Totally geodesic submanifolds and curvature

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Abstract

The purpose of these notes (under construction) is to provide an introduction to totally geodesic submanifolds and their relationship with curvature. The primary references are:

- Jürgen Berndt, Sergio Console, Carlos Olmos, *Submanifolds and Holonomy.* Chapman & Hall/CRC Research Notes in Mathematics, 434. Chapman & Hall/CRC, Boca Raton, FL, 2003.
- John M. Lee, *Introduction to Riemannian Manifolds.* Second edition of [MR1468735]. Graduate Texts in Mathematics, 176. Springer, Cham, 2018.

1 Introduction

Euclid's Elements ($\Sigma\tau o\iota\chi\epsilon\widetilde{\iota}\alpha$), written around 300 BC, dominated the study of geometry for over 2000 years. One of the primary goals of Euclid's work was to establish an axiomatic framework for planar geometry, where the fifth and final axiom, known as the parallel postulate is stated as follows:

Given a line ℓ and a point $p \notin \ell$, there exists exactly one line ℓ' parallel to ℓ with $p \in \ell'$.

It was not until the 19th century that Bernhard Riemann delivered his famous lecture *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen* ("On the Hypotheses on which Geometry is Based"), which laid out a new framework for studying geometry, far more general than Euclid's. This new framework is what today we called **Riemannian geometry**, and in it, the role of straight lines is played by geodesics. In this way, Riemannian geometry allows spaces like the sphere S^2, where two geodesics always intersect; or like the hyperbolic plane $\mathbb{R}\mathsf{H}^2$, where given a geodesic γ and a point p not on it, there are infinitely many geodesics passing through p that do not intersect γ, thus negating the parallel postulate. This phenomenon is an example of the implications of curvature, a key concept in Riemannian geometry, which is zero in Euclidean geometry. Thus, Riemannian geometry allows us to study spaces with non-zero curvature.

Additionally, it also permits the generalization of the number of dimensions in the space under study. In this way, we can generalize geodesics to higher-dimensional objects, which leads to what are called **totally geodesic submanifolds**. Intuitively, a submanifold of a Riemannian manifold is totally geodesic if it curves as the ambient space where it lives. Totally geodesic submanifolds play a fundamental role in Riemannian geometry. They are ubiquitous in the field and appear in many important theorems, such as the Soul Theorem (see [8]) or the Connectedness Principle, see [31]. However, despite their frequent appearance in Riemannian geometry, they are rather special and uncommon. Unlike geodesics, whose local existence is always guaranteed given a point and a direction, totally geodesic submanifolds of higher dimensions do not generally exist.

In fact, generic Riemannian manifolds do not have any proper totally geodesic submanifolds besides geodesics, see [23].

We will see that totally geodesic submanifolds are closely related to the existence of symmetries within the ambient space, which makes the study of these objects particularly fruitful in the case of homogeneous spaces, where Lie theory will play a central role.

Some general objectives of these lectures will be:

- Understand the basic properties of totally geodesic submanifolds.
- Obtain classifications in simple ambient spaces (as those with constant sectional curvature).
- Study Cartan's local existence theorem, Hermann's global existence theorem, and their consequences.
- Study the interplay between totally geodesic submanifolds and positively curved manifolds such as Frankel's Theorem, or the Connectedness Principle.
- Learn some tools and concepts regarding totally geodesic submanifolds in homogeneous spaces such as: extrinsically vs. intrinsically homogeneous totally geodesic submanifolds, or Lie triple systems in symmetric spaces.

2 Basics of Riemannian geometry

In this section we will recall some basic facts about Riemannian geometry which will be needed to develop all the theory about totally geodesic submanifolds. For a much more detailed approach we refer the reader to [22, 29].

2.1 Riemannian metrics

Throughout these notes, we will treat smooth manifolds as topological manifolds that are Hausdorff, locally Euclidean, second countable, and equipped with a C^∞-atlas.

Let M be a smooth manifold of dimension n. Recalll that the tangent bundle TM is a vector bundle of rank n equal to $\bigsqcup_{p\in M} T_pM$, where T_pM denotes the tangent space of M at $p \in M$. The sections of this vector bundle are vector fields, and the space of vector fields of M will be denoted by $\mathfrak{X}(M)$.

Now, let us take $\varphi\colon U \subset M \to \mathbb{R}^m$, $\varphi = (x^1, \dots, x^m)$ a chart of M around a point $p \in U \subset M$. The chart φ induces a local frame for the tangent bundle TM denoted by $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m})$ that are called the **coordinate fields**. We will usually abreviate as follows $\frac{\partial}{\partial x^i} \equiv \partial_i$. Using the coordinate fields we can also define a local frame $(dx^1, \dots, dx^m)$ for the cotangent bundle T^*M determined by $dx^i(\frac{\partial}{\partial x^j}) = \delta_{ij}$.

Definition 2.1 (Riemannian metric)**.** *A* ***Riemannian metric*** *g on a smooth manifold M is a section of the vector bundle $T^*M \otimes T^*M$ satisfying:*

- *$g(X,Y) = g(Y,X)$ for vector fields $X, Y \in \mathfrak{X}(M)$,*
- *$g(X,X) > 0$ whenever the vector field $X \in \mathfrak{X}(M)$ is not zero.*

In other words g defines a positive definite, symmetric inner product on each tangent space T_pM which depends smoothly on $p \in M$. If we take a coordinate frame a Riemannian metric takes the form

$$g = g_{ij} dx^i \otimes dx^j,$$

where we are using Einstein summation convention. The matrix (g_{ij}) is a positive definite symmetric matrix at each point $p \in M$.

Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds. We say that a map $\phi\colon M_1 \to M_2$ is an **isometry** if ϕ is a diffeomorphism satisfying

$$g_1(v,w) = g_2(\phi_{*p}v, \phi_{*p}w), \text{ for all } v,w \in T_pM_1, \text{ and for all } p \in M_1.$$

It can be shown that if M_1 and M_2 are connected, the isometry ϕ is uniquely determined by its value $\phi(p)$ and its differential ϕ_{*p} at a single poin $p \in M$, see [29, Proposition 62]. If the map ϕ is a local diffeomorphism instead of a diffeomorphism, we say that ϕ is a local isometry. Moreover, we say that (M_1, g_1) and (M_2, g_2) are **(locally) isometric** if there exists a (local) isometry $\phi\colon M_1 \to M_2$. It is easy to see that the set of isometries of a Riemannian manifold (M, g) is a group, and we will denote it by $\mathrm{Isom}(M)$. In fact $\mathrm{Isom}(M)$ is a Lie group, see [24].

Some examples of Riemannian manifolds are discussed in the following lines.

Example 2.2 (Euclidean spaces)**.** *Let us consider $\mathbb{R}^n$ with its Euclidean metric g, which is just the usual inner product of $\mathbb{R}^n$ through the identification $T_p\mathbb{R}^n \cong \mathbb{R}^n$. If we take standard coordinates $(x^1, \ldots, x^n)$, then we can write the Euclidean metric g as $g = \delta_{ij}dx^i \otimes dx^j$, where δ_{ij} denotes the Kronecker delta.*

Example 2.3 (Real hyperbolic spaces)**.** *We define the real hyperbolic space $\mathbb{R}\mathsf{H}^n$ as the smooth manifold $\mathbb{R}^{n-1} \times (0, +\infty)$ equipped with the Riemannian metric g given by*

$$g = \frac{dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n}{(x^n)^2},$$

where (x^i) denote the standard coordinates on $\mathbb{R}^{n-1} \times (0, +\infty)$.

A good source of Riemannian manifolds is provided by considering smooth inmersions and restricting the ambient metric to the immersed submanifold.

Example 2.4 (Isometric immersions)**.** *Let (M, g) be a Riemannian manifold, Σ a smooth manifold, and $f\colon \Sigma \to M$ a smooth immersion. Then, we can endow Σ with a Riemannian metric f^*g defined by*

$$f^*g(X,Y) = g(f_*X, f_*Y), \quad \textit{for all } X, Y \in \mathfrak{X}(M).$$

*When Σ is equipped with this metric we say that $f\colon (\Sigma, g_\Sigma) \to (M, g)$ is an **isometric immersion**.*

A nice example for this construction are round spheres. Let us consider the n-dimensional sphere $\mathsf{S}^n := \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$. Then, the inclusion map $i\colon \mathsf{S}^n \to \mathbb{R}^{n+1}$ defines an embedding of S^n into $\mathbb{R}^{n+1}$, and the inherited metric $i^*g_{=}g_{\mathsf{S}^n}$ of S^n is called the round metric.

Example 2.5 (Real hyperbolic spaces revisited)**.** *Another way to visualize the real hyperbolic space $\mathbb{R}\mathsf{H}^n$ is by considering it as an isometric immersion in the Minkowski space $\mathbb{R}^{n,1}$. The Minkowski space $\mathbb{R}^{n,1}$ is the real vector space of dimension $n+1$, equipped with the metric given by*

$$g = dx^1 \otimes dx^1 + \cdots + dx^{n-1} \otimes dx^{n-1} - dx^n \otimes dx^n.$$

Then $\mathbb{R}^{n,1}$ is not a Riemannian manifold but a pseudo-Riemannian manifold of signature $(n,1)$. Some aspects of Riemannian geometry can be extrapolated to the pseudo-Riemannian case, however, one has to be cautious. Then, one can define the real hyperbolic space by

$$\mathbb{R}\mathsf{H}^n = \{(x_1,\dots,x_{n+1}) \in \mathbb{R}^{n,1} : x_1^2 + \cdots + x_n - x_{n+1}^2 = -1\}.$$

Then, the inclusion map $i\colon \mathbb{R}\mathsf{H}^n \to \mathbb{R}^{n,1}$ defines an embedding of $\mathbb{R}\mathsf{H}^n$ into $\mathbb{R}^{n,1}$, and $\mathbb{R}\mathsf{H}^n$ equipped with the inherited metric i^*g is isometric to the Riemannian manifold described in Example 2.3, see [22, Theorem 3.7] for a proof.

2.2 The Levi-Civita connection and parallel transport

Let X and Y be vector fields on M. The Levi-Civita connection offers a natural way to define a new vector field on the Riemannian manifold (M,g) denoted by $\nabla_X Y \in \mathfrak{X}(M)$. This vector field measures, at each point $p \in M$, the rate of change of Y in the direction of $X_p \in T_pM$, while preserving important properties of the metric g.

Before introducing the Levi-Civita connection, let us define arbitrary connections on smooth manifolds.

Definition 2.6. *An **affine connection** ∇ on a smooth manifold M is a map $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, which maps (X,Y) to $\nabla_X Y$ and satisfies the following properties:*

i) $\nabla_{f_1X_1+f_2X_2}Y = f_1\nabla_{X_1}Y + f_2\nabla_{X_2}Y$, *where* $f_i \in \mathcal{C}^\infty(M)$ *and* $X_i, Y \in \mathfrak{X}(M)$ *for* $i = 1,2$,

ii) $\nabla_X(\lambda_1Y_1+\lambda_2Y_2) = \lambda_1\nabla_XY_1+\lambda_2\nabla_XY_2$, *where* $\lambda_i \in \mathcal{C}^\infty(M)$ *and* $X, Y_i \in \mathfrak{X}(M)$ *for* $i = 1,2$,

iii) $\nabla_X(fY) = X(f)Y + f\nabla_XY$, *where* $f \in \mathcal{C}^\infty(M)$ *and* $X, Y \in \mathfrak{X}(M)$.

Remark 2.7. *Although an affine connection ∇ is defined for pairs of globally defined vector fields of M, it turns out that the value of $(\nabla_XY)_p$ depends only on the values of Y in a small neighborhood of $p \in M$, and in the value of X at p, that is on $X_p \in T_pM$, see [22, Proposition 4.5]. Thus, we can write with no fear $(\nabla_XY)_p = \nabla_{X_p}Y$. Even more, let us consider a smooth curve $\gamma\colon (-\varepsilon,\varepsilon) \to M$ and denote its velocity at $\gamma(t) \in M$ by $\dot\gamma(t)$. Then, if $\gamma(0) = p$ and $\dot\gamma(0) = X_p$, the value of $(\nabla_XY)_p$ only depends on X_p and on the values of Y on the curve γ, see [22, Proposition 4.26].*

In a different vein, for computations, it is interesting to see how to express an affine connection in terms of a local frame. Let (E_i) be a smooth local frame defined on an open subset U of M. Then, we can expand the vector field $\nabla_{E_i}E_j$ in terms of this frame introducing some coefficient functions Γ_{ij}^k defined on U:

$$\nabla_{E_i}E_j = \Gamma_{ij}^kE_k.$$

These n^3 functions are called the **Christoffel symbols** of ∇ with respect to the local frame $(E_i)_{i=1}^n$. This is enough to determine the connection on U, as if we are given two vector fields X and Y defined on U, we can express them as $X = X^iE_i$ and $Y = Y^jE_j$. Then, using the properties in Definition 2.6, we can compute:

$$\nabla_XY = \nabla_{X^iE_i}(Y^jE_j) = X^i\nabla_{E_i}(Y^jE_j) = X^i(E_i(Y^j)E_j + \nabla_{E_i}E_j) = (X(Y^k) + X^iY^j\Gamma_{ij}^k)E_k.$$

Moreover, it can be checked that given n^3 smooth functions Γ_{ij}^k, one can define a connection using the previous equation. Consequently, there is a one to one correspondence between afine connections on M and choices of n^3 real valued smooth functions Γ_{ij}^k, see [22, Lemma 4.10].

Example 2.8 (The Euclidean connection)**.** *Let us take the Euclidean space $\mathbb{R}^n$, the Euclidean connection is given by $(\nabla_X Y)f = X(Y(f))$, where X and Y are smooth vector fields and $f \in \mathcal{C}^\infty(M)$. Take a standard chart $(x^1, \dots, x^n)$, and denote the associated coordinate fields by $(\partial_1, \dots, \partial_n)$. Observe that $\nabla_{\partial_i}\partial_j(x^l) = \partial_i\partial_j(x^l) = 0$. Then, $\nabla_{\partial_i}\partial_j = 0$ and all the Christoffel symbols with respect to this frame vanish.*

An affine connection induces connections in other vector bundles, see [22, Proposition 4.15]. This will be particularly useful in the case of (p, q)-tensors, i.e. sections of the vector bundle $TM^{\otimes p} \otimes T^*M^{\otimes q}$ that we will denote by $\Gamma(TM^{\otimes p} \otimes T^*M^{\otimes q})$. Indeed, given a tensor $T \in \Gamma(TM^{\otimes p} \otimes T^*M^{\otimes q})$, we can take its covariant derivative by means of the formula:

$$\begin{aligned}(\nabla_X T)(\omega^1, \dots, \omega^p, Y_{p+1}, \dots, Y_{p+q)}) = {} & X(T(\omega^1, \dots, \omega^p, Y_1, \dots, Y_q)) \\ & - \sum_{i=1}^{p} T(\omega^1, \dots, \nabla_X \omega^i, \dots, \omega^p, Y_{p+1}, \dots, Y_{p+q}) \\ & - \sum_{i=p+1}^{p+q} T(\omega_1, \dots, \nabla_X \omega_i, \dots, \omega_p, Y_{p+1}, \dots, \nabla_X Y_i, \dots, Y_{p+q}).\end{aligned}$$

The $(p, q + 1)$ tensor ∇T defined by

$$\nabla T(\omega^1, \dots, \omega^p, Y_{p+1}, \dots, Y_{p+q}, X) = (\nabla_X T)(\omega^1, \dots, \omega^p, Y_{p+1}, \dots, Y_{p+q})$$

is called the **total covariant derivative of** T.

As previously mentioned, there is a special connection that one can define in a Riemannian manifold (M, g), that is the Levi-Civita connection.

Definition 2.9. *The **Levi-Civita connection** is the unique connection on a Riemannian manifold which satisfies the following two properties:*

i) $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ *for all* $X, Y, Z \in \mathfrak{X}(M)$*,*

ii) $[X, Y] = \nabla_X Y - \nabla_Y X$ *for all* $X, Y \in \mathfrak{X}(M)$*.*

The existence and uniqueness is provided by the Koszul formula (see [22, Theorem 5.10]) which gives the Christoffel symbols of the Levi-Civita connection for a given coordinate frame in terms of the Riemannian metric:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}(\partial_i g_{kl} + \partial_j g_{il} - \partial_l g_{ij}). \qquad \text{(Koszul formula)}$$

A relevant notion in Riemannian geometry is that of parallel transport which will be explained in the following lines. A vector field $X \in \Gamma(TM)$ is **parallel** if $\nabla_Z X = 0$ for every $Z \in \Gamma(TM)$.

Exercise 2.10. *Prove that parallel vector fields of* $\mathbb{R}^n$ *are the constant vector fields.*

Throughout these notes when we say **path** we will understand a continous path $\gamma\colon [a, b] \to M$ which is piecewise smooth, meaning that it is not smooth just at a finite number of points $\{a_0, \dots, a_k\}$. Now consider a path $\gamma\colon [0, 1] \to M$ from $\gamma(0) = p \in M$ to $\gamma(1) = q \in M$. Given a tangent vector $v \in T_pM$, we say that $P_\gamma(v) \in T_qM$ is the **parallel transport of** v **along** γ if there exists a parallel vector field X along γ that satisfies $X(0) = v$ and $X(1) = P_\gamma(v)$. It turns out that $P_\gamma(v)$ always exists and it is unique (see [22, Theorem 4.32]), then it defines a map $P_\gamma\colon T_pM \to T_qM$.

Exercise 2.11. *Prove that* P_γ *is a linear isometry.*

2.3 Geodesics and the exponential map

Let $I \subset \mathbb{R}$ be a real interval, (M, g) a Riemannian manifold, and ∇ its Levi-Civita connection.

Definition 2.12. *A smooth curve $\gamma\colon I \to M$ is a* ***geodesic*** *if $\nabla_{\dot\gamma}\dot\gamma = 0$ for all $t \in I$.*

This definition can be extended to a path γ by requiring that γ satisfies $\nabla_{\dot\gamma}\dot\gamma = 0$ for all times when it is smooth. In this case, we will say that the path γ is a **geodesic path**. Notice that this property implies that $\frac{d}{dt}\langle\dot\gamma, \dot\gamma\rangle = 2\langle\nabla_{\dot\gamma}\dot\gamma, \dot\gamma\rangle = 0$. Thus, geodesics always have constant speed, and it is sometimes convenient to reparametrize them, which can always be done, so they have unit speed. In terms of a coordinate frame $(x^1, \dots, x^n)$, we can rewrite the previous equations as:

$$\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\Gamma^k_{ij}(x(t)) = 0, \quad \text{where } k \in \{1, \dots, n\}.$$

This is a system of second-order ordinary differential equations (ODEs) for certain real valued functions. It can be proven that we have existence and uniqueness given certain intial conditions, see [22, Theorem 4.27]. More precisely, given a point $p \in M$, and a tangent vector at $v \in T_pM$, there is a unique maximal geodesic $\gamma\colon I \to M$ such that $\gamma(0) = p$, and $\dot\gamma(0) = v$.

Exercise 2.13. *Show that the maximal geodesics of the Euclidean space are straight lines.*

For each $v \in T_pM$, let us denote by γ_v, the maximal geodesic with initial conditions $\gamma(0) = p$ and $\dot\gamma(0) = v$. Then, the assignment $v \mapsto \gamma_v$, and the fact that $\gamma_{\lambda v}(t) = \gamma_v(\lambda t)$ for every $\lambda \in \mathbb{R} \setminus \{0\}$, allows us to define a map called the Riemannian exponential map. Let us consider the following subset of the tangent bundle $\mathcal{E} := \{v \in TM : \gamma_v \text{ is defined in } [0, 1]\}$. Now we define the **Riemannian exponential map** as the map $\exp\colon \mathcal{E} \to M$ which takes $v \in TM$ to $\gamma_v(1) \in M$. We will denote the restriction of the exponential map to T_pM by $\exp_p$. It can be proved that the exponential map is smooth, and that $d(\exp_p)_0$ is "the identity map", and thus a local diffeomorphism at $0 \in T_pM$. Indeed, let us consider the curve $\alpha(t) = tw$ in T_pM for some $w \in T_pM$. Then, $d(\exp_p)_0(w) = \frac{d}{dt}_{|t=0}\left(\exp_p(tw)\right) = \frac{d}{dt}_{|t=0}\gamma_w(t) = w$. This implies that one can use $\exp_p$ to define smooth coordinates around $p \in M$. These are called **normal coordinates**. The open neighborhood of $p \in M$ defined by this chart is called **normal neighborhood** around $p \in M$.

Exercise 2.14. *Show that the Christoffel symbols at $p \in M$ vanish when using normal coordinates centered at $p \in M$.*

This set defines a partition of the interval $[a, b]$ into open intervals, where γ is smooth. We can use the Riemannian metric to define a functional, which gives the length of a path $\gamma\colon [a, b] \to M$ as $L(\gamma) = \int_a^b \sqrt{\langle\dot\gamma, \dot\gamma\rangle}dt$. It follows that geodesics have a strong relationship with the critical points of the length functional. Let us to introduce some calculus of variations terminology.

Definition 2.15. *Let $\gamma\colon [a, b] \to M$ be a path and $\varepsilon > 0$. A* ***variation*** *of γ is a continous map $\Gamma\colon (-\varepsilon, \varepsilon) \times [a, b] \to M$ such that*

i) *$\Gamma(0, t) = \gamma(t)$ for all $t \in [a, b]$.*

ii) *Γ is smooth in the rectangles $(-\varepsilon, \varepsilon) \times (a_i, a_{i+1})$, where $\{a = a_0, \dots, b = a_k\}$ defines the partition for which the path γ is smooth.*

We write $\Gamma_s(t) = \Gamma(s, t)$, and we say that Γ is **proper** if it fixes the endpoints, i.e. $\Gamma(s, a) = \Gamma(0, a)$, and $\Gamma(s, b) = \Gamma(0, b)$ for all $s \in (-\varepsilon, \varepsilon)$. We say that Γ is a **variation by geodesics** or **geodesic variation** if $\Gamma(s_0, t)$ is a geodesic for each fixed $s_0 \in (-\varepsilon, \varepsilon)$. Let us introduce the notation

$$S(s, t) = \partial_s\Gamma(s, t), \quad T(s, t) = \partial_t\Gamma(s, t).$$

The **variation field along** γ is the vector field along γ given by $V(t) = S(0,t) = \partial_{s_{|s=0}}\Gamma(s,t)$. If Γ is proper, then $V(a) = V(b) = 0$. Moreover, given a vector field Y along γ, one constructs a variation Γ with variational field Y by setting $\Gamma(s,t) = \exp_{\gamma(t)}(sY(t))$. Indeed, in this case, $\partial_{s_{|s=0}}\Gamma(s,t) = d(\exp_{\gamma(t)})_0(Y(t)) = Y(t)$, where we have used that $d(\exp_p)_0 = \mathrm{Id}$. An important property that will be used very often is $\partial_t S(s,t) = \partial_s T(s,t)$, see [22, Lemma 6.2].

Now we are interested in obtaining the critical points for the length functional. To make sense of this we take the length of a variation Γ of a curve γ, and we differentiate with respect to s. The first variation for the length functional of a unit speed path γ is

$$\frac{d}{ds}_{|s=0} L(\Gamma(s,t)) = -\int_a^b g(V, \nabla_{\dot\gamma}\dot\gamma)dt + g(V(b),\dot\gamma(b)) - g(V(a),\dot\gamma(a)) - \sum_{i=1}^{k-1} g(V(a_i), \dot\gamma(a_i^+) - \dot\gamma(a_i^-)),$$

where $\dot\gamma(a_i^\pm) = \lim_{t\to a_i^\pm}\dot\gamma(t)$ for each $i \in \{1,\ldots,k-1\}$. One can check [22, Theorem 6.3] for the details of this computation. As a consequence of the first variation for the length functional, every unit-speed path is a critical point of the length functional if and only if it is a geodesic, see [22, Corollary 6.5]. In particular, minimizing paths must be geodesics, up to reparametrization. Another functional which is intimately related with the length functional is the **energy functional**. This is defined as follows:

$$E(\gamma) = \frac{1}{2}\int_a^b g(\dot\gamma,\dot\gamma)dt.$$

The first variation for the energy functional of an arbitrary path γ is

$$\frac{d}{ds}_{|s=0} E(\Gamma(s,t)) = -\int_a^b g(V, \nabla_{\dot\gamma}\dot\gamma)dt + g(V(b),\dot\gamma(b)) - g(V(a),\dot\gamma(a)) - \sum_{i=1}^{k-1} g(V(a_i), \dot\gamma(a_i^+) - \dot\gamma(a_i^-)). \tag{1}$$

Exercise 2.16. *Prove the first variation formula for the energy functional.*

Solution. *Let us prove this formula in the case that γ is a smooth curve, since the proof in the general case is done just by decomposing the integral into sums where the path is smooth. Let Γ be a smooth variation of the curve γ with variational vector field V. Then*

$$\frac{d}{ds}E(\Gamma(s,t)) = \frac{1}{2}\int_a^b \partial_s g(\Gamma(s,t),\Gamma(s,t))dt = \int_a^b g(\partial_s T, T)dt = \int_a^b g(\partial_t S, T)dt, \tag{2}$$

where we have used the symmetry property $\partial_s T = \partial_t S$. Now using the compatibility of ∇ with the metric

$$\frac{d}{ds}E(\Gamma(s,t)) = \int_a^b \partial_t g(S,T) - g(S,\partial_t T)dt = [g(S,T)]_{t=a}^{t=b} - \int_a^b g(S,\partial_t T)dt.$$

Evaluating at $s=0$, we get

$$\frac{d}{ds}_{|s=0} E(\Gamma(s,t)) = -\int_a^b g(V,\nabla_{\dot\gamma}\dot\gamma)dt + g(V(b),\dot\gamma(b)) - g(V(a),\dot\gamma(a)).$$

Observe that by Cauchy-Schwarz inequality on $L^2([a,b])$, we have

$$L(\gamma)^2 = \left(\int_a^b \sqrt{g(\dot\gamma,\dot\gamma)}dt\right)^2 \le \left(\int_a^b 1^2 dt\right)\left(\int_a^b g(\dot\gamma,\dot\gamma)dt\right) = (b-a)E(\gamma), \tag{3}$$

and the equality is attached if and only if 1 and $|\dot{\gamma}|$ are linearly dependent, i.e. γ has constant speed. A path γ is said to be **minimizing** if $L(\gamma) \leq L(\widetilde{\gamma})$ for every path $\widetilde{\gamma}$ with the same endpoints. The inequality (3) implies that a path γ from $\gamma(a) = p$ to $\gamma(b) = q$ minimizes the energy functional if and only if it has constant speed and minimizes the length functional.

Given two points p and q in M, one can define a distance in a Riemannian manifold in the following way:

$$d(p, q) = \inf \{L(\gamma) : \gamma \text{ is a path in } M \text{ such that } \gamma(0) = p \text{ and } \gamma(1) = q\}.$$

It can be shown that this is a well-defined distance and that the metric topology coincides with the manifold topology, see [22, Theorem 2.55]. It turns out that we can characterize geometrically the completeness of this metric space in terms of geodesics. This is the content of Hopf-Rinow theorem, see [22, Theorem 6.19] for a proof.

Theorem 2.17 (Hopf-Rinow theorem)**.** *Let (M, g) be a connected Riemannian manifold. The following statements are equivalent:*

i) *(M, g) is complete as a metric space.*

ii) *The map $\exp_p$ is defined for every $v \in T_pM$ for a certain $p \in M$.*

iii) *The map $\exp$ is defined for every $v \in TM$.*

iv) *Every maximal geodesic γ of M is defined for all $t \in \mathbb{R}$.*

Also, if (M, g) is a connected complete Riemannian manifold, one can always connect two points p and q using a minimizing geodesic, see [22, Corollary 6.20]. If U is a normal neighborhood of $p \in M$, every point $q \in U$ can be connected to p by a unique minimizing geodesics, see [22, Proposition 6.11].

2.4 The curvature tensor

Let (M, g) be a Riemnanian manifold and denote by ∇ its Levi-Civita connection. We define the **curvature tensor** as

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad \text{where } X, Y, Z \in \mathfrak{X}(M).$$

This is a tensor of type $(1, 3)$. Thus, if we take a frame, we can write the curvature as:

$$R = R^l_{ijk} dx^i \otimes dx^j \otimes dx^k \otimes \partial_l, \quad \text{where } R(\partial_i, \partial_j, \partial_k) = R^l_{ijk} \partial_l.$$

We can use the metric to turn it into a tensor of type $(0, 4)$:

$$R(X, Y, Z, U) = g(\nabla_X \nabla_Y Z, U) - g(\nabla_Y \nabla_X Z, U) - g(\nabla_{[X,Y]} Z, U), \quad \text{where } X, Y, Z, U \in \mathfrak{X}(M).$$

This tensor has exactly the same information, and we denote it also by R, in a frame, we can write it as:

$$R = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l, \quad \text{where } R_{ijkl} = R^m_{ijk} g_{lm}.$$

The curvature tensor has the following symmetries, which can be written in a coordinate frame as follows:

$$R_{ijkl} = -R_{jikl}, \quad R_{ijkl} = -R_{ijlk}, \quad R_{ijkl} = R_{klij}, \quad R_{ijkl} + R_{jkil} + R_{kijl} = 0.$$

A Riemannian manifold is **flat** if it is locally isometric to the Euclidean space. It can be proved that flatness is equivalent to the equation $R = 0$, see [22, Theorem 7.10].

A simpler tensor that can be defined using the curvature tensor is the **Ricci tensor**. This is a $(0, 2)$-tensor defined as follows:

$$\mathrm{Ric}(X, Y) = \mathrm{tr}(Z \mapsto R(X, Y)Z), \quad \text{where } X, Y \in \mathfrak{X}(M).$$

The components of the Ricci tensor are denoted by R_{ij} and they can be computed by the following formula $R_{ij} = R_{kijl}g^{kl}$. The Ricci tensor is symmetric i.e. $R_{ij} = R_{ji}$, and the metrics that satisfy $\mathrm{Ric} = \lambda g$, for some constant $\lambda \in \mathbb{R}$ are called **Einstein metrics**. Finally another object that can be constructed using the curvature tensor is the **sectional curvature**. This is defined at a point $p \in M$ as follows:

$$\sec_p(v, w) = \frac{R(v, w, w, v)}{g(v, v)g(w, w) - g(v, w)^2}, \quad \text{where } v, w \in T_pM \text{ are linearly independent.}$$

The geometric interpretation of $\sec_p(v, w)$ is the following. Take the 2-plane $V \subset T_pM$ spanned by the tangent vectors $v, w \in T_pM$ and consider the surface S spanned by throwing arbitrarily small geodesics with initial velocities in V. Then, $\sec_p(v, w)$ is precisely the Gaussian curvature of S at $p \in S \subset M$.

Exercise 2.18. *Check that the curvature tensor of the Riemannian manifold (M, g) is given by the following expression in each of the following cases:*

$$R(X, Y)Z = \varepsilon(\langle X, Z\rangle Y - \langle Y, Z\rangle X), \quad \textit{where} \quad \epsilon := \begin{cases} 1 & \textit{if } M = \mathsf{S}^n \\ 0 & \textit{if } M = \mathbb{R}^n, \\ -1 & \textit{if } M = \mathbb{R}\mathsf{H}^n. \end{cases}$$

2.5 Jacobi fields

Let us consider a geodesic $\gamma\colon [0, \ell] \to M$ and a geodesic variation Γ of γ. Let us consider V the variation field of Γ. Then, since Γ is a geodesic for each fixed s, we have

$$0 = \partial_t T = \partial_s \partial_t T = \partial_t \partial_s T + R(S, T)T.$$

Thus, evaluating at $s = 0$, we get the Jacobi equation:

$$\partial_t^2 V + R(V, \dot\gamma)\dot\gamma = 0. \qquad \text{(Jacobi equation)}$$

There is an insightful heuristic interpretation for the previous equation. If we think of a geodesic variation Γ of γ as a one-parameter family of freely falling particles, the variation vector field V represents the position of particles arbitrarily close to γ. In this context, the derivative V' corresponds to their relative velocity, while V'' represents their relative acceleration. By assigning these particles a unit mass, the Jacobi equation can be interpreted as Newton's second law, where the curvature vector acts as the force.

Definition 2.19. *A **Jacobi field** along a geodesic γ is a smooth vector field Y along γ such that $\partial_t^2 Y + R(Y, \dot\gamma)\dot\gamma = 0$.*

Since Jacobi fields are solutions to an ordinary differential equation of order two, it can be proved that they are completely determined by the initial conditions $Y(0)$ and $Y'(0)$, thus the set of Jacobi fields along γ is a vector space of dimension $2n$ that we will denote by $\mathfrak{J}(\gamma)$. It can be checked that $Y_0(t) = \dot\gamma(t)$ and $Y_1(t) = t\dot\gamma(t)$ are Jacobi fields along γ, and the subspace of Jacobi fields along γ vanishing at the endpoints of γ is equal to the kernel of I. If $Y \in \mathfrak{J}(\gamma)$, then $\langle \dot\gamma(s), Y(s)\rangle = \lambda s + \mu$ for certain $\lambda, \mu \in \mathbb{R}$. Thus, $\mathfrak{J}(\gamma) = \mathbb{R}Y_0 \oplus \mathbb{R}Y_1 \oplus \mathfrak{J}^\perp$, where $\mathfrak{J}^\perp$ is the subspace of Jacobi fields always orthogonal to $\dot\gamma$.

We have seen that every variation vector field associated with a geodesic variation is a Jacobi field. Conversely, let us assume that V is a Jacobi vector field along an arbitrary geodesic $\gamma\colon [0,\ell] \to M$. Now take a curve $\sigma\colon (-\varepsilon,\varepsilon) \to M$ such that $\sigma(0) = \gamma(0)$ and $\dot\sigma(0) = V(0)$. Moreover, take parallel vector fields X_0 and X_1 along σ such that $X_0(0) = \dot\gamma(0)$ and $X_1(0) = V'(0)$. Then, define the smooth variation $\Gamma(s,t) = \exp_{\sigma(s)}(tX(s))$ where $X(s) = X_0(s) + sX_1(s)$. Clearly, $\Gamma(0,t) = \exp_{\gamma(0)}(t\dot\gamma(0)) = \gamma(t)$, and $\Gamma(s,t)$ is a geodesic for each fixed $s \in (-\varepsilon,\varepsilon)$. Thus, Γ is a geodesic variation, and its associated variation field is Jacobi. Let us check that it is exactly V. To this end, we compute $\partial_{s_{|s=0}}\Gamma(s,0) = \frac{d}{ds}{}_{s_{|s=0}}\sigma(s) = \dot\sigma(0) = V(0)$. Moreover,

$$\partial_{t_{|t=0}}\partial_{s_{|s=0}}\Gamma(s,t) = \partial_{s_{|s=0}}\partial_{t_{|t=0}}\Gamma(s,t) = \partial_{s_{|s=0}}d(\exp_{\sigma(s)})_0(X(s)) = \partial_{s_{|s=0}}X(s) = X_1(0) = V'(0),$$

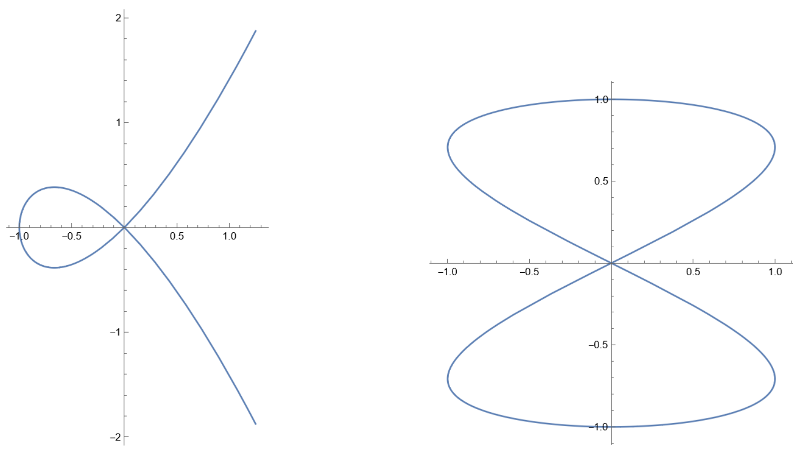
where the last equality holds because X_0 and X_1 are parallel. Hence, we have proved that $V(t) = \partial_{s_{|s=0}}\Gamma(s,t)$, and thus every Jacobi vector field arises as the variation field of a geodesic variation.

2.6 Basics of submanifold geometry

Recall that given a smooth immersion $f\colon \Sigma \to M$, we can consider several notions of submanifold attending to the relationship of the topology of Σ with that of the ambient space M. In particular, it is good to keep in mind the following 1-dimensional examples.

Example 2.20 (The nodal cubic)**.** *Consider the immersion $f\colon \mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = (t^2-1, t(t^2-1))$ is a* ***non-injective immersion****, as $\gamma(1) = \gamma(-1) = (0,0)$.*

Example 2.21 (The figure eight)**.** *Consider the immersion $\gamma\colon (-\pi,\pi) \subset \mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = (\sin(2t), \sin(t))$ is an* ***injective immersion which is not an embedding****, since the image of γ is not homeomorphic to $(-\pi,\pi)$.*



(a) The nodal cubic (b) The figure eight

Throughout these notes, when we use the word submanifold, we will mean embedded submanifold unless otherwise is stated. However, the following are local concepts. Therefore, they hold for immersed manifolds, as every immersed manifold is locally embedded. We will speak of *isometric immersions* when considering on our immersed manifold the metric given by the restriction of the ambient manifold's metric.

From the point of view of submanifold geometry, there is no sense in distinguishing between two isometric immersions (M, g) and (M', g') in $\overline{M}$ that differ by an isometry of the ambient space $\overline{M}$. We say that two isometric immersions $f\colon M \longrightarrow \overline{M}$ and $f'\colon M' \longrightarrow \overline{M}$ are *congruent* if there is an isometry $\varphi \in \mathrm{Isom}(\overline{M})$, such that $f' = \varphi \circ f$. In this case,

$$g' = f'^{*}\overline{g} = f^{*}(\varphi^{*}\overline{g}) = f^{*}\overline{g} = g.$$

Let $\overline{M}$ be a Riemannian manifold and M a Riemannian submanifold of $\overline{M}$. Each tangent space T_pM is equipped with a metric g_p. Therefore, we can consider the bundle of vectors orthogonal to the tangent space. We will call this set the *normal bundle of* M and denote it by νM. At each point $p \in M$, we have the decomposition $T_p\overline{M} = T_pM \oplus \nu_pM$, where $\oplus$ denotes the direct sum. Given a vector field $X \in \Gamma(T\overline{M})$, we will denote by $X^{\top}$ the orthogonal projection of X onto TM and by $X^{\perp}$ the orthogonal projection onto νM. If V is a vector space with a symmetric bilinear form $\langle\cdot,\cdot\rangle$ and $W \subset V$ is a vector space, we will denote by $V \ominus W := v \in V : \langle v, w\rangle = 0\ \forall w \in W$. If $\langle\cdot,\cdot\rangle$ is positive definite, then $V \ominus W$ is the orthogonal complement of W in V. We will use this notation for distributions on M and subbundles of $\overline{M}$ restricted to M.

Let $\overline{R}$ and R denote the curvature tensors of $\overline{M}$ and M, respectively. We can decompose $\overline{\nabla}_X Y$ into its tangential component $(\overline{\nabla}_X Y)^{\top}$ and normal component $(\overline{\nabla}_X Y)^{\perp}$, for each $X, Y \in \Gamma(TM)$. Then, the Levi-Civita connection on M is given by $\nabla_X Y = (\overline{\nabla}_X Y)^{\top}$. Moreover, we can define the *second fundamental form*, which is bilinear and symmetric, by $\mathrm{II}(X, Y) = (\overline{\nabla}_X Y)^{\perp}$. Therefore, we have an orthogonal decomposition known as the *Gauss formula* given by

$$\overline{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X, Y), \qquad \text{(Gauss formula)}$$

for each $X, Y \in \Gamma(TM)$. Additionally, there exists a vector field ξ, unitary and normal to M. We define the *shape operator of* M associated with ξ as the self-adjoint operator $\mathcal{S}_\xi$ satisfying the relation $\langle \mathcal{S}_\xi X, Y\rangle = \langle \mathrm{II}(X, Y), \xi\rangle$, where $X, Y \in \Gamma(TM)$. The eigenvalues and eigenspaces of $\mathcal{S}_\xi$ are called *principal curvatures* and *principal curvature spaces of* M *with respect to* ξ, respectively. Additionally, we denote by $\nabla^{\perp}$ the *normal connection of* M, which is defined by $\nabla^{\perp}_X \xi = (\overline{\nabla}_X \xi)^{\perp}$, for each $X \in \Gamma(TM)$ and $\xi \in \Gamma(\nu M)$. Thus, we have an orthogonal decomposition given by

$$\nabla^{\perp}_X \xi = -\mathcal{S}_\xi X + \nabla^{\perp}_X \xi, \qquad \text{(Weingarten formula)}$$

where $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\nu M)$, which we call the *Weingarten formula*. These two formulas are known as the *first-order fundamental equations*.

Definition 2.22. *An isometric immersion $f\colon M \to \bar{M}$ is* ***totally geodesic*** *if* $\mathrm{II} = 0$.

In what follows we will examine some simple examples of isometric immersion and we will compute their second fundamental forms.

Example 2.23 (The standard embedding of the sphere in $\mathbb{R}^n$)**.** *Let us consider the metric on $\mathsf{S}^n \subset \mathbb{R}^{n+1}$ induced by the Euclidean metric of $\mathbb{R}^n$. We will compute its second fundamental form. First of all, we compute the tangent space of S^n at an arbitrary point $x \in \mathsf{S}^n$. Let $\alpha\colon (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = x$. Then $\langle \alpha(t), \alpha(t)\rangle = 1$. Thus, $0 = \frac{d}{dt}_{|t=0}\langle \alpha(t), \alpha(t)\rangle = 2\langle \dot{\alpha}(0), \alpha(0)\rangle = 2\langle \dot{\alpha}(0), x\rangle$. Hence, $T_x\mathsf{S}^n = x^{\perp}$. Thus, a normal vector field of S^n is given by $\xi = x^i\partial_i$, where (x^i) denote the standard coordinates of $\mathbb{R}^{n+1}$. Let $X = \lambda^j\partial_j$ be a tangent vector field of S^n. Then, using the Levi-Civita connection $\bar{\nabla}$ of $\mathbb{R}^{n+1}$ (the Euclidean connection), we have for each $j \in \{1, \ldots, n\}$,*

$$(\bar{\nabla}_X \xi)_x(x^j) = X_x(\xi)(x^j) = X_x(x^i\partial_i(x^j)) = X_x(x^i\delta_{ij}) = X_x(x^j) = \lambda^k\partial_k(x^j) = \lambda^k\delta_{jk} = \lambda_j.$$

Then, $(\bar{\nabla}_X \xi) = X$. Consequently, $\langle \mathrm{II}(X,Y), \xi \rangle = -(\bar{\nabla}_X \xi), Y\rangle = -\langle X, Y \rangle$, and this yields

$$\mathrm{II}(X,Y) = -\langle X, Y \rangle \xi \quad ,\text{for every } X, Y \in \mathfrak{X}(M),$$

where ξ is the unit normal vector field of S^n pointing outwards.

Observe that by means of the Gauss formula, one also obtains the Levi Civita connection ∇ of the sphere S^n, which is given by

$$\nabla_X Y = \bar{\nabla}_X Y + \langle X, Y \rangle \xi \text{ for every } X, Y \in \mathfrak{X}(\mathsf{S}^n).$$

Exercise 2.24 (Intersections of affine hyperplanes of $\mathbb{R}^{n+1}$ with S^n)**.** *Compute the second fundamental form for a hypersurface of S^n obtained by intersecting an affine hyperplane of $\mathbb{R}^{n+1}$ with S^n itself, and determine when it is totally geodesic.*

Let's keep differentiating! Now, let $X, Y, Z \in \Gamma(TM)$. Using the definition of the curvature tensor, and Gauss and Weingarten formulas, we can see that

$$\overline{R}(X,Y)Z = R(X,Y)Z + (\nabla^{\perp}_X \mathrm{II})(Y,Z) - (\nabla^{\perp}_Y \mathrm{II})(X,Z) + \mathcal{S}_{\mathrm{II}(X,Z)}Y - \mathcal{S}_{\mathrm{II}(Y,Z)}X.$$

If we consider the tangential part, we get the *Gauss equation*

$$(\overline{R}(X,Y)Z)^{\top} = R(X,Y)Z - \mathcal{S}_{\mathrm{II}(Y,Z)}X + \mathcal{S}_{\mathrm{II}(X,Z)}Y. \tag{Gauss equation}$$

If we consider the normal part, we obtain the *Codazzi equation*

$$(\overline{R}(X,Y)Z)^{\perp} = (\nabla^{\perp}_X \mathrm{II})(Y,Z) - (\nabla^{\perp}_Y \mathrm{II})(X,Z). \tag{Codazzi equation}$$

Now consider $\xi \in \Gamma(\nu M)$ and $X, Y \in \Gamma(TM)$. Similarly as before, we can see that

$$\overline{R}(X,Y)\xi = R^{\perp}(X,Y)\xi + (\nabla_Y \mathcal{S})_{\xi} X - (\nabla_X \mathcal{S})_{\xi} Y + \mathrm{II}(Y, \mathcal{S}_{\xi} X) - \mathrm{II}(X, \mathcal{S}_{\xi} Y),$$

where $R^{\perp}$ denotes the curvature tensor associated with the normal connection. If we take the normal part, we get the *Ricci equation*

$$(\overline{R}(X,Y)\xi)^{\perp} = R^{\perp}(X,Y)\xi + \mathrm{II}(Y, \mathcal{S}_{\xi} X) - \mathrm{II}(X, \mathcal{S}_{\xi} Y). \tag{Ricci equation}$$

These three equations constitute the *second-order fundamental equations*.

3 Some properties of totally geodesic submanifolds

Let $\bar{M}$ and M be connected Riemannian manifolds and $f\colon M \to \bar{M}$ an isometric immersion. We denote by $\bar{\nabla}$ and ∇ the Levi-Civita connections of $\bar{M}$ and M, respectively. Recall that $f\colon M \to \bar{M}$ is a totally geodesic immersion in $\bar{M}$ if its second fundamental form II vanishes identically.

Lemma 3.1. *Let M be a connected immersed submanifold of $\bar{M}$. Then, the following statements are equivalent:*

i) M is totally geodesic.

ii) If $\alpha\colon I \to M$ is a curve in M and $v \in T_{\alpha(0)}M$, the parallel transport of v along α in $\bar{M}$ belongs to $T_{\alpha(t)}M$ for each $t \in I$.

iii) Every geodesic of M is a geodesic of $\bar{M}$.

iv) The geodesic γ_v of $\bar{M}$ with initial conditions $\dot{\gamma}(0) = p$ and $\dot{\gamma}_v(0) = v \in T_pM$ satisfies that $\gamma_v(t) \in M$ for every $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

Proof. Let α be a curve in M and assume that $v = \dot{\alpha}(0) \in T_{\alpha(0)}M$. Let $V \in \mathfrak{X}(\alpha)$ be the parallel transport of v in M along α. However, by Gauss formula, V is also parallel in $\bar{M}$. Thus, *i)* implies *ii)*.

Let γ be a geodesic of M. Then, $\dot{\gamma}$ is parallel in M. If we assume that *ii)* holds, we have that the parallel transport of $\dot{\gamma}(0)$ along γ in $\bar{M}$ is a vector field V along γ which takes values in TM. However $0 = \bar{\nabla}_{\dot{\gamma}} V = (\bar{\nabla}_{\dot{\gamma}} V)^{\top} = \nabla_{\dot{\gamma}} V$. Thus, V is also parallel along γ in M, and by uniqueness $V = \dot{\gamma}$, and consequently, $\bar{\nabla}_{\dot{\gamma}}\dot{\gamma} = 0$, yielding *ii)* implies *iii)*.

Let $\widetilde{\gamma}_v$ and γ_v be the geodesics of $\bar{M}$ and M, respectively, with initial conditions $p = \gamma(0)$ and $v = \dot{\gamma}(0) \in T_pM$. If we assume that *iii)* holds, γ_v is a geodesic of $\bar{M}$, and by the uniqueness of geodesics $\gamma_v(t) = \widetilde{\gamma}_v(t)$ for small values of t. Thus, *iii)* implies *iv)*.

Finally, let us prove that *iv)* implies *i)*. Since II is a symmetric tensor it suffices to prove that $\mathrm{II}(v, v) = 0$, for every $v \in T_pM$ and $p \in M$. Let γ_v be the geodesic of $\bar{M}$ with initial conditions $p = \gamma_v(0)$ and $v = \dot{\gamma}(0) \in T_pM$. By *iv)*, γ_v is contained in M, and then is also a geodesic of M. Consequently, Gauss formula yields the desired result. □

Notice that item *iv)* in the previous lemma implies the following:

Corollary 3.2. *Let Σ_1 and Σ_2 be totally geodesic submanifold of (M, g) passing through $p \in M$ with $T_p\Sigma_1 = T_p\Sigma_2$. Then, there is some $\varepsilon > 0$ such that $\Sigma_1 \cap B(p, \varepsilon) = \Sigma \cap B(p, \varepsilon)$, where $B(p, \varepsilon)$ denotes the open ball of center p and radius $\varepsilon > 0$ in (M, g). In particular if Σ_1 and Σ_2 are complete totally geodesic submanifolds of M and there is some $p \in \Sigma_1 \cap \Sigma_2$ such that $T_p\Sigma_1 = T_p\Sigma_2$, then $\Sigma_1 = \Sigma_2$.*

Notice that geodesics of $\mathbb{R}^n$ are straight lines, so affine subspaces of $\mathbb{R}^n$ are totally geodesic. Moreover, given a complete totally geodesic submanifold Σ of $\mathbb{R}^n$ with $p \in \Sigma$ and $T_p\Sigma = V$, we can consider $\Sigma_V = \{p + v : v \in V\}$ that is a totally geodesic submanifold of $\mathbb{R}^n$ passing through p with $T_p\Sigma_V = V$. Thus, by Corollary 3.2, $\Sigma = \Sigma_V$. Consequently, totally geodesic submanifolds of $\mathbb{R}^n$ are exactly the affine subspaces.

The following theorem gives the classification of totally geodesic submanifolds in the case of round spheres.

Theorem 3.3. *Let Σ be a complete submanifold of S^n. Then Σ is totally geodesic in S^n if and only if $\Sigma = V \cap \mathsf{S}^n = \mathsf{S}^k$, where V is a linear subspace of $\mathbb{R}^{n+1}$ of dimension $k+1$. Moreover, all totally geodesic submanifolds are embedded, and two complete totally geodesic submanifolds of S^n are congruent in S^n if and only if they have the same dimension.*

Proof. Let $V \subset \mathbb{R}^{n+1}$ be a linear subspace and consider $\Sigma_V = V \cap \mathsf{S}^{p,q}$. We will prove that Σ_V is a complete totally geodesic submanifold.

Clearly the intersection $V \cap \mathsf{S}^n$ is transverse at every point since V is a linear subspace. Thus, $\Sigma_V = \{v \in V : \langle v, v\rangle = 1\}$ is an embedded submanifold isometric to S^k, where $k+1$ is the dimension of V. Moreover, $T_x\Sigma_V = V \cap T_x\mathsf{S}^k = \{v \in V : \langle v, x\rangle = 0\}$, and thus it is isometric to $\mathbb{R}^k$. The geodesics of Σ_V are given by taking intersections of Σ_V with linear planes of $\mathbb{R}^{n+1}$, and thus they are also geodesics of S^n. This proves that Σ_V is a complete totally geodesic submanifold of S^n.

Conversely, let Σ be a complete totally geodesic submanifold of S^n. By homogeneity of S^n we can assume that it passes through $x \in \mathsf{S}^n$. Let us consider the subspace $V = \operatorname{span}\{p, T_x\Sigma\}$. Then, Σ_V is a totally geodesic submanifold with the same initial conditions as Σ. Consequently, since Σ_V is complete, by Corollary 3.2, $\Sigma = \Sigma_V$. By the preceeding discussion this implies that all totally geodesic submanifolds of S^n are embedded.

Finally, let us see that two complete totally geodesic submanifolds of S^n are congruent if and only if they have the same dimension. Let Σ_1 and Σ_2 be complete totally geodesic submanifolds. Notice that if the dimension is not the same, clearly, there is no $T \in \mathsf{O}(n+1)$ mapping Σ_1 to Σ_2. Let us assume that Σ_1 and Σ_2 have the same dimension. After some rotation, we can assume without loss of generality that there exists $x \in \Sigma_1 \cap \Sigma_2$. Now we can clearly find $T' \in \mathsf{O}(n)$ mapping $T_x\Sigma_1$ to $T_x\Sigma_2$, and we can extend T' to an isometry $T \in \mathsf{O}(n+1)$ such that $T(\Sigma_1) = \Sigma_2$ by Corollary 3.2. $\square$

Exercise 3.4. *Classify totally geodesic submanifolds of* $\mathbb{R}\mathsf{H}^n$.

Let us denote by $\overline{\exp}$ the exponential map of $\bar{M}$. Let us consider two totally geodesic submanifolds M_1 and M_2 of $\bar{M}$. Moreover, assume that $T_pM_1 = T_pM_2$ for some $p \in M_1 \cap M_2$. Then, by Lemma 3.1, there exists some open neighborhood U of $0 \in T_pM_1 = T_pM_2$ such that $\overline{\exp}_p(U) \subset M_1 \cap M_2$. Moreover, if M_i is complete for each $i \in \{1,2\}$, we have that $M_1 = \overline{\exp}_p T_pM_1 = \overline{\exp}_p T_pM_2 = M_2$ (since every geodesic of M_i is a geodesic in $\bar{M}$). This proves the following useful lemma.

Lemma 3.5. *Let M_i be a totally geodesic submanifold of $\bar{M}$, where $i \in \{1,2\}$. If $T_pM_1 = T_pM_2$ for some $p \in M_1 \cap M_2$, then M_1 and M_2 coincide around a neighborhood of $p \in \bar{M}$. Furthermore, if M_1 and M_2 are complete, then*

$$M_1 = \overline{\exp}_p T_pM_1 = \overline{\exp}_p T_pM_2 = M_2.$$

Let us consider two totally geodesic submanifolds M_1 and M_2 of $\bar{M}$ intersecting at $p \in M_1 \cap M_2$. By Lemma 3.1, we can find a small neighborhood U_i of 0 in $T_pM_i \subset T_p\bar{M}$ such that $\overline{\exp}_p(U_i) \subset M_i$, for each $i \in \{1,2\}$. Thus,

$$\overline{\exp}_p(U_1 \cap U_2) \subset \overline{\exp}_p(U_1) \cap \overline{\exp}_p(U_2) \subset M_1 \cap M_2$$

is a chain of inclusions of open subsets of $M_1 \cap M_2$. This shows that the intersection of totally geodesic submanifolds is again totally geodesic.

Proposition 3.1. *Let M_i be a totally geodesic submanifold of $\bar{M}$, where $i \in \{1,2\}$. Then, for any $p \in M_1 \cap M_2$, there is an open neighborhood of p in $M_1 \cap M_2$ that is an embedded totally geodesic submanifold of $\bar{M}$. In particular, every connected component of $M_1 \cap M_2$ is a totally geodesic submanifold of $\bar{M}$.*

The next result tells us that a way to construct totally geodesic submanifolds is by using the isometry group of the ambient space $\bar{M}$.

Theorem 3.6. *Let $\bar{M}$ be a Riemannian manifold and let $S \subset \mathrm{Isom}(\bar{M})$ be a subset. Then, every connected component of*

$$\mathrm{Fix}(S) := \{p \in \bar{M} : \varphi(p) = p \ \textit{ for every } \varphi \in S\}$$

is a totally geodesic closed submanifold of $\bar{M}$.

Proof. Let $p \in \mathrm{Fix}(S)$ and take $V = \{v \in T_p\bar{M} : (d\varphi)_pv = v \ \text{ for every } \varphi \in S\}$. Now choose a small normal neighborhood U of $\bar{M}$ around p such that $U \cap \mathrm{Fix}(S)$ is connected. We claim that $U \cap \mathrm{Fix}(S) = \overline{\exp}_p(\overline{\exp}_p^{-1}(U) \cap V)$. Notice that this implies that every connected component of $\mathrm{Fix}(S)$ is an embedded submanifold of $\bar{M}$, since V is a linear subspace of $T_p\bar{M}$, and thus the intersection $\overline{\exp}_p^{-1}(U) \cap V$ is transverse.

On the one hand, let us consider a geodesic γ_v starting at $p=\gamma(0)$ with $\dot\gamma(0)=v\in\overline{\exp}_p^{-1}(U)\cap V$. Thus, since $v\in V$, the uniqueness of geodesics, and the fact that isometries map geodesics to geodesics imply that $\varphi\circ\gamma=\gamma$ for every $\varphi\in S$. This proves that $\overline{\exp}_p(\overline{\exp}_p^{-1}(U)\cap V)\subset U\cap\mathrm{Fix}(S)$, since U is a normal neighborhood.

On the other hand, assume that $q\in U\cap\mathrm{Fix}(S)$ and that there is not any geodesic γ_v starting at p with initial velocity in $\overline{\exp}_p^{-1}(U)\cap V$ reaching q. However, since U is a normal neighborhood there exists a unique minimizing-length geodesic γ in $U\cap\mathrm{Fix}(S)$ joining p and q. Since γ cannot have initial velocity in $\overline{\exp}_p^{-1}(U)\cap V$, there is some isometry $\varphi\in S$ such that $\varphi\circ\gamma$ is a geodesic different from γ but connecting p and q, as one can check that has initial values $\varphi(\gamma(0))=p$ and $(d\varphi)_p\dot\gamma(0)=\dot\gamma(0)$. Then, we get a contradiction with the uniqueness of γ, and $U\cap\mathrm{Fix}(S)\subset\overline{\exp}_p(\overline{\exp}_p^{-1}(U)\cap V)$.

Thus, for every $p\in\mathrm{Fix}(S)$ there is a neighborhood U of $p\in\bar M$ such that $U\cap\mathrm{Fix}(S)$ is an embedded submanifold of $\bar M$. Moreover, every geodesic of $\bar M$ with intial conditions in $\mathrm{Fix}(S)$ stays for a while in $\mathrm{Fix}(S)$. The reason for this is that given a geodesic γ with initial conditions in $\mathrm{Fix}(S)$, then $\varphi\circ\gamma$ is a geodesic with the same initial conditions as γ, and by uniqueness $\gamma=\varphi\circ\gamma$ for some values of time. Thus, γ belongs to $\mathrm{Fix}(S)$ for some values of time. Thus, $\mathrm{Fix}(S)$ consists of an union of totally geodesic submanifolds. By definition, $\mathrm{Fix}(S)$ is closed. Indeed, for each $\varphi\in S$, the set $\mathrm{Fix}(\varphi)=\{p\in\bar M:\varphi(p)=p\}$ is closed, as it is the preimage of the diagonal of $\bar M\times\bar M$ by the continous map $p\in\bar M\mapsto(\varphi(p),p)\in\bar M\times\bar M$. Therefore, every connected component of $\mathrm{Fix}(S)$ is a totally geodesic closed embedded submanifold of $\bar M$. □

Although the above result shows that the existence of totally geodesic submanifolds is linked to the existence of isometries, there are totally geodesic submanifolds that are not fixed points of a set of isometries. For instance, one can take the round sphere S^3 and the totally geodesic submanifold obtained by intersecting S^3 with the linear subspace defined by the equation $x_4=0$ gives a totally geodesic surface $\mathsf{S}^2\subset\mathsf{S}^3$, which is the fixed point set of reflection φ with respect to the hiperplane defined by the equation $x_4=0$. Moreover, this is the only non trivial isometry fixing S^2. Now if we perturb the metric of the sphere near $(0,0,0,1)\in\mathsf{S}^3$, the map φ will not longer be an isometry, so S^2 will keep being a totally geodesic submanifold (as this is a local property) but it does not arises as a fixed point set of any isometry.

4 Existence theorems for totally geodesic submanifolds

In this section we will tackle the following problem. Given a point $p\in M$, and a subspace $V\subset T_pM$, when does exist a totally geodesic submanifold Σ of M passing through $p\in M$ with $T_p\Sigma=V$. The results of this section were mainly obtained from [12] and [5].

Theorem 4.1 (Cartan local existence theorem)**.** *Let (M,g) be a Riemannian manifold and consider $V\subset T_pM$ a linear subspace for a fixed point $p\in M$. Then, there exists a totally geodesic submanifold Σ of M passing through $p\in M$ with $T_p\Sigma=V$ if and only if there exists some $\ell>0$ such that $\mathrm{inj}(p)>\ell$, and for every unit vector $v\in V$,*

$$R(P_{\gamma(t)}X,P_{\gamma(t)}Y,P_{\gamma(t)}Z)\in P_{\gamma(t)}V,\quad \textit{for all } X,Y,Z\in V \textit{ and } t\leq\ell,$$

where γ denotes the geodesic $\gamma\colon[0,\ell]\to M$ with $\dot\gamma(0)=v$.

Proof. First of all, assume there is a totally geodesic submanifold Σ of M passing through $p\in M$ and such that $T_p\Sigma=V$. As Σ is totally geodesic, by Gauss equation we have

$$R(X,Y,Z,W)=R^{\Sigma}(X,Y,Z,W)\ \text{ for all } X,Y,Z,W\in\mathfrak{X}(M).$$

Moreover, by Codazzi equation we have

$$R(X,Y,Z,\xi)=0 \text{ for all } X,Y,Z\in\mathfrak{X}(M), \text{ and } \xi\in\mathfrak{X}^{\perp}(M).$$

Thus $R(X,Y,Z)=R^{\Sigma}(X,Y,Z)$ for every $X,Y,Z\in\mathfrak{X}(M)$. Now since Σ is totally geodesic, it follows by Lemma 3.1, the parallel transport $P_{\gamma(t)}T_p\Sigma=T_{\gamma(t)}\Sigma$. Consequently, we have that $R(P_{\gamma(t)}X,P_{\gamma(t)}Y,P_{\gamma(t)}Z)\in P_{\gamma(t)}T_{\gamma(t)}\Sigma$ as desired.

Now, we prove the converse. To this end we will prove that $\Sigma=\exp_p(B(0,\ell)\cap V)$ is a totally geodesic submanifold of M for some small ℓ. Notice that Σ is an embedded submanifold of M since by hypothesis exp is a local diffeomorphism and $B(0,\ell)\cap V$ is an embedded submanifold of $T_pM\equiv\mathbb{R}^n$. Now we prove that $\Sigma=\exp_p(B(0,\ell)\cap V)$ is a totally geodesic submanifold.

We first prove that for each unit vectors $u,v\in V$, the Jacobi vector field J along $\gamma(t)=\exp_p(tv)$ with initial conditions $J(0)=0$ and $J'(0)=u\in V$ satisfies that $J(t)\in T_{\gamma(t)}\Sigma$ for every $t\in[0,\ell]$. Let us first construct this Jacobi vector field. Take the variation through geodesics $\Omega(s,t)=\exp_p(t(v+su))$. Recall that $J(t)=\partial_{s_{|s=0}}\Omega(s,t)=(d\exp_p)_{tv}(tu)$ is a Jacobi vector field since it is the variational field of a variation through geodesics. We have that $J(0)=0$. Moreover,

$$J'(0)=\partial_{t_{|s=0}}\partial_{s_{|s=0}}\Omega(s,t)=\partial_{s_{|s=0}}\partial_{t_{|t=0}}\Omega(s,t)=\partial_{s_{|s=0}}(d\exp_p)_0(v+su)=\partial_{s_{|s=0}}(v+su)=u\in V.$$

Now take an orthonormal frame $(E_1=\dot\gamma,E_2,\dots,E_n)$ parallel along γ where $E_1(0),\dots,E_k(0)\in V$ and $E_{k+1}(0),\dots,E_n(0)$ in the orthogonal complement of V in T_pM. Thus, we can write $J(t)=\sum_{j=1}^n a_i(t)E_i(t)$. Observe that $a_j(0)=0$ for all $j\in\{1,\dots,n\}$ since $J(0)=0$, and $a_j'(0)=0$ for every $j>k$. Since J is a Jacobi vector field, we have for each $i\in\{k+1,\dots,n\}$,

$$\begin{aligned}0&=g(J''(t)+R(J(t),E_1(t),E_1(t)),E_i(t))=a_i''(t)+\sum_{j=1}^n a_j(t)R(E_j(t),E_1(t),E_1(t),E_i(t))\\&=a_i''(t)+\sum_{j=k+1}^n a_j(t)R(E_j(t),E_1(t),E_1(t),E_i(t)),\end{aligned}$$

since by hypothesis $R(E_j(t),E_1(t),E_1(t),E_i(t))=0$ for every $j\le k$. Moreover, by the symmetries of the curvature tensor, we have that $R(E_j(t),E_1(t),E_1(t),E_i(t))=0$ for every $j>k$

If we define $\omega_{ij}(t)=R(E_j(t),E_1(t),E_1(t),E_i(t))$ for each $i,j\in\{1,\dots,n\}$, we have a system of linear ODE's given by

$$a_i''(t)+\sum_{j=k+1}^n\omega_{ij}(t)a_j(t)=0,\quad a_i(0)=a_i'(0)=0,\quad \text{for each } i\in\{k+1,\dots,n\}.$$

But the only solution to this system is $\alpha_i\equiv0$ for each $i\in\{k+1,\dots,n\}$. Thus, we have proved that $J(t)\in P_{\gamma(t)}V$. Moreover, since $\exp_{p_{|B(0,\ell)\cap V}}$ is a diffeomorphism onto Σ, for each $q\in\Sigma$, the tangent space $T_q\Sigma=(d\exp_p)_w(V)$ where $\exp_p(w)=q$. Consequently, $J(t)=(d\exp_p)_{tv}(tu)\in T_{\gamma(t)}\Sigma$ for each $t\in[0,\ell]$.

Now let $\eta\colon(-\delta,\delta)\to\Sigma$ be an arbitrary curve starting at $q=\eta(0)\in\Sigma$. We are going to prove that Σ is invariant under parallel transport in M along η, and thus totally geodesic in M.

Consider the smooth variation $\Gamma(s,t)=\exp_p(\frac{t}{\ell}v(s))$, where $s\in(-\delta,\delta)$, $t\in[0,\ell]$, and $v(s)=\exp_p^{-1}(\eta(s))$, i.e., for each $s\in(-\delta,\delta)$, $v(s)$ is the position vector from $p\in\Sigma$ for $\eta(s)$.

Thus, $\Gamma(s,0)=p$, and $\Gamma(s,\ell)=\eta(s)$. Also, by definition of Σ, we have that $\Gamma(s,t)\in\Sigma$. Notice that $T(s,t)=\partial_t\Gamma(s,t)=(d\exp_{p\,\frac{tv(s)}{\ell}})(\frac{v(s)}{\ell})\in T_{\Gamma(s,t)}\Sigma$. Now for each fixed $s\in(-\delta,\delta)$, let

$J_s(t) = \partial_s \Gamma(s,t)$ be the Jacobi vector field along the geodesic $\gamma_s(t) = \Gamma(s,t)$. Let us compute the initial conditions of the Jacobi field J_s. Firstly, $J_s(0) = \partial_s \Gamma(s,0) = (d\exp_p)_{tv(s)}(t\partial_s v(s))_{|t=0} = 0$. Secondly,

$$J_s'(0) = \partial_{t_{|t=0}} \partial_s \Gamma(s,t) = \partial_s \partial_{t_{|t=0}} \Gamma(s,t) = \frac{1}{\ell} \partial_s (d\exp_p)_0(v(s))$$
$$= \frac{1}{\ell} \partial_s v(s) = \frac{1}{\ell} (d\exp_p^{-1})_{\eta(s)}(\dot{\eta}(s)) = \frac{1}{\ell} \dot{\eta}(s) \in T_{\eta(s)}\Sigma.$$

Hence, since for each $s \in (-\delta, \delta)$, J_s is a Jacobi vector field along the radial geodesic $\gamma_s(t) = \Gamma(s,t)$ with initial conditions $J_s(0) = 0$ and $J_s'(0) = \frac{\dot{\eta}(0)}{\ell} \in T_q\Sigma$. By the preceeding discussion, for every s,t, we deduce that $J_s(t) \in T_{\Gamma(s,t)} \in V_{s,t}$, where $V(s,t) := P_{\gamma_s(t)}V$, and $J_s(t) \in T_{\Gamma(s,t)}\Sigma$ and $T_{\Gamma(s,t)}\Sigma = V(s,t)$.

Let $z \in T_q\Sigma = V_{0,1}$ be arbitrary where $q = \eta(0) = \Gamma(0,1)$. We parallel translate z along γ_0 from q to p to obtain $z_0 \in V = V_{0,0}$, and then we parallel translate z_0 along γ_s from p to each $\eta(s)$ to obtain a vector field Z along Γ. By the discussion in the preceeding paragraph, we know that $Z(s,t) \in V_{s,t}$. Now $R(\partial_t, \partial_s)Z = \partial_t\partial_s Z - \partial_s\partial_t Z - \nabla_[\partial s, \partial_t]Z = \partial_t\partial_s Z$, since Z is parallel along each geodesic γ_s and $[\partial_s, \partial_t] = 0$. Let $w \in T_pM$ be orthogonal to V and extend it to a vector field W along Γ parallel along each γ_s. Thus, $g(W(s,t), V(s,t)) = 0$ for all s,t, since $g(W(s,0), Z(s,0)) = 0$. Thus,

$$0 = g(R(\partial_t, \partial_s Z, W)) = g(\partial_t\partial_s Z, W) = \partial_t g(\partial_s Z, W) - g(\partial_s Z, \partial_t W) = \partial_t g(\partial_s Z, W),$$

where we have used that W is paralell along each γ_s. Hence, $g(\partial_s Z, W)$ is constant along each γ_s. However, $g(\partial_s Z(s,0), W(s,0)) = 0$, as $\Gamma(s,0)$ is constantly equal to $p \in \Sigma$, thus $g(\nabla_{\dot{\eta}(s)} Z, W) = g(\partial_s Z(s,1), W(s,1)) = 0$. This implies that $T_{\eta(s)}\Sigma = V_{s,1}$ is a family of subspaces invariant under parallel along $\eta(s)$. Since η is an arbitrary curve of Σ, this proves that Σ is totally geodesic in M by Lemma 3.1. $\square$

Notice that this result is local as it only guarantees existence of a totally geodesic submanifold in a neighborhood around a given point $p \in M$. If we assume some stronger hypothesis, we are able prove a global version of the previous result proved by Hermann in [16]. We need to introduce the following notions.

Definition 4.2. *A **once-broken geodesic** is a piecewise smooth geodesic $\gamma\colon (-\varepsilon, \varepsilon) \to M$, which is only non-smooth at a single time $t_1 \in (-\varepsilon, \varepsilon)$. In that case we say that γ is **broken** at t_0. Given a point $p \in M$ and a vector subspace $V \subset M$, we say that a geodesic $\gamma\colon [0, \ell] \to M$, with $\ell > 0$, is V-admissible if it satisfies:*

i) $P_{\gamma(t)}\dot{\gamma}(t) \in P_{\gamma(t)}V$ for all $t \in [0, \ell]$.

ii) If γ is broken at t_0, then $\gamma([t_0, \ell])$ is contained in a convex neighborhood of $\gamma(t_0)$.

Notice that a geodesic with initial velocity in V is clearly V-admissible. Moreover, not every once-broken with initial velocity in V is V-admissible.

Theorem 4.3 (Hermann's theorem)**.** *Let (M,g) be a complete Riemannian manifold, $p \in M$ and $V \subset T_pM$ a linear subspace. Then, there exists a complete immersed totally geodesic submanifold Σ of M with $p \in \Sigma$ and $T_p\Sigma = V$ if and only if for all V-admissible once broken geodesic γ of M starting at p,*

$$R(P_\gamma X, P_\gamma Y, P_\gamma Z) \in P_\gamma V, \quad \textit{for all } X, Y, Z \in V.$$

Proof. If Σ is totally geodesic, the proof is the same as in Cartan's theorem, see proof of Theorem 4.1. Let us prove the converse. By Cartan's theorem (Theorem 4.1), we have that there is a totally geodesic submanifold passing through p and with tangent space at p equal to V. Without loss of generality, we can take Σ inextendable, this means that Σ is the largest totally geodesic submanifold passing through $p \in M$ with tangent space $T_p\Sigma = V$. Notice that Σ is not necesarilly embedded but only immersed. We will prove that Σ is complete. We will argue by contradiction.

If Σ is not complete, there is a geodesic $\alpha \in [0,1) \to \Sigma$ for which $\lim_{t\to 1} \alpha(t)$ does not exist in Σ. However, since M is complete $\lim_{t\to 1} \alpha(t) = q \in M$ such that $q \notin \Sigma$. Moreover, since Σ is totally geodesic in M, there exists a geodesic $\widetilde{\alpha}\colon [0,1] \to M$ such that $\widetilde{\alpha}(t) = \alpha(t)$ for each $t \in [0,1)$.

Let $\widetilde{V}$ be the parallel transport of V in M along $\widetilde{\alpha}$ from p to $q = \widetilde{\gamma}(1)$., i.e. $\widetilde{V} = P_{\widetilde{\alpha}(1)}V$. Let $\sigma\colon [0,\delta) \to M$ be an arbitrary geodesic starting at $q = \sigma(0)$ with initial velocity in $\widetilde{V}$, and define the piecewise smooth curve $\beta\colon [0, 1+\delta] \to M$ where

$$\beta(t) = \begin{cases} \alpha(t) & t \in [0,1) \\ \sigma(t-1) & t \in [1, 1+\delta]. \end{cases}$$

Then, β is a once-broken geodesic starting at p, and broken at $\gamma(1) = q$. Moreover, $\dot{\beta} \in P_\beta \widetilde{V}$, by definition of $\widetilde{V}$, and $\delta > 0$ can be taken sufficiently small so $\sigma([1, 1+\delta])$ is contained in a convex neighborhood of $q \in M$, and β is a once-broken geodesic V-admissible geodesic. Now, combining the hypothesis with Cartan's theorem (Theorem 4.1) there is a totally geodesic submanifold $\widetilde{\Sigma}$ of M with $T_q\widetilde{\Sigma} = \widetilde{V}$.

Now consider the once-broken geodesic $\gamma\colon [0, 1+\varepsilon) \to M$ where

$$\gamma(t) = \begin{cases} \widetilde{\alpha}(t) & t \in [0,1) \\ \alpha(1-t) & t \in [1, 1+\varepsilon), \end{cases}$$

for sufficiently small $\varepsilon > 0$. By construction the parallel transpor ot $V = T_p\Sigma$ to α from p to $\alpha(1-t_0)$ coincides with the parallel transport of $W = T_q\widetilde{\Sigma}$ along γ from $q = \gamma(1)$ to $\gamma(1+t_0) = \alpha(1-t_0)$ for all $t_0 \in (0,\varepsilon)$. It follows that the tangent space of Σ and $\widetilde{\Sigma}$ coincides at all points on $\gamma((1, 1+\varepsilon)) = \beta((1-\varepsilon, 1))$. Thus, by uniqueness of totally geodesic submanifolds (see Corollary 3.2), since Σ is inextendable $\widetilde{\Sigma}$ is contained in Σ but this yields a contradiction as $q \in \widetilde{\Sigma}$ but $q \notin \Sigma$. $\square$

The previous condition is not particularly practical when studying totally geodesic submanifolds within a given ambient space. However, by assuming analyticity, we can achieve the following useful algebraic characterization of totally geodesic submanifolds of analyici Riemannian manifolds. Let us first introduce some notation.

Let k be a non-negative integer. The k-th covariant derivative of the curvature tensor $\bar{R}$ denoted by $\bar{\nabla}^k \bar{R}$ is a $(1, k+3)$-tensor define inductively from $\bar{\nabla}^{k-1}\bar{R}$. A subspace $V \subset T_p\bar{M}$ is invariant under $(\bar{\nabla}^k \bar{R})_p$ if

$$(\bar{\nabla}^k \bar{R})(U_1, \ldots, U_k, X, Y, Z) \in V$$

for every $X, Y, Z, U_1, \ldots, U_k \in V$.

Theorem 4.4. *Let $\bar{M}$ be an analytic complete Riemannian manifold, $p \in \bar{M}$ and V a linear subspace of $T_p\bar{M}$. Then, the following statements are equivalent:*

i) $(\bar{\nabla}^k \bar{R})(U, \ldots, U, X, Y, Z) \in V$, *where* $U, X, Y, Z \in V$.

ii) $(\bar\nabla^k \bar R)_p$ *leaves* V *invariant for every* $k \geq 0$.

iii) *There exists a complete immersed totally geodesic submanifold* M *of* $\bar M$ *such that* $p \in M$ *and* $\overline{\exp}_p(V) = M$.

There exists a complete totally geodesic submanifold M *of* $\bar M$ *such that* $p \in M$ *and* $\overline{\exp}_p(V) = M$ *if and only if* $(\bar\nabla^k \bar R)_p$ *leaves* V *invariant for every* $k \geq 0$.

Proof. Firstly, we will prove that *i)* implies *iii)*. Let us extend $X, Y, Z \in V$ and $\xi \in V^\perp := T_p\bar M \ominus V$ to parallel vector fields along an arbitrary once-broken V-admissible geodesic $\gamma\colon [0,1] \to \bar M$ starting at $p \in \bar M$ with initial velocity $\dot\gamma(0) = U \in V$. Then, by *i)*, we have

$$\frac{d}{dt}_{|t=0}^{k} \bar R(X(t), Y(t), Z(t), \xi(t)) = 0, \quad \text{for every } k \geq 0.$$

By the analiticity of $\bar M$, this shows that $\bar R(X(t), Y(t), Z(t), \xi(t)) = 0$ for every $t \in [0,1]$. Then, by Theorem 4.3, there exists a complete totally geodesic submanifold N of $\bar M$ with $T_pN = V$ defined locally around $p \in N$, yielding *iii)*.

Moreover, if M is a totally geodesic submanifold of $\bar M$, Gauss formula together with Gauss and Codazzi equations imply that T_pM is invariant under $(\bar\nabla^k \bar R)_p$ for every $k \geq 0$ and $p \in M$. This proves that *iii)* implies *ii*. Finally, *ii)* implies *i)* trivially. ☐

The previous theorem gives us another proof for the classification of totally geodesic submanifolds of spaces with constant curvature.

Example 4.5 (Totally geodesic submanifolds of spaces with constant curvature)**.** *Let* M *be the Euclidean space* $\mathbb{R}^n$*, the round sphere* S^n *or the hyperbolic space* $\mathbb{R}\mathsf{H}^n$*. Notice that all subspaces of* T_pM *are* R*-invariant. Moreover, one can check that* $\nabla^k R = 0$ *for all* $k > 0$*. Thus, for every subspace* $V \subset T_pM$*, we have that* $\Sigma_V = \exp_p(V)$ *is always a complete totally geodesic submanifold. A unit speed geodesic* γ_v *starting from a point* $p \in M$ *and initial velocity* $v \in T_pM$ *are given by* $\gamma_v\colon \mathbb{R} \to M \subset \bar M$*, where* $\bar M$ *is either* $\mathbb{R}^{n+1}$*,* $\mathbb{R}^n$*, or* $\mathbb{R}^{n,1}$*, in each case, with*

$$\gamma_v(t) = \begin{cases} \cos(t)p + \sin(t)v & \text{if } M = \mathsf{S}^n, \\ p + tv & \text{if } M = \mathbb{R}^n, \\ \cosh(t)p + \sinh(t)v & \text{if } M = \mathbb{R}\mathsf{H}^n. \end{cases}$$

Thus, $\Sigma_V = \exp_p(V) = M \cap \operatorname{span}\{p, V\}$*, proving that totally geodesic submanifolds are affine subspaces when* $M = \mathbb{R}^n$ *or intersections of linear spaces with* M*, when* M *is equal to* S^n *or* $\mathbb{R}\mathsf{H}^n$*.*

5 Totally geodesic submanifolds and positive curvature

A central topic in Riemannian geometry is the relationship between curvature and topology. In particular, the study of manifolds with positive curvature remains a deeply intriguing and mysterious area, largely due to the apparent scarcity of known examples. A very natural question is the following:

What smooth manifolds M admit a metric g with $\sec > 0$?

Let us assume that (M, g) is a compact Riemannian manifold of dimension n. If $n = 2$, then Gauss-Bonnet theorem states that

$$\int_M K dA = 2\pi\chi(M),$$

where K denotes the Gaussian curvature and $\chi(M)$ is the Euler characteristic of M . As we have explained the Gaussian curvature and the sectional curvature coincide for a manifold of dimension two. Thus, if (M, g) has positive curvature, $\int_M K dA > 0$. However, by the classification of compact surfaces, we know that the only compact surfaces with postive Euler characteristic are the sphere S^2 with $\chi(\mathsf{S}^2) = 2$, and the real projective space $\mathbb{R}\mathsf{P}^2$ with $\chi(\mathbb{R}\mathsf{P}^2) = 1$.

If $n = 3$, the situation is much more involved. In the eighties, Hamilton introduced the Ricci flow. The Ricci flow of a metric g on M consist on a family of metrics $g(t)$ of M that depend on a parameter $t \in \mathbb{R}$ that satisfy the equation

$$\partial_t g(t) = -2\mathrm{Ric}_{g(t)}, \quad g(0) = g.$$

Hamilton proved that if one starts with a simply connected manifold M equipped with a metric g with $\mathrm{Ric} > 0$ (in particular with $\sec > 0$), one can take a normalization of the flow above in such a way that the initial Riemannian manifold converges to a round sphere, see [14]. Thus, in particular, he proved that every compact 3-manifold with $\mathrm{Ric} > 0$ (in particular with $\sec > 0$) is diffeomorphic to S^3. Hence, every compact 3-manifold with $\mathrm{Ric} > 0$ (in particular with $\sec > 0$) is a quotient of a round 3-sphere by a finite subgroup acting freely and isometric on S^3. Surprisingly, Perelmann proved, using again the Ricci flow, that one could get rid of the $\mathrm{Ric} > 0$-hypothesis, proving Poincaré's conjecture, so far the only solved Millenium problem. When $n = 4$, our understanding is quite limited. In fact, even a seemingly simple question, such as Hopf's conjecture, remains one of the most important open problems in Riemannian geometry.

Conjecture 5.1 (Hopf's conjecture)**.** *The manifold* $\mathsf{S}^2 \times \mathsf{S}^2$ *does not admit a metric with* $\sec > 0$.

The fact that for this simple 4-manifold, we cannot even determine whether it admits a metric with $\sec > 0$ highlights the complexity of this topic and how far we are from addressing the question posed at the beginning of this section. However, there is a partial answer to this question, which essentially indicates that if a positively curved metric exists on $\mathsf{S}^2 \times \mathsf{S}^2$, it must exhibit very limited symmetry.

Theorem 5.2 (Hsiang and Kleiner, [17])**.** *Let (M, g) be an orientable and compact 4-manifold. Then, if it exists a non-trivial isometric action by S^1, then M is homeomorphic to S^4 or the complex projective plane $\mathbb{C}\mathsf{P}^2$.*

In particular, this shows that the isometry group of a positively curved metric on $\mathsf{S}^2 \times \mathsf{S}^2$ is finite. For higher dimensions, the problem gets even more complicated.

Another issue, previously mentioned at the beginning of this section, is the lack of examples. We know very few compact manifolds which admit a metric with $\sec > 0$, and all the ones we know have a lot of symmetry. In particular, simply connected homogeneous spaces with $\sec > 0$ were classified by Berger3 [4], Wallach [30], and Bèrard-Bergery [3]. This are listed as follows:

1) the compact rank one symmetric spaces: S^n, $\mathbb{C}\mathsf{P}^n$, $\mathbb{H}\mathsf{P}^n$, and $\mathbb{O}\mathsf{P}^2$.

2) the Wallach flag manifolds: $W^6 = \mathsf{SU}_3/\mathsf{T}^2$, $W^{12} = \mathsf{Sp}_3/\mathsf{Sp}_1^3$, and $W^{24} = \mathsf{F}_4/\mathsf{Spin}_8$.

3) the Berger space $B^{13} = \mathsf{SU}_5/\mathsf{Sp}_2\mathsf{U}_1$.

4) the Alloff-Wallach spaces: $W^7_{p,q} = \mathsf{SU}_3/\mathrm{diag}(z^p, z^q, \overline{z}^{p+q})$, where $(p,q) \geq 0$ and p,q are coprime.

5) the Berger space $B^7 = \mathsf{SO}_5/\mathsf{SO}_3^{\max}$.

Regarding inhomogeneous metrics, some of them are biquotients. A **biquotient** is a manifold obtained as the quotient of a homogeneous space G/H by a free action of a subgroup $\mathsf{L} < \mathsf{G}$. Eschenburg [10] and Bazaikin [2] constructed biquotients with positive curvature. Then, one more example is a cohomogeneity one metric on the connected sum $T_1\mathsf{S}^4\#\Sigma^7$, where $T_1\mathsf{S}^4$ is the unit tangent bundle of S^4, and Σ^7 is an exotic sphere. This was constructed independently by Dearricott [9] and Grove, Verdiani, and Ziller [13]. So far, we do not know more examples of manifolds admitting metrics with $\sec > 0$. Excluding compact rank one symmetric spaces, they all happen in dimension 6, 7, 12, 13, and 24.

One could ask if there are general topological obstructions for the existence of metrics with $\sec > 0$. The most important (and maybe the only) such result is the following:

Theorem 5.3 (Synge's theorem)**.** *Let (M,g) be compact with* $\sec > 0$*. Then, the following statements hold:*

- *If* $\dim M$ *is even, then* $\pi_1(M)$ *is either* 0 *or* $\mathbb{Z}_2$*.*
- *If* $\dim M$ *is odd, then* M *is orientable.*

For the proof of the forthcoming theorem we will need to derive the second variation for the energy functional.

Let γ be a geodesic and $\Gamma\colon (-\varepsilon,\varepsilon)\times[a,b]\to M$ be a smooth variation of γ with variational field V. Then, using an intermediate step of the computation for Equation (1), we have

$$\begin{aligned}\frac{d^2}{ds^2}E(\Gamma(s,t)) &= \int_a^b \partial_s g(\partial_t S, T)dt = \int_a^b g(\partial_s\partial_t S, T) + g(\partial_t S, \partial_s T)dt\\ &= \int_a^b ||\partial_t S||^2 + R(\partial_s,\partial_t,\partial_s,\partial_t) + \partial_t g(\partial_s S, T) - g(\partial_s S, \partial_t T)dt,\end{aligned}$$

where we have used that ∂_s, ∂_t commute as they are the coordinate fields of the domain of Γ. Now, evaluating at $s=0$, using that γ is a geodesic and the identities of the curvature tensor R, we obtain the **second variation for the energy functional for geodesics**

$$\frac{d^2}{ds^2}_{|s=0}E(\Gamma(s,t)) = \int_a^b ||\partial_t V(t)||^2 - R(\partial_s,\partial_t,\partial_t,\partial_s)dt + \Big[g(\nabla_{\frac{\partial}{\partial s}_{|s=0}} S(s,t), \dot\gamma(t))\Big]_{t=a}^{t=b}. \tag{4}$$

Observe that the second variation formula i.e. (Equation (4)) defines a quadratic form on the space of piecewise smooth vector fields along a geodesic $\gamma\colon[a,b]\to M$ vanishing at the endpoints. The associated symmetric bilinear form I is called the **index form** and it is given by

$$I(X,Y) = \int_a^b \langle \partial_t X, \partial_t Y\rangle + R(\dot\gamma, X, Y, \dot\gamma).$$

Finally, notice that if Γ is a proper normal variation of γ, and V is the associated variational field, the second variation of the energy is equal to $I(V,V)$. Now if γ is minimizing, then $I(V,V)\geq 0$ for every proper normal vector field V along γ.

As it was earlier commented, the development of Riemannian geometry was significantly influenced by going beyond of Euclid's fifth postulate, which states: given a line ℓ and a point

$p \notin \ell$, there exists exactly one line ℓ' parallel to ℓ that passes through p. In contrast, for one of the simplest 2-dimensional geometries, the round sphere S^2, there are no parallel lines (geodesics). The following theorem extends this phenomenon to higher dimensions and Riemannian manifolds with positive curvature.

Theorem 5.4 (Frankel theorem, [11])**.** *Let (M, g) be complete and connected with* $\sec > 0$ *and let Σ_1 and Σ_2 be compact totally geodesic submanifolds of M. If $\dim \Sigma_1 + \dim \Sigma_2 \geq \dim M$, then Σ_1 and Σ_2 intersect.*

To prove this theorem we need the following lemma.

Lemma 5.5. *Let Σ be a submanifold of (M, g), and consider $\gamma\colon [a, b] \to M$ a minimizing geodesic from $\gamma(a) = p \notin \Sigma$ to $\gamma(b) = q \in \Sigma$ with $L(\gamma) = d(p, \Sigma)$. Then, $\dot\gamma(b) \perp T_q\Sigma$.*

Proof. Take $v \in T_q\Sigma$, and $\sigma\colon (-\varepsilon, \varepsilon) \to \Sigma$ with $\sigma(0) = q$ and $\dot\sigma(0) = v \in T_q\Sigma$. Let $\Gamma\colon (-\varepsilon, \varepsilon) \to M$ be a variation of γ with $\Gamma(s, 0) = p$, and $\Gamma(s, b) = \sigma(s)$. Then, $V(a) = S(0, a) = \frac{\partial}{\partial s}_{|s=0}\Gamma(s, t) = 0$, and $V(b) = S(0, b) = \frac{\partial}{\partial}_{|s=0}\Gamma(s, t) = v \in T_q\Sigma$. Now, since γ is a geodesic, the first variation formula for the length yields

$$0 = \frac{d}{ds}_{|s=0}L(\Gamma(s, t)) = -\int_a^b \langle V, \nabla_{\dot\gamma}\dot\gamma\rangle dt + \langle V(b), \dot\gamma(b) - \langle V(a), V(b)\rangle = \langle v, \dot\gamma(b)\rangle.$$

Thus, $\dot\gamma \perp T_q\Sigma$, and the lemma follows. □

Proof of Frankel's theorem. Assume that Σ_1 and Σ_2 do not intersect. By compactness there exists a shortest unit-speed geodesic connecting Σ_1 and Σ_2. Thus, let $\gamma\colon [a, b] \to M$ be such that $\gamma(a) = p \in \Sigma_1$, $\gamma(b) = q \in \Sigma_2$ and $d(\Sigma_1, \Sigma_2) = d(\Sigma_1, q) = d(\Sigma_2, p)$. Since γ is a unit speed minimizing geodesic, it must be a minimum for the length functional, and for every smooth variation Γ of γ, we have $\frac{d^2}{ds^2}_{|s=0}E(\Gamma(s, t)) \geq 0$.

Let us parallel transport $T_p\Sigma_1$ along γ to get a subspace W in T_qM. By the properties of parallel transport $W \perp \dot\gamma(b)$, thus

$$\dim(W \cap T_q\Sigma_2) = \dim \Sigma_1 + \dim \Sigma_2 - \dim M + 1 \geq 1$$

Then we can choose some non-zero $v \in T_p\Sigma_1$, whose parallel transport along γ is tangent to $T_p\Sigma_2$. Let us denote this parallel vector field along γ by V. Now for some small $\varepsilon > 0$, one defines the smooth variation given by $\Gamma\colon (-\varepsilon, \varepsilon) \times [a, b] \to M$, where $(s, t) \mapsto \Gamma(s, t) = \exp_{\gamma(t)}(sV(t))$. This is clearly a smooth variation of γ and $S(0, t) = \frac{\partial}{\partial_s}_{|s=0}\Gamma(s, t) = d(\exp_{\gamma(t)})_0(V(t)) = V(t)$, as the differential of $\exp_{\gamma(t)}$ at $0 \in T_{\gamma(t)}M$ is the identity.

Moreover, $\sigma_{t_0}(s) = (s, t_0)$ is a geodesic of M for every fixed $t_0 \in [a, b]$. Notice that $\sigma_a(0) = p$. $\sigma_b(0) = q$, and $\dot\sigma_{t_0}(0) = V(t)$ for all $t \in [a, b]$. Thus, by Lemma 3.1, as Σ_1 and Σ_2 are totally geodesic, the curves σ_a and σ_b belong to Σ_1 and Σ_2 for small values of s. Thus, by shriking the interval $(-\varepsilon, \varepsilon)$ if necessary, we can assume that for each fixed $s_0 \in (-\varepsilon, \varepsilon)$, we have $\Gamma(s_0, a) \in \Sigma_1$ and $\Gamma(s_0, b) \in \Sigma_2$. Now the second variation of the energy yields

$$\frac{d^2}{ds^2}_{|s=0}E(\Gamma(s, t)) = \int_a^b ||\partial_t V(t)||^2 - R(\partial_s, \partial_t, \partial_t, \partial_s)dt + \Big[g(\nabla_{\frac{\partial}{\partial s}_{|s=0}}S(s, t), \dot\gamma(t))\Big]_{t=a}^{t=b}.$$

However, notice that the term $\int_a^b ||\partial_t V(t)||^2 dt$ vanishes since V is parallel, and that $\nabla_{\frac{\partial}{\partial s}_{|s=0}}S(s, t) = \nabla_{\dot\sigma_{t_0}}\dot\sigma_{t_0} = 0$, as σ_{t_0} is a geodesic for every $t_0 \in [a, b]$. Thus, $\frac{d^2}{ds^2}_{|s=0}E(\Gamma(s, t)) < 0$, as M has positive sectional curvature. However, since γ is a minimizing unit speed geodesic it must be a minimum for the energy functional, and then a minimum for the length functional. □

An extension of the previous theorem is the connectedness principle proved by Wilking in 2002. This states that the topology of a totally geodesic submanifold of a positively curved manifold "coincides" up to a certain degree with the topology of the ambient space where it lives.

Before stating the theorem, recall that a continous map $f\colon M \to N$ is said **k-connected** if its induced maps in the homotopy groups $f_*\colon \pi_j(M) \to \pi_j(N)$ are isomorphisms for $j < k$ and is an epimorphism for $j = k$.

Theorem 5.6 (Connectedness principle, [31])**.** *Let (M, g) be compact with $\sec > 0$ and dimension n, and let Σ be a compact totally geodesic submanifold of M of dimension k. Then, the inclusion map $i\colon \Sigma \to M$ is $(2k - n + 1)$-connected.*

This has allowed to obtain rigidity results for manifolds with $\sec > 0$ abundant isometries. The naive idea would be to take fixed point sets of isometries, that are totally geodesic submanifolds by Theorem 3.6, and then use the connectedness principle to bound the topology of the ambient space.

Moreover, it is immediate to check that the Connectedness principle implies Frankel theorem when there is a totally geodesic submanifold of dimension $k \geq n/2$. Appart from the totally geodesic submanifolds in compact rank one symmetric spaces, it is hard to find totally geodesic submanifolds in positively curved ambient spaces with dimension $k \geq n/2$. Indeed, one has the following result, see [26, p. 42] for a proof.

Theorem 5.7. *Let (M, g) be compact with $\sec > 0$. If M contains a totally geodesic hypersurface, then M is either homeomorphic to S^n or $\mathbb{R}\mathsf{P}^n$.*

It is an open question if one can replace the word homeomorphic by diffeomorphic in the previous theorem. Also, it is an open question if one can replace the codimension one condition by codimension two, just by adding the complex projective space $\mathbb{C}\mathsf{P}^n$.

6 Totally geodesic submanifolds in homogeneous spaces

Homogeneity is a central notion in Mathematics. The origin of homogeneous spaces dates back to the emergence of non-Euclidean geometry in the mid-19th century. The geometry of these spaces is quite different from that of the Euclidean spaces that we are accustomed to studying in high school. At this point, the need arises to clarify how to define geometry. Erlangen's program answers this question. This was proposed by Felix Klein in 1872. Basically, geometry was defined as the study of those properties in a space that are invariant under a given transformation group.

Intuitively, a homogeneous space is a space that looks the same at each point. For this reason, homogeneous spaces serve as a model space for various types of geometric structures. In particular, our interest lies in those homogeneous spaces that arise from isometric actions, that is, actions preserving the metric.

For a more complete introduction to the theory of homogeneous spaces, one can consult [1] or [19, Chapter X].

6.1 Some basic facts about homogeneous spaces

Let us begin by introducing some notation relative to Lie groups. For a complete introduction to Lie groups one can consult [20]. Recall that a **Lie group** G is a group equipped with a smooth manifold structure in such a way that the multiplication and the inversion are smooth. Moreover, a closed subgroup H of G is a Lie group, see [21, Theorem 15.29]. To each G we can associate a Lie algebra $\mathfrak{g}$ given by its left-invariant vector fields, that we will always write in gothic letters. Recall that the Lie **exponential map** is defined by $\operatorname{Exp}\colon \mathfrak{g} \to \mathsf{G}$, where $\operatorname{Exp}(X) = \gamma_X(1)$, where γ_X is the integral curve of the left-invariant vector field $X \in \mathfrak{g}$ with initial conditions $\gamma_X(0) = e \in \mathsf{G}$. Moreover, for each $h \in \mathsf{G}$, one can consider the map $C_h\colon \mathsf{G} \to \mathsf{G}$, such that $g \in \mathsf{G} \mapsto C_h(g) = hgh^{-1}$. Then, we can define the **the adjoint representation** of G by $\operatorname{Ad}\colon \mathsf{G} \to \operatorname{Aut}(\mathfrak{g})$ where $\operatorname{Ad}(g) = (dC_g)_e$. The adjoint representation of $\mathfrak{g}$ is defined by $\operatorname{ad}\colon \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$, where $\operatorname{ad}_X = (d\operatorname{Ad})_e(X)$. It can be checked that $\operatorname{ad}_X(Y) = [X, Y]$, where $[\cdot,\cdot]$ denotes the Lie bracket of $\mathfrak{g}$. Another relevant thing is that Lie groups admits the structure of a real-analytic manifold in one and only one way such that multiplication and inversion are real analytic. In this case the exponential function is real analytic, see [20, Proposition 1.117].

Let G be a Lie group and consider a closed subgroup K. Then, we construct the smooth manifold $\mathsf{G}/\mathsf{K} := \{gK : g \in \mathsf{G}\}$. Moreover, G acts on G/K as follows:

$$g \cdot (hK) = (gh)K.$$

This action is clearly transitive, and one has the following (see [18, Proposition 4.2]):

Proposition 6.1. *Let G be a Lie group and K be a closed subgroup of G. Then, there is a unique real analytic structure on G/K such that the canonical projection $\pi\colon \mathsf{G} \to \mathsf{G}/\mathsf{K}$ is an analytic submersion.*

Alternatively, every smooth manifold admitting a transitive action can be seen as a quotient of Lie groups, see [21].

Proposition 6.2. *Let G be a Lie group acting transitively on a smooth manifold M, and let $\mathsf{K} = \{g \in G : go = o\}$ for some point $o \in M$. Then, the map $\mathsf{G}/\mathsf{K} \to M$, such that $gK \in \mathsf{G}/\mathsf{K} \mapsto go \in M$ is a diffeomorphism.*

In view of this discussion, is equivalent to consider smooth manifolds with a transitive G-action, and quotients of G by closed subgroups. These spaces are what we call homogeneous spaces. Since our interest relies on Riemannian geometry, the natural group G to consider in our case is the isometry group.

Definition 6.1. *A **Riemannian homogeneous space** $(M, \langle\cdot,\cdot\rangle)$ is a Riemannian manifold for which the isometry group $\operatorname{Isom}(M)$ acts transitively.*

Equivalently, a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ is Riemannian homogeneous if it admits a transitive G-action and the metric $\langle\cdot,\cdot\rangle$ is G-invariant that is

$$\langle (dg)_e X, (dg)_e Y \cdot\rangle_{gp} = \langle X, Y\rangle_p, \quad \text{for all } X, Y \in T_pM \text{ and for all } g \in \mathsf{G}.$$

Sometimes, we will omit the word Riemannian as the only homogeneous spaces that we will consider are Riemannian. In the following we will consider some examples:

Example 6.2 (Round spheres)**.** *Let us consider the action of SO_{n+1} on S^n. This action is clearly transitive. Indeed, if $p, q \in \mathsf{S}^n$, one can define a transformation $T \in \mathsf{SO}_{n+1}$, such that $T(p) = q$, and that maps an arbitrary orthonormal basis of $p^\perp$ to $q^\perp$. Moreover, the isotropy subgroup for $(1, 0, \ldots, 0) \in \mathsf{S}^n$ is*

$$\mathsf{K} = \left\{ \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array}\right) : A \in \mathsf{SO}_n \right\} \simeq \mathsf{SO}_n.$$

Thus, $\mathsf{S}^n = \mathsf{SO}_{n+1}/\mathsf{SO}_n$.

Example 6.3 (Real projective spaces). *Let us consider the action of* SO_{n+1} *on* $\mathbb{R}\mathsf{P}^n$*, which is the space of real lines of* $\mathbb{R}^{n+1}$ *through the origin. This action is clearly transitive by similar reasons as in Example 6.2. Moreover, the isotropy subgroup for the line* $[1:0:\cdots:0] \in \mathbb{R}\mathsf{P}^n$ *is*

$$\mathsf{K} = \left\{ \left(\begin{array}{c|c} \det(A) & 0 \\ \hline 0 & A \end{array} \right) : A \in \mathsf{O}_n \right\} \cong \mathsf{O}_n.$$

Thus, $\mathbb{R}\mathsf{P}^n = \mathsf{SO}_{n+1}/\mathsf{O}_n$.

Example 6.4 (Berger spheres). *Let us consider the standard action of* SU_{n+1} *on* $\mathbb{C}^{n+1}$*. This induces an action on the unit sphere*

$$\mathsf{S}^{2n+1} = \{z \in \mathbb{C}^{n+1} : \langle z, z \rangle = 1\},$$

where $\langle \cdot, \cdot \rangle$ *denotes the Hermitian inner product of* $\mathbb{C}^{n+1}$*. Since* SU_{n+1} *sends any Hermitian orthonormal basis into any other Hermitian basis, a similar argument as in Example 6.2, shows that the action of* SU_{n+1} *is transitive on* S^{2n+1}*. The isotropy of* $(1, 0, \cdots, 0) \in \mathsf{S}^{2n+1}$ *is*

$$\mathsf{K} = \left\{ \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array} \right) : A \in \mathsf{SU}_n \right\} \cong \mathsf{SU}_n.$$

Example 6.5 (Complex projective spaces). *Let us consider the standard action of* SU_{n+1} *on* $\mathbb{C}^{n+1}$*. This induces an action on the complex projective space*

$$\mathbb{C}\mathsf{P}^n = \{\mathbb{C}\text{-lines passing through the origin of } \mathbb{C}^{n+1}\}.$$

Since SU_{n+1} *sends any Hermitian orthonormal basis into any other Hermitian basis, a similar argument as in Example 6.2, shows that the action of* SU_{n+1} *is transitive.The isotropy of* $[1 : 0 : \cdots : 0] \in \mathbb{C}\mathsf{P}^n$ *is*

$$\mathsf{K} = \left\{ \left(\begin{array}{c|c} e^{i\theta} & 0 \\ \hline 0 & A \end{array} \right) : A \in \mathsf{U}_n,\ \theta \in [0, 2\pi],\ \text{where } e^{i\theta} \det A = 1 \right\} \cong \mathsf{S}(\mathsf{U}_1 \times \mathsf{U}_n) \cong \mathsf{U}_n.$$

Exercise 6.6. *Prove that the real hyperbolic space* $\mathbb{R}\mathsf{H}^n$ *can be seen as homogeneous space as* $\mathsf{SO}_{n,1}/\mathsf{SO}_n$*, where* $\mathsf{SO}_{n,1}$ *are the linear transformations of determinant one preserving the inner product of the Minkowski space* $\mathbb{R}^{n,1}$.

6.1.1 Fundamental vector fields and isotropy representation

To have a better understanding of homogeneous spaces we need to make Lie algebras come into play. In order to that, first let us compute the differential of the submersion $\pi \colon \mathsf{G} \to \mathsf{G}/\mathsf{K}$, which maps every $g \in \mathsf{G}$ to $g\mathsf{K} \in \mathsf{G}/\mathsf{K}$. Then, the differential of π at $e \in \mathsf{G}$ is $d\pi_e \colon \mathfrak{g} \to T_o\mathsf{G}/\mathsf{K}$, where $o = eK$. Let $X \in \mathfrak{g}$, then $(d\pi)_e(X) = \frac{d}{dt}_{|t=0}(\pi(\mathrm{Exp}(tX))) = \frac{d}{dt}_{|t=0}(\mathrm{Exp}(tX)o)$. Observe that given $X \in \mathfrak{g}$, we obtain a vector field X^* defined on G/K given by

$$X^*(p) = \frac{d}{dt}_{|t=0}(\mathrm{Exp}(tX)p) \quad \text{for every } p \in \mathsf{G}/\mathsf{K}.$$

The vector field X^* is called **fundamental vector field associated with** $X \in \mathfrak{g}$. Let us denote by $\theta_t^X(p) = \mathrm{Exp}(tX) \cdot p$ the flow of X^*. Notice that this flow is by isometries so X^* is a Killing vector field. Thus in this case one has $[X^*, Y^*] = -[X,Y]^*$. Indeed, by the definition of the Lie derivative,

$$\begin{aligned}
[X^*, Y^*]_p &= \frac{d}{dt}_{|t=0}(d\theta^X_{-t})_{\theta^X_t(p)} Y^*_{\theta^X_t(p)} = \frac{d}{dt}_{|t=0}\frac{d}{ds}_{|s=0} \operatorname{Exp}(-tX)(\operatorname{Exp}(sY)(\operatorname{Exp}(tX)\cdot p)) \\
&= \frac{d}{dt}_{|t=0}\frac{d}{ds}_{|s=0} C(\operatorname{Exp}(-tX))(\operatorname{Exp}(sY)\cdot p) = \frac{d}{dt}_{|t=0}(\operatorname{Ad}(\operatorname{Exp}(-tX))Y)^*_p \\
&= \frac{d}{dt}_{|t=0}(e^{-t\operatorname{ad}_X}Y)^*_p = -[X,Y]^*_p.
\end{aligned}$$

Furthermore, from this, we obtain that $(d\pi)_e(\mathfrak{k}) = 0$, where $\mathfrak{k}$ is the Lie algebra of K. Thus, $\dim \mathsf{G}/\mathsf{K} = \dim \mathsf{G} - \dim \mathsf{K}$, since $(d\pi)_e$ is surjective by Proposition 6.1.

One of the special things of Riemannian homogeneous spaces G/K among general homogeneous spaces is that they admit a reductive decomposition [6, Proposition 1.48]. A **reductive decomposition** of G/K is a splitting of the Lie algebra $\mathfrak{g}$ as follows:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \text{with } \operatorname{Ad}(k)\mathfrak{p} \subset \mathfrak{p},$$

where $\mathfrak{p}$ is some linear subspace of $\mathfrak{g}$ and $\mathfrak{k}$ is the Lie algebra of K. Notice that this implies that the restriction of the adjoint representation of G to K defines a representation of K in the subspace $\mathfrak{p}$.

Now we will introduce a representation which is key in the understanding of homogeneous spaces. Let $o \in \mathsf{G}/\mathsf{K}$ and take $k \in \mathsf{K}$. We will also denote by k the map $k\colon \mathsf{G}/\mathsf{K} \to \mathsf{G}/\mathsf{K}$ which is defined via $g\mathsf{K} \mapsto (kg)\mathsf{K}$.

Definition 6.7. *The* ***isotropy representation*** *is the homomorphism of groups* $\mathsf{K} \to \mathsf{GL}(T_o\mathsf{G}/\mathsf{K})$ *given by*

$$k \in \mathsf{K} \mapsto (dk)_o \colon T_o\mathsf{G}/\mathsf{K} \to T_o\mathsf{G}/\mathsf{K}.$$

Thus, we have just defined two representations of K in $\mathfrak{p}$. It turns out that they are equivalent. This will allows us to identify the subspace $\mathfrak{p}$ of a reductive decomposition of $M = \mathsf{G}/\mathsf{K}$ with T_oM where $o = eK$. Let $X \in \mathfrak{p}$, then

$$\begin{aligned}
(d\pi)_e(\operatorname{Ad}(k)X) &= \frac{d}{dt}_{|t=0}(\operatorname{Exp}(t\operatorname{Ad}(kY)K)) = \frac{d}{dt}_{|t=0}(k\operatorname{Exp}(tK)k^{-1}K) = \frac{d}{dt}_{|t=0}(k\operatorname{Exp}(tK)K) \\
&= (dk)_0(d\pi)_e(X).
\end{aligned}$$

In other words the map $(d\pi)_e$ gives an equivalence between the adjoint representation of G restricted to $\mathfrak{K}$ on $\mathfrak{p}$ and the isotropy representation of $\mathfrak{p}$ and we have proved.

Proposition 6.3. *Let* $M = \mathsf{G}/\mathsf{K}$ *be a homogeneous space with reductive decomposition* $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$. *Then, the adjoint representation of* G *restricted to* K *on* $\mathfrak{p}$ *and the isotropy representation of* $\mathfrak{p}$ *are equivalent.*

A first consequence of the previous equivalency is that we have an easy way to parametrize the G-invariant metrics of a homogeneous space, see [1, Proposition 5.1].

Proposition 6.4. *Let* $M = \mathsf{G}/\mathsf{K}$ *be a Riemannian homogeneous space with reductive decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. *Then, there is a one to one correspondence between* G*-invariant metrics on* M *and* $\operatorname{Ad}(\mathsf{K})$*-invariant inner products on* $\mathfrak{p}$.

In particular, from the proof of the previous proposition one deduces that $\langle (d\pi)_e(X), (d\pi)_e(Y)\rangle = \langle (d\pi)_e(\operatorname{Ad}(k)X), (d\pi)_e(\operatorname{Ad}(k)Y)\rangle = \langle X, Y\rangle$ for every $X, Y \in \mathfrak{p}$ and $k \in \mathsf{K}$.

Exercise 6.8. *Consider the Berger sphere $\mathsf{S}^{2n+1} = \mathsf{SU}_{n+1}/\mathsf{SU}_n$ as described in Example 6.4, and take as reductive complement the matrices given by*

$$\mathfrak{p} = \left\{ \left(\begin{array}{c|c} ix & v \\ \hline -v^* & -ix \end{array} \right) : v \in \mathbb{C}^n, x \in \mathbb{R} \right\} \subset \mathfrak{su}_n,$$

where $(\cdot)^*$ denotes the conjugate transpose of a matrix with coefficients in $\mathbb{C}$.

Prove that the isotropy representation of this space splits as a direct sum of two irreducible representations of dimension 1 and $2n$, respectively. Use Proposition 6.3 to prove that there is exactly a 1-parameter family of SU_{n+1}-invariant metrics on Berger spheres (up to homothety).

The following lemma gives an infinitesimal characterization of Riemannian homogeneous spaces in terms of Killing vector fields.

Lemma 6.9. *Let M be a connected Riemannian manifold. Then, the following statements are equivalent:*

i) M is homogeneous.

ii) There exists some point $p \in M$ such that T_pM is spanned by Killing vector fields of M evaluated at p.

iii) For every $p \in M$, T_pM is spanned by Killing vector fields of M evaluated at p.

Proof. Let $M = \mathsf{G}/\mathsf{K}$ be a homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ corresponding to some point $o \in M$. Notice that $\mathfrak{p}$ is identified with T_oM. Thus, the fundamental vector fields induced by elements of $\mathfrak{p}$, when evaluated at o, span T_oM.

If M is a Riemannian manifold and there is a point $p \in M$ such that T_pM is spanned by the Killing vector fields evaluated at p, then there exists some open neighborhood U of p such that every point in U lies on an integral curve of a Killing vector field. This implies that the orbit of p by the action of the isometry group is open. However, since it is also closed, we have that the isometry group acts transitively on M. $\square$

6.1.2 The Levi-Civita of a homogeneous space and the canonical connection

In this section, we will compute the Levi-Civita connection of a Riemannian homogeneous space. This will allow us to compute the curvature tensor which is essential for the understanding of totally geodesic submanifolds.

Before moving to the computation of the Levi-Civita connection, recall that a Killing vector field $X \in \mathfrak{X}(M)$ is characterized by the equation

$$\langle \nabla_Y X, Z \rangle + \langle \nabla_Z X, Y \rangle = 0 \quad \text{for every } Y, Z \in \mathfrak{X}(M).$$

Then if $X \in \mathfrak{X}(M)$ is a Killing vector field we have

$$2\langle \nabla_X Y, Z \rangle = \langle [X,Y], Z \rangle + \langle [X,Z], Y \rangle + \langle X, [Y,Z] \rangle \quad \text{for every } Y, Z \in \mathfrak{X}(M). \tag{5}$$

Let $M = \mathsf{G}/\mathsf{K}$, fix a basepoint $o = eK \in M$, and a reducrtive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let us consider the tensor $U \colon \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$, determined by

$$2\langle U(X,Y), Z \rangle = \langle [Z,X], Y \rangle + \langle [Z,Y], X \rangle \quad \text{for every } X, Y, Z \in \mathfrak{p}.$$

A reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is said to be **naturally reductive** if $U \equiv 0$. Moreover, a Riemannian homogeneous space (M, g) equipped with a transitive action by G is said naturally reductive if it admits a naturally reductive decomposition. Then, the Levi-Civita connection at o is given by

$$(\nabla^*_X Y^*)_o = (-\frac{1}{2}[X,Y]_{\mathfrak{p}} + U(X,Y))^*_o \quad \text{for every } X, Y \in \mathfrak{p}, \tag{6}$$

where $\cdot_{\mathfrak{p}}$ denotes the orthogonal projection to $\mathfrak{p}$. Let us prove Equation (6). Let $X, Y, Z \in \mathfrak{p}$. Recall that fundamental vector fields associated are Killing. Then, using Equation (5), and the fact that $[X^*, Y^*] = -[X,Y]^*$, we have

$$\begin{aligned} 2\langle \nabla^*_X Y^*, Z^* \rangle &= -\langle [X,Y]^*, Z^* \rangle - \langle [X,Z]^*, Y^* \rangle - \langle X^*, [Y,Z]^* \rangle \\ &= -\langle [X,Y], Z \rangle - \langle [X,Z], Y \rangle - \langle X, [Y,Z] \rangle \\ &= -\langle [X,Y], Z \rangle + \langle [Z,X], Y \rangle + \langle [Z,Y], X \rangle = -\langle [X,Y], Z \rangle + 2\langle U(X,Y), Z \rangle, \end{aligned}$$

where $X, Y, Z \in \mathfrak{p}$. Consequently, one gets Equation (6).

Another connection that plays an important role in the study of homogeneous spaces is the canonical connection. Let $M = \mathsf{G}/\mathsf{K}$ be a homogeneous space with basepoint $o = eK \in M$ and reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then, we define the **canonical connection** as the unique G-invariant connection on M such that its value at $o \in M$ is given by

$$(\nabla^c_{X^*} Y^*)_o = (-[X,Y]_{\mathfrak{p}})^*_o \quad X, Y \in \mathfrak{p}.$$

It is worth to compare the Levi-Civita connection with the canonical connection. To do so, one defines the difference tensor $D = \nabla - \nabla^c$. This is a tensor, as it is the difference of two connections.

Exercise 6.10. *For a Riemannian homogeneous space $M = \mathsf{G}/\mathsf{K}$ with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, the following conditions are equivalent:*

i) the reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is naturally reductive,

ii) the difference tensor D is skew-symmetric,

iii) the geodesics of ∇ and ∇^c coincide.

Finally, one can compute the curvature tensor by using Equation 6. We have the following expression

$$R_o(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]_{\mathfrak{p}}} Z - [[X,Y]_{\mathfrak{p}}, Z], \quad \text{where } X, Y, Z \in \mathfrak{p}. \tag{7}$$

Exercise 6.11. *Consider Berger spheres $\mathsf{S}^{2n+1} = \mathsf{SU}_{n+1}/\mathsf{SU}_n$ with the reductive complement $\mathfrak{p}$ given in Exercise 6.8. Let $\mathfrak{p}_0$ and $\mathfrak{p}_1$ be the SU-invariant subspaces of $\mathfrak{p}$ of dimensions 1 and $2n$. For each $\tau > 0$, we define the $\mathrm{Ad}(\mathsf{SU}_n)$-invariant inner product on $\mathfrak{p}$ given by*

$$\langle X, Y \rangle_\tau = \tau \mathcal{B}(X^0, Y^0) + \mathcal{B}(X^1, Y^1), \text{ for all } X, Y \in \mathfrak{p},$$

where $(\cdot)^0$, and $(\cdot)^1$ denote the orthogonal projection onto $\mathfrak{p}_0$, and $\mathfrak{p}_1$, respectively; and $\mathcal{B}$ is the scalar product given by $\mathcal{B}(X,Y) = -\mathrm{tr}(XY)$, for all $X, Y \in \mathfrak{su}_{n+1}$. Find $\tau > 0$ so that the metric $\langle \cdot, \cdot \rangle_\tau$ corresponds to the round metric of S^{2n+1}.

6.2 Totally geodesic submanifolds in homogeneous spaces

In this section, we prove some results concerning totally geodesic submanifolds in homogeneous spaces. The first result shows that totally geodesic submanifolds of homogeneous spaces are again homogeneous spaces.

Proposition 6.5. *Let $\bar{M}$ be a homogeneous Riemannian manifold and let M be a complete totally geodesic submanifold of $\bar{M}$. Then, M is homogeneous.*

Proof. Let X be a Killing vector field of $\bar{M}$. Thus, for each $p \in M$, we have the orthogonal decomposition

$$X(p) = X(p)_{T_pM} + X(p)_{\nu_pM} \quad \text{for every } p \in M,$$

where $X(p)_{T_pM}$ and $X(p)_{\nu_pM}$ denote the orthogonal projections of $X(p)$ to T_pM and ν_pM, respectively. Since X is a Killing vector field of $\bar{M}$, for each $Y \in \Gamma(TM)$, we have

$$0 = \langle \bar{\nabla}_Y X, Y\rangle = \langle \bar{\nabla}_Y X_{TM}, Y\rangle + \langle \bar{\nabla}_Y X_{\nu M}, Y\rangle = \langle \nabla_Y X_{TM}, Y\rangle,$$

since M is totally geodesic. Thus, the tangential projection of a Killing vector field of $\bar{M}$ to M, when restricted to M, is a Killing vector field of M, since M is complete. The tangent space of $\bar{M}$ at every point of $\bar{M}$ is generated by Killing fields of $\bar{M}$, implying that the tangent space of M at every point is generated by projecting these Killing vector fields. Hence, by Lemma 6.9, M is homogeneous. □

A submanifold Σ of a Riemannian manifold M is **extrinsically homogeneous** if given two points $p, q \in \Sigma$ there is always an isometry $g \in \mathrm{Isom}(M)$ such that $gp = q$ and $g\Sigma = \Sigma$.

Notice that complete totally geodesic submanifolds of $\bar{M}$ are intrinsically homogeneous, but they are not necessarily extrinsically homogeneous. However, the connected components of the fix point set of any collection of isometries are extrinsically homogeneous submanifolds, see [5, Lemma 9.1.1]. This shows once again that the fixed points sets of isometries provide the most natural and well-behaved examples of totally geodesic submanifolds. We will give a proof in the case of homogeneous spaces.

Proposition 6.6. *Let $M = \mathsf{G}/\mathsf{K}$ be a Riemannian homogeneous space and $S \subset G$ a subset. Then, every connected component of* $\mathrm{Fix}(S)$ *is an extrinsically totally geodesic submanifold of M. In particular, if $p \in \Sigma$, then $\Sigma = G_\Sigma \cdot p$, where $\mathsf{G}_\Sigma = \{g \in \mathsf{G} : g\psi = \psi g$ for all $\psi \in S\}$.*

Proof. First of all, by Theorem 3.6, we have that connected components of $\mathrm{Fix}(S)$ are totally geodesic in M.

Now, notice that G_Σ acts on Σ. Indeed, let $g \in \mathsf{G}_\Sigma$ and $p \in \Sigma$, then $gp = g\psi p = \psi g p$ for all $\psi \in S$. Thus, $gp \in \Sigma$, and $\mathsf{G}_\Sigma \cdot p \subset \Sigma$. Now we will prove that $\mathsf{G}_\Sigma \cdot p = \Sigma$.

In order to do so, it is enough to check that $T_p\Sigma = T_p(\mathsf{G}_\Sigma \cdot p)$. We claim that $T_p\Sigma = \{v \in T_pM : (d\psi)_p v = v$ for all $\psi \in S\}$. Let us assume that $v \in T_pM$ is such that $(d\psi)_p v = v$ for all $\psi \in S$. Take a geodesic γ of M starting at $p \in M$ with velocity $v \in T_pM$. Then, for each $\psi \in S$. $\psi\gamma$ is a geodesic with initial point $\psi\gamma(0) = \psi p = p$ and with initial velocity $\frac{d}{dt}_{|t=0}\psi\gamma(t) = (d\psi)_p(\dot{\gamma}(0)) = (d\psi)_p(v) = v$. Thus, by uniqueness $\psi\gamma = \gamma$ for every $\psi \in S$, and thus γ belongs to Σ, so $v \in T_p\Sigma$. Conversely, if $v \in T_p\Sigma$, then there is a smooth curve $\alpha\colon (-\varepsilon, \varepsilon) \to \Sigma$ such that $\alpha(0) = p$ and $\dot{\alpha}(0) = v$. Differentiating on both sides we get

$$v = \dot{\alpha}(0) = \frac{d}{dt}_{|t=0}(\psi\alpha(t)) = (d\psi)_p(\dot{\alpha}(0)) = (d\psi)_p v \text{ for all } \psi \in S.$$

Now we claim that the Lie algebra of G_Σ is $\mathfrak{g}_\Sigma = \{X \in \mathfrak{g} : \mathrm{Ad}(\psi)X = X \quad \text{for all } \psi \in S\}$. Indeed, let $\gamma(t) = \mathrm{Exp}(tX)$ where $X \in \mathfrak{g}_\Sigma$. Then, $\mathrm{Exp}(tX) = C_\psi \mathrm{Exp}(tX)$. Differentiating on both sides, we get $X = \frac{d}{dt}_{|t=0}\mathrm{Exp}(tX) = \frac{d}{dt}_{|t=0}(C_\psi \mathrm{Exp}(tX)) = \mathrm{Ad}(\psi)X$. Finally, $(d\pi)_e$ restricted to $\mathfrak{p}$ gives an isomorphism onto T_pM. Then, for each $v \in T_p\Sigma$, we can find some

$X_v \in \mathfrak{p}$ such that $(d\pi)_e(X_v) = v$. Let us check that $X_v \in \mathfrak{g}_\Sigma$, so then $\mathfrak{p} \cap \mathfrak{g}_\Sigma \cong T_p\Sigma$.

$$\begin{aligned}(d\pi)_e(X_v) &= v = (d\psi)_p v = (d\psi)_p(d\pi)_e(X_v) = (d\psi)_p \frac{d}{dt}_{|t=0} \operatorname{Exp}(tX_v) \cdot p \\ &= \frac{d}{dt}_{|t=0} \psi \operatorname{Exp}(tX_v)\psi^{-1} \cdot p = \frac{d}{dt}_{|t=0} \operatorname{Exp}(t \operatorname{Ad}(\psi)X_v) \cdot p = (d\pi)_e(\operatorname{Ad}(\psi)X_v).\end{aligned}$$

Since $(d\pi)_e$ restricted to $\mathfrak{p}$ is injective, we conclude that $X_v \in \mathfrak{g}_\Sigma$. Consequently, since $\mathfrak{p} \cap \mathfrak{g}_\Sigma \cong T_p\Sigma$, we have that $T_p\mathsf{G}_\Sigma \cdot p = T_p\Sigma$, and thus $\mathsf{G}_\Sigma \cdot p = \Sigma$, as $\Sigma \subset \mathsf{G}_\Sigma \cdot p$. □

The following gives a sufficient condition for a subspace of $\mathfrak{g}$ to exponentiate to a totally geodesic submanifold.

Proposition 6.7 (Olmos, Rodríguez-Vázquez, [25]). *Let $M = \mathsf{G}/\mathsf{K}$ be a Riemannian homogeneous space with base point $o = e\mathsf{K} \in M$ and reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{p}_\Sigma$ be a subspace of $\mathfrak{p}$ invariant under D and R. Then, there is a complete totally geodesic submanifold Σ of M such that $T_o\Sigma = \mathfrak{p}_\Sigma$ given by $\Sigma = \exp_o \mathfrak{p}_\Sigma$, where $\mathfrak{p}_\Sigma$ is identified with a subspace of T_oM in the standard way.*

Proof. Notice that Riemannian homogeneous spaces are complete, and real analytic Riemannian manifolds. The covariant derivatives of the curvature tensor can be expressed in terms of D and R. Indeed, by [19, Proposition 2.7, Chapter X] the curvature tensor R of a Riemannian homogeneous space $M = \mathsf{G}/\mathsf{K}$ satisfies $\nabla^c R = 0$, since it is G-invariant. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a reductive decomposition for $M = \mathsf{G}/\mathsf{K}$. Then, using the definition of the difference tensor and the identification of $\mathfrak{p}$ with T_oM we have

$$\begin{aligned}(\nabla_V R)(X,Y,Z) &= ((\nabla_V - \nabla^c_V)R)(X,Y,Z) \\ &= D_V R(X,Y)Z - R(D_V X,Y)Z - R(X,D_V Y)Z - R(X,Y)D_V Z,\end{aligned}$$

where $X, Y, Z, V \in \mathfrak{p}$.

Hence, every subspace $\mathfrak{p}_\Sigma$ of $\mathfrak{p}$ invariant under D and R is invariant under every covariant derivative of R, which implies by Theorem 4.4 that $\mathfrak{p}_\Sigma$ is the tangent space of some totally geodesic submanifold of M and $\Sigma = \exp_o \mathfrak{p}_\Sigma$ is a complete totally geodesic submanifold of M, where $\exp_o$ denotes the Riemannian exponential map of M at o, this yields the desired result. □

We remark that Proposition 6.7 provides a sufficient condition to obtain totally geodesic submanifolds in homogeneous spaces in terms of a linear algebraic property. However, this condition does not need to be necessary.

6.3 Symmetric spaces and totally geodesic submanifolds

In this section we will discuss symmetric spaces from the point of view of Riemannian geometry. Let us begin by introducing the definition of a symmetric space.

Symmetric spaces arise in a broad diversity of situations in both Mathematics and Physics. Their origin goes back to the following question posed by Cartan in 1926:

Which are the Riemannian manifolds whose curvature tensor R
is preserved by parallel transport along any curve?

This property is equivalent to the equation $\nabla R = 0$, and the spaces satisfying this property are intimately related to symmetric spaces. Indeed, every Riemannian manifold satisfying $\nabla R = 0$ is locally isometric to a symmetric space. Cartan achieved a complete classification of symmetric spaces in [7]. For a detailed exposition of the theory of symmetric spaces one can follow [15], [27], and [28].

6.3.1 Some basic properties of symmetric spaces

We start by introducing the notion of symmetric space.

Definition 6.12. *A Riemannian manifold M is a symmetric space if it is connected, and for each $p \in M$, there exists an isometry $\zeta_p \in \mathrm{Isom}(M)$ such that its differential at p is $-\mathrm{Id}$ on T_pM.*

The isometry ζ_p is called the **global symmetry of M at p**, as it reverses the geodesics passing through p.

We now show some examples of symmetric spaces.

Exercise 6.13.

1) *Euclidean spaces $\mathbb{R}^n$. For each point p, the map*

$$\begin{array}{rccc} \zeta_p\colon & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & x & \longmapsto & -x+2p, \end{array}$$

is clearly an isometry since it is the composition of a rotation and a translation, and its differential at p is $-\mathrm{Id}$. Therefore, ζ_p is a global symmetry of $\mathbb{R}^n$ at p.

2) *Spheres S^n. For each point $p \in \mathsf{S}^n \subset \mathbb{R}^{n+1}$, consider*

$$\begin{array}{rccc} \zeta_p\colon & \mathsf{S}^n & \longrightarrow & \mathsf{S}^n \\ & x & \longmapsto & -x+2\langle x,p\rangle p, \end{array}$$

where $\langle \cdot,\cdot\rangle$ denotes the usual scalar product in $\mathbb{R}^{n+1}$. Thus, it is an isometry since $\zeta_p \in O(n+1)$. Moreover, the differential of ζ_p at p is $-\mathrm{Id}$ on T_pS^n. Therefore, ζ_p is a global symmetry of S^n at p.

3) *Hyperbolic spaces $\mathbb{R}H^n$. For each point $p \in \mathbb{R}H^n \subset \mathbb{R}^{n+1}_1$, consider*

$$\begin{array}{rccc} \zeta_p\colon & \mathbb{R}H^n & \longrightarrow & \mathbb{R}H^n \\ & x & \longmapsto & -x-2\langle x,p\rangle p, \end{array}$$

where $\langle \cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^{n+1,1}$. The map ζ_p is an isometry since $\zeta_p \in O(1,n+1)$. Moreover, the differential of ζ_p at p is $-\mathrm{Id}$ on $T_p\mathbb{R}H^n$. Therefore, ζ_p is a global symmetry of $\mathbb{R}H^n$ at p.

4) *Compact Lie groups. Let G be a compact Lie group. It is known that every compact Lie group admits a bi-invariant metric. Let us equip G with a **bi-invariant metric**, meaning for each $g \in G$, we have $L_g, R_g \in \mathrm{Isom}(G)$. Then the map $\zeta_e(g) = g^{-1}, g \in G$, is a global symmetry at the neutral element e, as it is a diffeomorphism whose differential at e is $-\mathrm{Id}$ on T_eG. Given $g \in G$, consider the map*

$$\zeta_g := R_g \circ \zeta_e \circ L_{g^{-1}}.$$

On one hand, ζ_g is an isometry since it is a composition of isometries. On the other hand, $\zeta_g(g) = g$, and the differential of ζ_g at g is $-\mathrm{Id}$ on T_gG. Therefore, ζ_g is a global symmetry of G at g.

As we pointed out in the introduction of this section, symmetric spaces satisfy $\nabla R = 0$. Indeed, let $p \in M$, and $X, Y, Z, V \in T_pM$. Then,

$$\begin{aligned} -(\nabla R)_X(Y,Z,W) &= (d\zeta)_p(\nabla R)_X(Y,Z,W) = (\nabla R)_{(d\zeta)_pX}((d\zeta)_pY,(d\zeta)_pZ,(d\zeta)_pW) \\ &= (\nabla R)_{-X}(-Y,-Z,-W) = (\nabla R)_X(Y,Z,W). \end{aligned}$$

Furthermore, one can prove that a simply connected, complete, Riemannian manifold satisfying $\nabla R = 0$ is symmetric, see [29, p. 225].

The following theorem shows a very important property of symmetric spaces: their completeness.

Proposition 6.8. *Every symmetric space M is complete.*

Proof. Let M be a symmetric space, and suppose it is not complete. Then, there exist $p \in M$ and $v \in T_pM$ such that $\gamma_v\colon [0,b) \longrightarrow M$ with $\gamma_v(0) = p$, is a maximal geodesic for some $b \in \mathbb{R}$. Let $q = \gamma_v(\frac{3}{4}b)$. Combining $\widetilde{\gamma}(t) = \zeta_q \circ \gamma_v(t)$ with $\gamma_v(t)$, we obtain a geodesic that extends γ_v, contradicting its maximality. $\square$

A fundamental property of symmetric spaces is that they are homogeneous.

Proposition 6.9. *Every symmetric space M is a homogeneous space.*

Proof. Let $p, q \in M$. Since M is symmetric and complete, there exists a geodesic segment connecting p to q. Let $o \in M$ be the midpoint of the segment. We have $\zeta_o \in \mathrm{Isom}(M)$, which maps p to q. Therefore, M is a homogeneous space. $\square$

As symmetric spaces are homogeneous, they have right to a reductive decomposition, which turns out to be so special. Let us compute its reductive decomposition. Let (M, g) be a symmetric space and $G = \mathrm{Isom}(M)^0$ the connected component of $\mathrm{Isom}(M)$ that contains the neutral element. We have that G is a Lie subgroup of $\mathrm{Isom}(M)$. Let $p \in M$ and $\zeta_p \in \mathrm{Isom}(M)$, the geodesic symmetry of M at p. Let K be the isotropy group of G at p, which is compact. The set G/K is diffeomorphic to M via the map

$$\begin{array}{rccc} \Phi\colon & G/K & \longrightarrow & M \\ & gK & \longmapsto & g(p). \end{array}$$

Thus, if we consider the metric $h = \Phi^* g$ on G/K, Φ is an isometry and the metric h is **G-invariant**, that is, the map $gK \mapsto hgK$ is an isometry for each $h \in G$. The *isotropy representation of $M \cong G/K$ at p* is the orthogonal representation defined by $K \times T_pM \longrightarrow T_pM$, such that $(k, v) \mapsto dk_p v$. A symmetric space $M = G/K$ is **irreducible** if its isotropy representation restricted to K^0, the connected component of K that contains the neutral element, is an irreducible representation. Otherwise, a symmetric space is said to be **reducible**. It is known that M is irreducible if and only if $\widetilde{M}$, the universal covering of M, is irreducible.

The map $\sigma\colon G \longrightarrow G$, such that $g \mapsto \zeta_p g \zeta_p$ is an involutive automorphism of G. Moreover, $G_\sigma^0 \subset K \subset G_\sigma$, where $G_\sigma = \{g \in G : \sigma(g) = g\}$ and G_σ^0 is the connected component containing the neutral element of G_σ. If G is a connected Lie group, K a compact subgroup, and there exists an involutive automorphism σ of G such that $G_\sigma^0 \subset K \subset G_\sigma$, the pair (G, K) is called a **symmetric pair**. Furthermore, we say that the symmetric pair (G, K) is an **effective symmetric pair** if the action of G on $M \cong G/K$ is effective.

Let θ be the differential of σ at $e \in G$. The Lie algebra of K is given by

$$\mathfrak{k} = \{X \in \mathfrak{g} : \theta(X) = X\}$$

and we define

$$\mathfrak{p} = \{X \in \mathfrak{g} : \theta(X) = -X\}.$$

It follows that θ is a Cartan involution of $\mathfrak{g}$ and that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$ with respect to θ. The **rank of** M is said to be the maximum dimension of an abelian subspace of $\mathfrak{p}$. Since θ is an involutive automorphism of $\mathfrak{g}$ we have:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad \text{and} \quad [\mathfrak{p}, \mathfrak{p}].$$

This implies that U vanishes identically, so symmetric spaces are naturally reductive, and their Levi-Civita connection at fundamental vector fields at $o \in M$ is zero. Moreover, the difference tensor D also vanishes identically, thus using Equation (7), the curvature of a symmetric space is given by

$$R_o(X, Y)Z = -[[X, Y], Z], \quad \text{where } X, Y, Z \in \mathfrak{p}. \tag{8}$$

6.3.2 Totally geodesic submanifolds in symmetric spaces

Let Σ be a connected totally geodesic submanifold of a symmetric space $M = \mathsf{G}/\mathsf{K}$. By the homogeneity of M, we can assume without loss of generality that $o \in \Sigma$. By Theorem 4.4 and the fact that symmetric spaces have parallel curvature tensor, a totally geodesic submanifold Σ of M with $o \in \Sigma$ and $V = T_o\Sigma \subset T_oM$ exists if and only if $V \subset T_oM$ is **curvature invariant**. This means that $R_o(V, V)V \subset V$, where R is the Riemannian curvature tensor of M. By Equation (8) we can write the curvature tensor of M at o as

$$R_o(X, Y)Z = -[[X, Y], Z], \quad \text{for } X, Y, Z \in T_oM.$$

Thus, a subspace $V \subset \mathfrak{p}$ is curvature invariant if and only if $[[X, Y], Z] \in V$ for every $X, Y, Z \in V$. A subspace V of $\mathfrak{p}$ with this property is called a **Lie triple system** in $\mathfrak{p}$. Hence, there is a one-to-one correspondence between Lie triple systems V in $\mathfrak{p}$ and complete totally geodesic submanifolds Σ in M containing $o \in M$.

Exercise 6.14 (Totally geodesic submanifolds of the complex projective space $\mathbb{C}\mathsf{P}^n$)**.** *Prove that* $\mathbb{C}\mathsf{P}^n$ *is a symmetric space and use the characterization of totally geodesic submanifolds in terms of Lie triple systems to prove that their totally geodesic submanifolds are either:* $\mathbb{C}\mathsf{P}^k$*, or* $\mathbb{R}\mathsf{P}^k$ *with* $k \leq n$*.*

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