New examples of $\mathrm{Ric}_2 > 0$

Alberto Rodríguez-Vázquez

Department of Mathematics, KU Leuven

alberto.rodriguezvazquez@kuleuven.be

Joint work with: J. DeVito, M. Domínguez-Vázquez and D. González-Álvaro

What smooth manifolds M^n admit a metric g with $\sec > 0$?

By Gauss-Bonnet theorem, we have $M^2 \cong \mathbb{S}^2$ or $\mathbb{R}\mathbf{P}^2$.

By Hamilton's work [8] on Ricci flow, we have:
$M^3 \cong \mathbb{S}^3/\Gamma$, where $\Gamma \leq \mathsf{O}(3)$ is finite without fixed points.

For $n \geq 4$ the question is much harder. Indeed, an open problem is:

Hopf's conjecture:
$\mathbb{S}^2 \times \mathbb{S}^2$ does not admit a metric with $\sec > 0$.

So far we know very few examples of manifolds with $\sec > 0$.

- Homogeneous spaces (classified in [2, 3, 10]):

$\mathbb{S}^n$, $\mathbb{C}\mathbf{P}^n$, $\mathbb{H}\mathbf{P}^n$, $\mathbb{O}\mathbf{P}^2$ (CROSS'es),

$W^6 = \mathsf{SU}_3/T^2,\ W^{12} = \mathsf{Sp}_3/\mathsf{Sp}_1^3,\ W^{24} = \mathsf{F}_4/\mathsf{Spin}_8,$

$B^{13} = \mathsf{SU}_5/\mathsf{Sp}_2\mathsf{U}_1,$

$W^7_{p,q} = \mathsf{SU}_3/\mathrm{diag}(z^p, z^q, \bar{z}^{p+q}),\ \ (p,q) = 1,\ \ pq > 0,$

$B^7 = \mathsf{SO}_5/\mathsf{SO}_3^{\max}.$

- Inhomogeneous spaces (all known ones have low cohomogeneity):

Certain biquotients in [1] and [6], and one exotic $T_1\mathbb{S}^4$ in [4, 7].

What are other conditions weaker than $\sec > 0$?

- (M, g) has **almost positive curvature** if

(M, g) has $\sec \geq 0$ and $\exists \Omega \subset M$ open and dense with $\sec > 0$.

Examples: $\mathbb{S}^7 \times \mathbb{S}^6$ (in [11]), certain quotients of $\mathbb{S}^7 \times \mathbb{S}^7$ (in [9]).

- (M, g) has **positive intermediate Ricci curvature** ($\mathsf{Ric}_k > 0$) if

$$\sum_{i=1}^{k} \sec(x, y_i) > 0 \ \text{ for orthonormal } x, y_1, \ldots, y_k \in T_pM.$$

Note that $\mathsf{Ric}_1 > 0 \Leftrightarrow \sec > 0$, and $\mathsf{Ric}_{n-1} > 0 \Leftrightarrow \mathsf{Ric} > 0$,

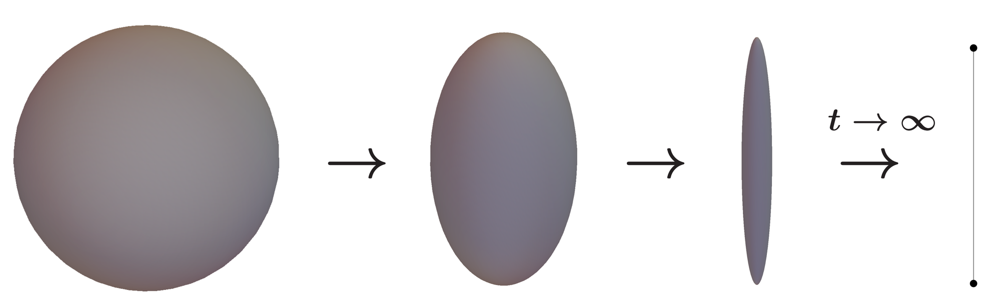
$\mathsf{Ric}_1 > 0 \Rightarrow \mathsf{Ric}_2 > 0 \Rightarrow \cdots \Rightarrow \mathsf{Ric}_{n-2} > 0 \Rightarrow \mathsf{Ric}_{n-1} > 0.$

Examples: $\mathbb{S}^3 \times \mathbb{S}^3$, $\mathbb{S}^3 \times \mathbb{S}^2$, and $\mathbb{S}^2 \times \mathbb{S}^2$ (see [5]).

Cheeger deformation

Let (M, g), consider $G \leq \mathrm{Isom}(M)$ and $G \curvearrowright M$.

A Cheeger deformation is the family of metrics $(g_t)_{t \geq 0}$ obtained by shrinking the metric of M in the direction of the G-tangent vectors.



An example of a Cheeger deformation by $\mathsf{SO}_2 \curvearrowright \mathbb{S}^2$.

Cheeger deformations preserve the condition $\sec \geq 0$.

What is a fat bundle?

A Riemannian submersion $\pi\colon M \to N$ is **fat** if:
- π has totally geodesic fibers.
- $\sec(X, U) > 0$ for every non-zero horizontal X, and vertical U.

Let $H \leq K \leq G$ be Lie groups, the **homogeneous bundle**
$K/H \to G/H \to G/K$ is fat if $[X, U] \neq 0$.

Wallach's theorem:
$\pi\colon G/H \to G/K$ fat, G/K CROSS $\Longrightarrow$ G/H admits a metric of $\sec > 0$

As a consequence $W^6, W^{12}, W^{24}, W^7_{p,q}$ admit $\sec > 0$.

Generalizing fatness

We define the coindex of fatness f of a homogeneous bundle as

$$f := \max\left\{ \max_{x \in T_pG/K} \dim Z_{T_oK/H}(x),\ \max_{y \in T_oK/H} \dim Z_{T_pG/K}(y) \right\}$$

This allows us to generalize Wallach's theorem.

Main Theorem:
Let $\mathsf{H} \leq \mathsf{K} \leq \mathsf{G}$ induce a homogeneous bundle with:
(*) $f \leq 1$, $\sec(\mathsf{K}/\mathsf{H}) > 0$, and $\sec(\mathsf{G}/\mathsf{K}) > 0$.
Then, $\exists$ a Cheeger deformed metric for $\mathsf{K} \curvearrowright \mathsf{G}/\mathsf{H}$ with $\mathsf{Ric}_2 > 0$.

By classifying all $\mathsf{H} \leq \mathsf{K} \leq \mathsf{G}$ satisfying (*), we found:

$$\mathsf{G}_2 \leq \mathsf{Spin}_7 \leq \mathsf{Spin}_8 \ \text{ and } \ \mathsf{SU}_3 \leq \mathsf{G}_2 \leq \mathsf{Spin}_7.$$

Thus, we have constructed homogeneous metrics with $\mathsf{Ric}_2 > 0$ in:

$$\mathbb{S}^7 \times \mathbb{S}^7 = \mathsf{Spin}_8/\mathsf{G}_2 \ \text{ and } \ \mathbb{S}^7 \times \mathbb{S}^6 = \mathsf{Spin}_7/\mathsf{SU}_3.$$

Also, we found inhomogeneous examples with $\mathrm{Ric}_2 > 0$:

(a) ∞-many spaces with the cohomology ring of $\mathbb{C}P^3 \times \mathbb{S}^7$.
(b) One space with the cohomology ring of $\mathbb{C}P^3 \times \mathbb{S}^6$.
(c) One space with the cohomology ring of $\mathbb{S}^7 \times \mathbb{S}^4$.
(d) One space with the cohomology ring of $\mathbb{S}^6 \times \mathbb{S}^4$.

In short, we have found the first examples of dimensions 10 to 14.

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