstruction

Reznikov compared the volume functional to the topology of an Euclidean hypersurface. Let M^{n+1} be a smooth closed oriented immersed hypersurface in $\mathbb{R}^{n+2}$, endowed with the induced metric, and let $\mathcal{S} = \sup_{x \in M} \|S_x\| = \sup_{x \in M} |\lambda_i(x)|$, where S_x is the second fundamental operator in $T_x M$, and $\lambda_i(x)$ are the principal curvatures.

Theorem (Reznikov)

For any unit vector field $\vec{v}$ on M we have

$$\mathrm{vol}(\vec{v}) - \mathrm{vol}(M) \geq \frac{\mathrm{vol}(\mathbb{S}^{n+1})}{\mathcal{S}} |\deg(\nu)|,$$

where $\deg(\nu)$ is the degree of the Gauss map $\nu : M \to \mathbb{S}^{n+1}$.

- A. G. Reznikov, Lower bounds on volumes of vector fields, *Arch. Math.* **58**, (1992) 509–513

A topological lower bound for the energy of a unit vector field on a closed Euclidean hypersurface

Theorem A

For a unit vector field on a closed oriented Euclidean hypersurface M^{2n+1},

$$E(\vec{v}) \geq C(n)\frac{|\deg(\nu)|\mathrm{vol}(\mathbb{S}^{2n+1})}{S^{[2n-1]}} + \frac{2n+1}{2}\mathrm{vol}(M^{2n+1})$$

where $S^{[2n-1]}$ and $C(n)$ are constants depending on the immersion of M and on n.

- the energy of a given vector field depends on the topology of the immersion

Definition

If $\{u_1, \ldots, u_{2n+1}\}$ *is an orthonormal basis at* $x \in M$*, then, for each* $1 \leq A \leq 2n+1$,

$$S^{[A]} = \sup_{1 \leq i_1, \ldots, i_A \leq 2n+1;\ x \in M} \{\|S(u_{i_1}) \wedge \cdots \wedge S(u_{i_A})\|_\infty\},$$

where $\|\cdot\|_\infty$ *denotes the maximum norm, naturally extended to* $\Lambda^A(M)$.

$$C(n) = \begin{cases} \frac{n}{2n-1}, & \text{if } M^{2n+1} = \mathbb{S}^{2n+1}(r), \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Theorem (Borreli, Brito and Gil-Medrano)

The infimum of E among all globally defined unit smooth vector fields of the sphere $\mathbb{S}^{2n+1}$ $(n \geq 2)$ *is*

$$\left(\frac{2n+1}{2} + \frac{n}{2n-1}\right) \mathrm{vol}(\mathbb{S}^{2n+1}). \tag{2}$$

This value is not attained by any globally defined unit smooth vector field.

- V. Borrelli, F. Brito, O. Gil-Medrano, The infimum of the energy of unit vector fields on odd-dimensional spheres, Ann. Glob. Anal. Geom., **23**, (2003) 129–140

Corollary

Let $\mathbb{S}^{2n+1}(r)$ *be the round sphere of radius* r *in* $\mathbb{R}^{2n+2}$*. Then*

$$E(\vec{v}) \geq \left(\frac{2n+1}{2} r^{2n+1} + \frac{n}{2n-1} r^{2n-1} \right) \mathrm{vol}(\mathbb{S}^{2n+1}).$$

Higher order total bending functionals

Let $\vec{v}$ be a unit vector field on a compact oriented Riemannian manifold M^m. For every $1 \leq k \leq m-1$, define

$$\mathcal{B}_k(\vec{v}) = \int_M \underbrace{\|\nabla \vec{v} \wedge \cdots \wedge \nabla \vec{v}\|^2}_{k\text{-times}}. \tag{3}$$

If σ_{2n} denotes the $2n$-th elementary symmetric function, and $\mathcal{V}$ is the restriction of $\nabla \vec{v}$ to $V^\perp$ then our last theorem reads

Higher order total bending functionals

Theorem B

Let M^{2n+1} be a compact oriented Riemannian manifold, and let $\vec{v}$ be a unit vector field on M. Then

$$\mathcal{B}_n(\vec{v}) \geq \binom{2n}{n} \int_M |\sigma_{2n}(\mathcal{V})|. \tag{4}$$

Furthermore, when M^{2n+1} is a closed Euclidean hypersurface,

$$\mathcal{B}_n(\vec{v}) \geq \frac{|\deg(\nu)|}{\mathcal{S}} \binom{2n}{n} \mathrm{vol}(\mathbb{S}^{2n+1}), \tag{5}$$

where $\mathcal{S}$ is the aforementioned constant.

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Corollary

Hopf vector fields minimize $\mathcal{B}_n$ on $\mathbb{S}^{2n+1}$.

- What about uniqueness?

Thank you!